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The rationality problem for four-dimensional linear actions

By

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Abstract

Let G be a finite subgroup of $GL(4, \mathbb{Q})$. Let G act on the rational function field $\mathbb{Q}(x_1, x_2, x_3, x_4)$ by $\mathbb{Q}$-automorphism defined by the linear action of variables. Linear Noether's problem asks whether the fixed field $\mathbb{Q}(x_1, x_2, x_3, x_4)^G$ is rational (i.e. purely transcendental) over $\mathbb{Q}$. So far some partial results have been known, but in this paper we will give the almost complete results of this problem. One of motivations of this problem is the relation to the inverse Galois problem.

1. Introduction

In this paper, we will show the almost complete results of the rationality problem for linear actions (Problem 2 below) for $n = 4$ and $K = \mathbb{Q}$. (See Theorem 1.1.) In this section, we will explain the background of this research. First, we state the rationality problem in general case:

Problem 1. Let K be a field and G be a finite subgroup of $Aut_K K(x_1, \ldots, x_n)$, where $K(x_1, \ldots, x_n)$ is the rational function field with n variables over K. Then is the fixed field $K(x_1, \ldots, x_n)^G$ rational over K ?

In case of $n = 1$, it is well known that Problem 1 is affirmative for arbitrary K and G by Lüroth's theorem. In case of $n = 2$, if K is an algebraically closed field of characteristic 0, then Problem 1 is affirmative for arbitrary G by Zariski-Castelnuovo's theorem. However, if $n \geq 3$, then this problem is very difficult and few general results are known. Note that there are many cases where the results for Problem 1 for two groups are different even if they are isomorphic to each other as abstract groups. For example, the following two actions are both of order 2:

$$\sigma : x_1 \mapsto -x_1, x_2 \mapsto -x_2, x_3 \mapsto -x_3,$$
$$\tau : x_1 \mapsto \frac{a_1}{x_1}, x_2 \mapsto \frac{a_2}{x_2}, x_3 \mapsto \frac{a_3}{x_3},$$

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and the fixed field $K(x_1, x_2, x_3)^\sigma$ is clearly rational over K. However, it is known that $K(x_1, x_2, x_3)^\tau$ is not rational over K if $[K(\sqrt{a_1}, \sqrt{a_2}, \sqrt{a_3}) : K] \geq 8$.[*1]

In this paper, we restrict ourselves to the significant special case of Problem 1, which is often called "linear Noether's problem" and is related to the inverse Galois problem.

Problem 2 (linear Noether's problem). Let K be a field and G be a finite subgroup of $GL(n, K)$. If we define a G-action on $K(x_1, \ldots, x_n)$ by

$$\sigma(x_j) = \sum_{i=1}^{n} a_{ij} x_i \quad (j = 1, \ldots n) \text{ for } \forall \sigma = (a_{ij}) \in G,$$

then is the fixed field $K(x_1, \ldots, x_n)^G$ rational over K ?

Whether Problem 2 has an affirmative answer or not depends on the pair (K, G). In this paper, we will consider the case $K = \mathbb{Q}$, because it is the most difficult and important case. Our main theorem is as follows.

Theorem 1.1. *Problem 2 for $K = \mathbb{Q}$ and $n = 4$ is affirmative for all groups except for 6 conjugacy classes of* (4,26,1), (4,33,2), (4,33,3), (4,33,6), (4,33,7) *and* (4,33,11) *in the GAP code. The problem is negative for* (4,26,1) *and* (4,33,2).

In general, for mutually conjugate subgroups G_1 and G_2 of $GL(n, \mathbb{Q})$, linear Noether's problems are the same by changing variables. So we have only to consider the conjugacy classes of finite subgroups of $GL(n, \mathbb{Q})$ and we have 227 ones in the case of $n = 4$. See [1]. So far some partial results have been known, but in this paper we will give the almost complete results of this problem.

We just mention some relevant results. First linear Noether's problem for $n \leq 3$ has been proven to be affirmative.

Theorem 1.2 (Oura, Rikuna [21]). *If $K = \mathbb{Q}$ and $n = 3$, Problem 2 for any G has an affirmative answer.*

For $K = \mathbb{Q}$ and $n = 4$ it is known that Problem 2 is negative for a group G which is isomorphic to the cyclic group C_8. (See [11].) So the results of the other groups attract the attention of many mathematicians.

Theorem 1.3 (Plans [22], Rikuna [23]). *Problem for $K = \mathbb{Q}$ and $n = 4$ has an affirmative answer if G belongs to the conjugacy class* (4,32,5) *or* (4,32,11) *in the GAP code, which is isomorphic to $SL(2, 3)$ or $GL(2, 3)$ respectively.*

Theorem 1.4 (Kang [12]). *Problem 2 for $K = \mathbb{Q}$ and $n = 4$ has an affirmative answer if G is a non-abelian group of order* 16.[*2]

[*1] This theorem has been proved in more general condition by Saltman [24] and Kang [13].
[*2] The results of this paper [12] contain the case other than $K = \mathbb{Q}$ and $n = 4$.

In the previous paper, we have shown the following result as a generalization of Theorem 1.4.

Theorem 1.5 (Kitayama [17]). *Problem for $K = \mathbb{Q}$ and $n = 4$ has an affirmative answer if G is a 2-group which is not isomorphic to C_8.*

Theorem 1.6 (Yamasaki [25]). *Problem for $K = \mathbb{Q}$ and $n = 4$ has an affirmative answer if G is a nonsolvable or reflection group.*

Our previous results, Theorem 1.5 and 1.6, are the partial results of linear Noether's problem for $K = \mathbb{Q}$ and $n = 4$, which show that the problem has an affirmative answer for 73 conjugacy classes out of 227 ones and a negative answer for only 1 class. In this paper, we will prove Theorem 1.1, which is the almost complete results of linear Noether's problem for $K = \mathbb{Q}$ and $n = 4$.

Finally, we explain one of motivations of Problem 2. It is an application to the inverse Galois problem, particularly an explicit construction of generic polynomials, which are defined as follows.

Definition 1.1 ([15], [19]). A polynomial $f(t_1, \ldots, t_m; X) \in K(t_1, \ldots, t_m)[X]$ is called a generic polynomial for G-extension over K if it satisfies the following conditions:
(1) $\mathrm{Gal}(f(t_1, \ldots, t_m; X)/K(t_1, \ldots, t_m)) \simeq G$,
(2) for every G-extension of infinte fields L/M with $M \supseteq K$,
$L = \mathrm{Spl}_M f(a_1, \ldots, a_m; X)$ for some $a_1, \cdots, a_m \in M$.

Explicit construction of generic polynomials has been studied by many mathematicians as the most important problem in constructive aspects of the inverse Galois problem with the aim of applying it to algebraic number theory (c.f. [11]). Whether a generic polynomial for G-extension over K exists or not depends on the pair (K, G). In fact, there are some cases where it doesn't exist. For example, if $K = \mathbb{Q}$ and G is abelian, then a generic polynomial exists if and only if G has no elements of order 8 (Lenstra [20]). On the other hand, general results for non-abelian groups are not known and few results for each individual groups are known.

In this situation, the explicit affirmative answer of linear Noether's problem for (K, G) produces a generic polynomial explicitly as follows. If linear Noether's problem is affirmative, that is, $K(x_1, \ldots, x_n)^G$ is purely transcendental over K, then there is a K-isomorphism $\Phi : K(x_1, \ldots, x_n)^G \longrightarrow K(t_1, \ldots, t_n)$. Let $f(X) \in K(x_1, \ldots, x_n)^G[X]$ be a defining polynomial of G-extension $K(x_1, \ldots, x_n)/K(x_1, \ldots, x_n)^G$ and define

$$F(t_1, \ldots, t_n; X) := \Phi(f(X)) \in K(t_1, \ldots, t_n)[X].$$

Theorem 1.7 (Kemper, Mattig [16]). *$F(t_1, \ldots, t_n; X)$ defined above is a generic polynomial for G-extension over K.*

Thus linear Noether's problem is the most effective way to construct generic polynomials for groups of small order.

Remark 1. Our motivation to Problem 2 is not only the existence proof and construction of generic polynomials. Note that an affirmative answer of Problem 2 is not a necessary condition for the existence of generic polynomials and it is possible that Problem 2 has different results for two groups which are not conjugate to each other, even if they are isomorphic as abstract groups. So, we should consider all of 227 conjugacy classes of finite subgroups of $GL(4,\mathbb{Q})$.

Notation.
We define the matrices as the following way.

$$c_2 := \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad m := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$c_3 := \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad c_4 := \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\gamma := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad g_1 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$c_5 := \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \quad \sigma_4 := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$s := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad h := \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} /2$$

$$c_2' := \gamma c_2 \gamma\ ,\ m' := \gamma m \gamma\ ,\ c_3' := \gamma c_3 \gamma\ ,\ c_4' := \gamma c_4 \gamma\ ,\ c_8 := \gamma c_4.$$

2. Preliminaries

We recall some results which are used repeatedly throughout this paper.

Theorem 2.1 (Kang, Hajja [8]). *Let L be a field and G a finite group acting on $L(x_1,\ldots,x_m)$, the rational function field of m variables over L. Suppose that*
(1) *for any $\sigma \in G, \sigma(L) \subset L$;*

(2) *the restriction of the action of G to L is faithful*;
(3) *for any $\sigma \in G$,*

$$\begin{pmatrix} \sigma(x_1) \\ \vdots \\ \sigma(x_m) \end{pmatrix} = A(\sigma) \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} + B(\sigma)$$

where $A(\sigma) \in GL(m, L)$ and $B(\sigma)$ is an $m \times 1$ matrix over L.

Then there exist $z_1, ..., z_m \in L(x_1, .., x_m)$ such that $L(x_1, ..., x_m)^G = L^G(z_1, ..., z_m)$.

Let K be a field. An element $f \in K(x_1, \ldots, x_n)$ is called homogeneous of degree d if it can be written as $f = g/h$ with $g, h \in K[x_1, \ldots, x_n]$ homogeneous and $deg(g) - deg(h) = d$. We write

$$K(x_1, \ldots x_n)_0 := \{f \in K(x_1, \ldots, x_n) | f \text{ is homogeneous of degree } 0\} \cup \{0\}.$$

Theorem 2.2 (Kemper [14]). *Let G be a finite subgroup of $GL(n, K)$. If $K(x_1, \ldots, x_n)_0^G$ is rational over K, then $K(x_1, \ldots, x_n)^G$ is rational over K.*

When we use Theorem 2.1 and Theorem 2.2, there are some cases where we can reduce the linear Noether's problem to the following problem.

Problem 2.1 (monomial Noether's problem). Let K be a field and G a finite subgroup of $GL(n, \mathbb{Z})$. Suppose G acts on $K(x_1, \ldots, x_n)$ by

$$\sigma(x_i) := \alpha_i(\sigma) \prod_{j=1}^{n} x_j^{a_{i,j}} \quad , \alpha_i(\sigma) \in K^\times \ , \quad i = 1, \ldots, n \ , \ \forall \sigma = (a_{i,j}) \in G.$$

Then is the fixed field $K(x_1, \ldots, x_n)^G$ rational over K ?

Particularly, the case $\alpha_i(\sigma) = 1$ for $\forall \sigma \in G$ *and* $\forall i$, is called purely monomial Noether's problem.

For this problem, the following theorem holds.

Theorem 2.3 (Hajja, Kang, Hoshi, Rikuna [4, 5, 6, 7, 10]).
(1) *All 2-dimensional monomial Noether's problems are affirmative.*
(2) *All 3-dimensional purely monomial Noether's problems are affirmative.*

The following theorems are often useful.

Theorem 2.4 (Hajja, Kang [7]). *Let K be any field and σ a K-automorphism of $K(x, y)$ defined by $\sigma(x) = a/x, \sigma(y) = b/y, (a, b \in K^\times)$. Then $K(x, y)^{\langle\sigma\rangle} = K(u, v)$ where*

$$u := \frac{x - \frac{a}{x}}{xy - \frac{ab}{xy}} \ , \ v := \frac{y - \frac{b}{y}}{xy - \frac{ab}{xy}}.$$

Let ζ be an n-th root of unity and σ a $\mathbb{Q}(\zeta)$-automorphism of $\mathbb{Q}(\zeta)(x_1,\ldots,x_n)$ defined by $x_1 \mapsto x_2 \mapsto \ldots x_n \mapsto x_1$. For $f \in \mathbb{Q}(\zeta)(x_1,\ldots,x_n)^{\langle\sigma\rangle}$, we define a set $[f]_{conj} := \{$all conjugates of f over $\mathbb{Q}(x_1,\ldots,x_n)^{\langle\sigma\rangle}\}$ and $\iota(f) := \sharp[f]_{conj}$.

Theorem 2.5 (Masuda [18], Hoshi [9])**.** *Suppose that there exist elements $a_1,\ldots,a_t \in \mathbb{Q}(\zeta)(x_1,\ldots,x_n)^{\langle\sigma\rangle}$ such that*
(1) $\mathbb{Q}(\zeta)(x_1,\ldots,x_n)^{\langle\sigma\rangle} = \mathbb{Q}(\zeta)([a_i]_{conj}|i=1,\ldots,t)$
(2) $\sum_{i=1}^{t}\iota(a_i) = n$.
Let $\omega_{i,1},\ldots,\omega_{i,\iota(a_i)}$ be a basis of $\mathbb{Q}(\zeta)\cap\mathbb{Q}(x_1,\ldots,x_n)^{\langle\sigma\rangle}(a_i)/\mathbb{Q}$. Then it is also a basis of $\mathbb{Q}(x_1,\ldots,x_n)^{\langle\sigma\rangle}(a_i)/\mathbb{Q}(x_1,\ldots,x_n)^{\langle\sigma\rangle}$ and if we write

$$a_i = \sum_{j=1}^{\iota(a_i)} m_{i,j}\omega_{i,j} \qquad (m_{i,j} \in \mathbb{Q}(x_1,\ldots,x_n)^{\langle\sigma\rangle}),$$

then $\mathbb{Q}(x_1,\ldots,x_n)^{\langle\sigma\rangle} = \mathbb{Q}(m_{i,j}|i=1,\ldots,t\ ,\ j=1,\ldots,\iota(a_i))$.

3. Main results

3.1. Groups of order 3×2^n

In this subsection, we consider subgroups of $GL(4,\mathbb{Q})$ of order 3×2^n. There are 104 conjugacy classes. Among them, generators of 3-Sylow groups are c_3-type for 79 classes, and c_3c_3'-type for 25 classes. See section 4.

First, we shall study on the former 79 classes.

Theorem 3.1. *Let $G_0 = \langle c_3, c_2, mc_4, c_4', m'\rangle$. If $\langle c_3\rangle \le G \le G_0$, then $\mathbb{Q}(x_1,x_2,x_3,x_4)^G$ is rational over $\mathbb{Q}$. (This theorem covers all of 56 conjugacy classes whose 3-Sylow groups are normal.)*

Proof. G_0 acts on $\mathbb{Q}(x_1,x_2,x_3,x_4)^{\langle c_3\rangle}$, since $\langle c_3\rangle$ is normal in G_0. First we shall determine $\mathbb{Q}(x_1,x_2,x_3,x_4)^{\langle c_3\rangle}$.

c_3 acts trivially on (x_3,x_4), so that $\mathbb{Q}(x_1,x_2,x_3,x_4)^{\langle c_3\rangle} = \big(\mathbb{Q}(x_1,x_2)^{\langle c_3\rangle}\big)(x_3,x_4)$. c_3 acts on (x_1,x_2) as $x_1 \mapsto -x_2 \mapsto -(x_1-x_2) \mapsto x_1$, so the following y_1 and y_2 belong to $\mathbb{Q}(x_1,x_2)^{\langle c_3\rangle}$.

$$y_1 = \frac{1}{x_1} - \frac{1}{x_2} - \frac{1}{x_1-x_2},\ y_2 = \frac{x_2}{x_1} - \frac{x_1-x_2}{x_2} + \frac{x_1}{x_1-x_2}$$

We have $\mathbb{Q}(x_1,x_2) = \mathbb{Q}(x_1,\frac{x_2}{x_1}) = \mathbb{Q}(y_1,\frac{x_2}{x_1})$, and y_2 is a rational function of $\frac{x_2}{x_1}$ with the denominator quadratic and the numerator cubic, so that $[\mathbb{Q}(\frac{x_2}{x_1}):\mathbb{Q}(y_2)] = 3$. Therefore $[\mathbb{Q}(x_1,x_2):\mathbb{Q}(y_1,y_2)] = [\mathbb{Q}(y_1,\frac{x_2}{x_1}):\mathbb{Q}(y_1,y_2)] = 3$, thus we get $\mathbb{Q}(y_1,y_2) = \mathbb{Q}(x_1,x_2)^{\langle c_3\rangle}$.

The action of G_0 on (y_1,y_2,x_3,x_4) is as follows. c_4' and m' act on (x_3,x_4) as usual, while c_2 and mc_4 act on (y_1,y_2) as $c_2 : y_1 \mapsto -y_1,\ y_2 \mapsto y_2$ and $mc_4 : y_1 \mapsto -y_1,\ y_2 \mapsto -y_2+3$ so that $y_2 - \frac{3}{2} \mapsto -(y_2-\frac{3}{2})$. Therefore G_0 acts linearly on $(y_1, y_2-\frac{3}{2}, x_3, x_4)$. It follows that $\mathbb{Q}(x_1,x_2,x_3,x_4)^G$ is rational over $\mathbb{Q}$ by applying the results of linear Noether's problem for order 2^n (Theorem 1.5). □

Theorem 3.2. *If generators of 3-Sylow groups are c_3-type, then $\mathbb{Q}(x_1,x_2,x_3,x_4)^G$ is rational over $\mathbb{Q}$ even when 3-Sylow groups are not normal.*

Proof. There are 23 conjugacy classes in question. The largest group is M_0, the group of all orthogonal matrices whose entries are integers. $\#(M_0) = 384$, and any other group is conjugate to a subgroup of M_0 in $GL(4,\mathbb{Q})$.

The smallest group is $\langle g_1, c_2', mm'\rangle$, which is isomorphic to $\mathfrak{A}_4$. Identifying them, we shall denote this group with $\mathfrak{A}_4$. 21 groups out of 23 ones contain $\mathfrak{A}_4$, but other 2 groups do not.

$\mathbb{Q}(x_1,x_2,x_3,x_4)^G$ is rational if $\mathbb{Q}(x_1,x_2,x_3,x_4)_0^G$ is rational (Theorem 2.2), so we shall discuss the rationality of $\mathbb{Q}(x_1,x_2,x_3,x_4)_0^G$. Note that c_2c_2' acts trivially on $\mathbb{Q}(x_1,x_2,x_3,x_4)_0$. M_0 acts on $\mathbb{Q}(x_1,x_2,x_3,x_4)_0^{\langle c_2',mm'\rangle}$, since $\langle c_2, c_2', mm'\rangle$ is normal in M_0. First we shall determine $\mathbb{Q}(x_1,x_2,x_3,x_4)_0^{\langle c_2',mm'\rangle}$.

Since $\mathbb{Q}(x_1,x_2,x_3,x_4)_0 = \mathbb{Q}(y_1,y_2,y_3)$ where $y_1 = \frac{x_2}{x_1}, y_2 = \frac{x_3}{x_1}, y_3 = \frac{x_4}{x_1}$, and since c_2' acts as $y_1 \mapsto y_1$, $y_2 \mapsto -y_2$, $y_3 \mapsto -y_3$ and mm' acts as $y_1 \mapsto -y_1$, $y_2 \mapsto y_2$, $y_3 \mapsto -y_3$, we have $\mathbb{Q}(x_1,x_2,x_3,x_4)_0^{\langle c_2',mm'\rangle} = \mathbb{Q}(z_1,z_2,z_3)$ where $z_1 = \frac{y_2y_3}{y_1} = \frac{x_3x_4}{x_1x_2}$, $z_2 = \frac{y_1y_3}{y_2} = \frac{x_2x_4}{x_1x_3}$, $z_3 = \frac{y_1y_2}{y_3} = \frac{x_2x_3}{x_1x_4}$.

The normalizer of $\mathfrak{A}_4$ in M_0 is $M_1 := \langle \mathfrak{A}_4, c_2c_2', mc_2', m'c_4'\rangle$ and we have $M_0 = \langle M_1, \gamma\rangle$. The action of M_0 on (z_1,z_2,z_3) is as follows.

$$
\begin{aligned}
g_1 &: \quad z_1 \mapsto z_2 \mapsto z_3 \mapsto z_1 \\
m'c_4' &: \quad z_1 \mapsto z_1,\ z_2 \leftrightarrow z_3 \\
mc_2' &: \quad z_1 \mapsto -z_1,\ z_2 \mapsto -z_2,\ z_3 \mapsto -z_3 \\
\gamma &: \quad z_1 \mapsto \tfrac{1}{z_1},\ z_2 \mapsto z_2,\ z_3 \mapsto \tfrac{1}{z_3}
\end{aligned}
$$

From this, we observe that M_1 acts linearly on (z_1,z_2,z_3). Thus if $\mathfrak{A}_4 \le G \le M_1$, then $\mathbb{Q}(x_1,x_2,x_3,x_4)^G$ is rational over $\mathbb{Q}$ by applying the results of 3-dimensional linear Noether's problem (Theorem 1.2).

Since $M_1/\mathfrak{A}_4 \simeq C_2 \times C_2 \times C_2$, there are 16 subgroups of M_1 containing $\mathfrak{A}_4$. Any two of them are not conjugate in $GL(4,\mathbb{Q})$, so that we have settled the problem for 16 conjugacy classes.

Let $M_2 := \langle \mathfrak{A}_4, \gamma\rangle$. We have $[M_2 : \mathfrak{A}_4] = 8$ and the 2-Sylow group of M_2 is obtained from that of $\mathfrak{A}_4$ by adding c_2c_2', γ and $g_1\gamma g_1^{-1}$. This 2-Sylow group is normal in M_0, so M_0 acts on $\mathbb{Q}(z_1,z_2,z_3)^{\langle \gamma, g_1\gamma g_1^{-1}\rangle}$.

We see that $g_1\gamma g_1^{-1}$ acts as $z_1 \mapsto \frac{1}{z_1}$, $z_2 \mapsto \frac{1}{z_2}$, $z_3 \mapsto z_3$, therefore $\mathbb{Q}(z_1,z_2,z_3)^{\langle \gamma, g_1\gamma g_1^{-1}\rangle} = \mathbb{Q}(\frac{w_2w_3}{w_1}, \frac{w_1w_3}{w_2}, \frac{w_1w_2}{w_3})$ where $w_1 = \frac{z_1-1}{z_1+1}, w_2 = \frac{z_2-1}{z_2+1}$ and $w_3 = \frac{z_3-1}{z_3+1}$. We can easily check that all of g_1, $m'c_4'$ and mc_2' act purely monomially on $(\frac{w_2w_3}{w_1}, \frac{w_1w_3}{w_2}, \frac{w_1w_2}{w_3})$, so that if $M_2 \le G \le M_0$, then $\mathbb{Q}(x_1,x_2,x_3,x_4)^G$ is rational over $\mathbb{Q}$. Since $M_0/M_2 \simeq C_2 \times C_2$, there are 5 subgroups of M_0 containing M_2. Any two of them are not conjugate in $GL(4,\mathbb{Q})$. Therefore we have 5 more conjugacy classes for which linear Noether's problem is affirmative. Now, the problem has been settled for 21 conjugacy classes.

Remaining 2 groups do not contain $\mathfrak{A}_4$, and isomorphic to $GL(2,3)$ and $SL(2,3)$ respectively. It is already known that linear Noether's problem is affirmative for these groups (Rikuna [23]). □

Next, we shall study the case where generators of 3-Sylow groups are c_3c_3'-type.

Theorem 3.3. *Let $G_1 = \langle c_3c_3', c_2, c_2', mc_4m'c_4', \gamma\rangle$. If $\langle c_3c_3'\rangle \leq G \leq G_1$, then $\mathbb{Q}(x_1, x_2, x_3, x_4)^G$ is rational over $\mathbb{Q}$.*

(This theorem covers 15 conjugacy classes out of 18 ones whose 3-Sylow groups are normal).

To prove this theorem, we need the following lemma.

Lemma 3.1 (Rikuna [23]). *Define $\mathbb{Q}$-automorphism ρ over $\mathbb{Q}(s,t)$, the rational function field of two variabes, by $\rho : s \mapsto t$, $t \mapsto \frac{1}{st}$. Then $\mathbb{Q}(s,t)^\rho = \mathbb{Q}(u,v)$ where*

$$u = \frac{t\big(s^3t^3 - s^2(3t-s)+1\big)}{s^2t^2(s^2-st+t^2) - st(s+t)+1}, \quad v = \frac{s\big(s^3t^3 - t^2(3s-t)+1\big)}{s^2t^2(s^2-st+t^2) - st(s+t)+1}.$$

Proof of Theorem 3.3. G_1 acts on $\mathbb{Q}(x_1,x_2,x_3,x_4)^{\langle c_3c_3'\rangle}$, since $\langle c_3c_3'\rangle$ is normal in G_1.

Just as in the proof of Theorem 3.1, we have $\mathbb{Q}(x_1,x_2,x_3,x_4) = \mathbb{Q}(y_1, y_3, \frac{x_2}{x_1}, \frac{x_4}{x_3})$ where y_1 is the same as in the proof of Theorem 3.1 and y_3 is the γ-image of y_1. Since c_3c_3' acts trivially on (y_1,y_3), we have $\mathbb{Q}(y_1, y_3, \frac{x_2}{x_1}, \frac{x_4}{x_3})^{\langle c_3c_3'\rangle} = \big(\mathbb{Q}(\frac{x_2}{x_1}, \frac{x_4}{x_3})^{\langle c_3c_3'\rangle}\big)(y_1,y_3)$.

c_3c_3' acts as $\frac{x_2}{x_1} \mapsto -\frac{x_1-x_2}{x_2} = 1 - \frac{x_1}{x_2}$ and similarly on $\frac{x_4}{x_3}$, so that putting $s = (\frac{x_2}{x_1})/(\frac{x_4}{x_3}) = \frac{x_2x_3}{x_1x_4}$, c_3c_3' maps s to $t = (1-\frac{x_1}{x_2})/(1-\frac{x_3}{x_4}) = \frac{1}{s}(\frac{x_2}{x_1}-1)/(\frac{x_4}{x_3}-1)$. From this, we see that $\mathbb{Q}(\frac{x_2}{x_1}, \frac{x_4}{x_3}) = \mathbb{Q}(s,t)$ and c_3c_3' acts as $s \mapsto t \mapsto \frac{1}{st} \mapsto s$. Using Lemma 3.1, we get

$$\mathbb{Q}(\frac{x_2}{x_1}, \frac{x_4}{x_3})^{\langle c_3c_3'\rangle} = \mathbb{Q}(u,v)$$

where u, v are given in Lemma 3.1.

It is easy to verify that G_1 acts on $\mathbb{Q}(y_1,y_3)$ and $\mathbb{Q}(u,v)$ separately, and the action is linear on $\mathbb{Q}(y_1,y_3)$, namely $c_2 : y_1 \mapsto -y_1, y_3 \mapsto y_3$, $c_2' : y_1 \mapsto y_1, y_3 \mapsto -y_3$, $mc_4m'c_4' : y_1 \mapsto -y_1, y_3 \mapsto -y_3$ and $\gamma : y_1 \leftrightarrow y_3$.

On the other hand, c_2 and c_2' act trivially on (u,v). The action of $mc_4m'c_4'$ and γ are rather complicated. Both of $mc_4m'c_4'$ and γ act as $s \mapsto \frac{1}{s}$. Since γ commutes with c_3c_3', γ acts as $t \mapsto \frac{1}{t}$, and since the conjugate of c_3c_3' by $mc_4m'c_4'$ is $(c_3c_3')^2$, $mc_4m'c_4'$ acts as $t \mapsto st$. Replacing s by $r = st$, the actions are $mc_4m'c_4' : r \leftrightarrow t$ and $\gamma : r \mapsto \frac{1}{r}, t \mapsto \frac{1}{t}$.

u and v are written as functions of r and t as follows

$$u = \frac{r^3t^3 - 3r^2t^2 + r^3 + t^3}{r^2(r^2 - rt^2 + t^4) - rt(r+t^2) + t^2}, \quad v = \frac{r^4t - 3r^2t^2 + rt^4 + rt}{r^2(r^2 - rt^2 + t^4) - rt(r+t^2) + t^2}.$$

From this we observe that $mc_4m'c_4'$ keeps $\frac{v}{u}$ invariant, γ maps $\frac{v}{u}$ to $\frac{u}{v}$ and $mc_4m'c_4'\gamma$ acts as $u \leftrightarrow v$. Especially $\langle c_2, c_2', mc_4m'c_4'\gamma\rangle$ acts linearly on (y_1, y_3, u, v), so that if $\langle c_3c_3'\rangle \leq G \leq \langle c_3c_3', c_2, c_2', mc_4m'c_4'\gamma\rangle$, then

$\mathbb{Q}(x_1,x_2,x_3,x_4)^G$ is rational over $\mathbb{Q}$ by the results of linear Noether's problem of order 2^n. This condition on G is satisfied by 8 conjugacy classes, for which the problem has been settled.

Next, we shall show that by a suitable change of generators of $\mathbb{Q}(u,v)$, γ acts linearly on the new generators. From the facts observed above, we see that $mc_4m'c'_4$ acts as $u\mapsto \frac{B}{v}, v\mapsto \frac{B}{u}$ and γ acts as $u\mapsto \frac{B}{u}, v\mapsto\frac{B}{v}$ for some $B\in\mathbb{Q}(u,v)$. By a computer calculation, we see that $B=\frac{uv}{u^2-uv+v^2}$.

Let $w=\frac{u-v}{u+v}$, then $\mathbb{Q}(u,v)=\mathbb{Q}(u,w)$ and γ maps w to $-w$. Since γ maps u to $\frac{B}{u}=\frac{1}{u}\frac{\frac{v}{u}}{1-\frac{v}{u}+(\frac{v}{u})^2}=\frac{1}{u}\frac{1-w^2}{1+3w^2}$, putting $U=\frac{1+3w^2}{w+1}u$, we have $\mathbb{Q}(u,w)=\mathbb{Q}(U,w)$ and γ maps U to $\frac{1+3w^2}{U}$.

Extend the coefficient field from $\mathbb{Q}$ to $\mathbb{Q}(\sqrt{-3})$, then since $3w^2+1=(\sqrt{-3}w+1)(-\sqrt{-3}w+1)$, we have $\gamma: V\mapsto -V$ for $V=\frac{U-\sqrt{-3}w-1}{U+\sqrt{-3}w+1}$. Denote the conjugate on $\mathbb{Q}(\sqrt{-3})/\mathbb{Q}$ with $\bar{\ }$, then $\overline{V}=\frac{U+\sqrt{-3}w-1}{U-\sqrt{-3}w+1}$ also satisfies $\gamma:\overline{V}\mapsto-\overline{V}$ and $(V,\overline{V})$ become generators of $\mathbb{Q}(\sqrt{-3})(U,w)$. Thus $V+\overline{V}$ and $\frac{V-\overline{V}}{\sqrt{-3}}$ become generators of $\mathbb{Q}(U,w)$ and γ acts as $V\pm\overline{V}\mapsto-(V\pm\overline{V})$.

From this we see that if $\langle c_3c'_3\rangle\le G\le\langle c_3c'_3,c_2,c'_2,\gamma\rangle$, then $\mathbb{Q}(x_1,x_2,x_3,x_4)^G$ is rational over $\mathbb{Q}$. This condition on G is satisfied by 2 more conjugacy classes, for which the problem has been settled.

Next, we shall study the case where $mc_4m'c'_4\in G$. G_1 acts on $\mathbb{Q}(y_1,y_3,u,v)^{\langle mc_4m'c'_4\rangle}$, since $\langle c_3c'_3,mc_4m'c'_4\rangle$ is normal in G_1. We shall determine $\mathbb{Q}(y_1,y_3,u,v)^{\langle mc_4m'c'_4\rangle}$.

Define $\alpha=u+\frac{B}{v}$ and $\beta=v+\frac{B}{u}$, then both α and β belong to $\mathbb{Q}(u,v)^{\langle mc_4m'c'_4\rangle}$ and $\frac{\beta}{\alpha}=\frac{v}{u}$, so that $\mathbb{Q}(u,v)=\mathbb{Q}(u,\frac{\beta}{\alpha})$. Since $\alpha=u+\frac{1}{v}\frac{uv}{u^2-uv+v^2}=u+\frac{1}{u}\frac{1}{1-\frac{v}{u}+(\frac{v}{u})^2}$, we have $[\mathbb{Q}(u,\frac{\beta}{\alpha}):\mathbb{Q}(\alpha,\frac{\beta}{\alpha})]=2$ so that $[\mathbb{Q}(u,v):\mathbb{Q}(\alpha,\beta)]=2$. This implies $\mathbb{Q}(\alpha,\beta)=\mathbb{Q}(u,v)^{\langle mc_4m'c'_4\rangle}$, therefore $\mathbb{Q}(y_1,y_3,u,v)^{\langle mc_4m'c'_4\rangle}=\mathbb{Q}(a,b,\alpha,\beta)$ where $a=y_1(u-\frac{B}{v}), b=y_3(v-\frac{B}{u})$.

G_1 acts linearly on (a,b,α,β) as follows. c_2 keeps b,α and β invariant and maps a to $-a$. c'_2 keeps a,α and β invariant and maps b to $-b$. γ acts as $\alpha\leftrightarrow\beta$ and $a\leftrightarrow -b$.

Therefore if $\langle c_3c'_3,mc_4m'c'_4\rangle\le G\le G_1$, then $\mathbb{Q}(x_1,x_2,x_3,x_4)^G$ is rational over $\mathbb{Q}$. This condition on G is satisfied by 4 more conjugacy classes, for which the problem has been settled. The number of solved classes is now $8+2+4=14$.

The only remained class is $G=\langle c_3c'_3,\gamma c_2,mc_4m'c'_4\gamma\rangle$. Since $c_2c'_2\in G$, G acts on $\mathbb{Q}(u,v,y_1,y_3)^{\langle c_2c'_2\rangle}$ which is equal to $\mathbb{Q}(u,v,z_1,z_2)$ where $z_1=y_1^2+y_3^2$, $z_2=\frac{y_3}{y_1}$. $mc_4m'c'_4\gamma$ acts as $u\leftrightarrow v$, $y_1\mapsto -y_3$, $y_3\mapsto -y_1$ so that it keeps z_1 invariant and maps z_2 to $\frac{1}{z_2}$. Put $u_1=\frac{u-v}{u+v}$, $v_1=u+v$, $\zeta_2=\frac{z_2-1}{z_2+1}$, then $mc_4m'c'_4\gamma$ acts as $u_1\mapsto -u_1$, $v_1\mapsto v_1$, $\zeta_2\mapsto-\zeta_2$, so that we have $\mathbb{Q}(u,v,z_1,z_2)^{\langle mc_4m'c'_4\gamma\rangle}=\mathbb{Q}(u_1^2,v_1,z_1,u_1\zeta_2)$.

Now, we shall check the action of γc_2 on $(u_1^2,v_1,z_1,u_1\zeta_2)$. γc_2 acts as $u_1\mapsto -u_1$ so that $u_1^2\mapsto u_1^2$, $v_1\mapsto\frac{B}{u}+\frac{B}{v}=\frac{u+v}{uv}B=\frac{u+v}{u^2-uv+v^2}=\frac{A}{v_1}$ where $A=\frac{4}{3u_1^2+1}$, $z_1\mapsto z_1$, $\zeta_2\mapsto-\frac{1}{\zeta_2}$ so that $u_1\zeta_2\mapsto\frac{u_1}{\zeta_2}=\frac{u_1^2}{u_1\zeta_2}$. Therefore γc_2 acts monomially on $\big(\mathbb{Q}(u_1^2,z_1)\big)(v_1,u_1\zeta_2)$, the rational function field over $\mathbb{Q}(u_1^2,z_1)$

with two variables $v_1, u_1\zeta_2$. This assures the rationality of $\mathbb{Q}(x_1, x_2, x_3, x_4)^G$ by Theorem 2.3(1). $\square$

Together with 7 conjugacy classes whose 3-Sylow groups are not normal, there remain 10 conjugacy classes not yet studied. We shall discuss on these groups later in subsection 3.4.

3.2. Groups of order 9×2^n

In this subsection, we shall consier the subgroups of $GL(4, \mathbb{Q})$ of order 9×2^n. There are 45 conjugacy classes. (See Chapter 4).

Theorem 3.4. *Let* $G_2 = \langle c_3, c'_3, c_2, c'_2, mc_4, m'c'_4, \gamma\rangle$. *If* $\langle c_3, c'_3\rangle \leq G \leq G_2$, *then* $\mathbb{Q}(x_1, x_2, x_3, x_4)^G$ *is rational over* $\mathbb{Q}$. (*This theorem covers all of* 39 *conjugacy classes whose* 3*-Sylow groups are normal*).

Proof. G_2 acts on $\mathbb{Q}(x_1, x_2, x_3, x_4)^{\langle c_3, c'_3\rangle}$, since $\langle c_3, c'_3\rangle$ is normal in G_2. First we shall determine $\mathbb{Q}(x_1, x_2, x_3, x_4)^{\langle c_3, c'_3\rangle}$.

Just as in the proof of Theorem 3.1, we have $\mathbb{Q}(x_1, x_2, x_3, x_4)^{\langle c_3\rangle} = \mathbb{Q}(y_1, y_2, x_3, x_4)$, where y_1 and y_2 are the same as in the proof of Theorem 3.1. Since c'_3 acts on (x_3, x_4) in the same way as c_3 on (x_1, x_2), we obtain $\mathbb{Q}(x_1, x_2, x_3, x_4)^{\langle c_3, c'_3\rangle} = \mathbb{Q}(y_1, y_2, y_3, y_4)$ where y_3 and y_4 are the γ-image of y_1 and y_2 respectively.

c_2 and mc_4 act trivially on (y_3, y_4) and act on (y_1, y_2) as $c_2 : y_1 \mapsto -y_1, y_2 \mapsto y_2$ and $mc_4 : y_1 \mapsto -y_1, y_2 \mapsto -y_2 + 3$. c'_2 and $m'c'_4$ act trivially on (y_1, y_2) and act on (y_3, y_4) in the same way as c_2 and mc_4 on (y_1, y_2). γ acts as $y_1 \leftrightarrow y_3, y_2 \leftrightarrow y_4$.

Thus G_2 acts linearly on $(y_1, y_2 - \frac{3}{2}, y_3, y_4 - \frac{3}{2})$. It follows that $\mathbb{Q}(x_1, x_2, x_3, x_4)^G$ is rational over $\mathbb{Q}$ by applying the results of linear Noether's problem for the groups of order 2^n (Theorem 1.5). $\square$

We shall study later in subsection 3.4 on the remaining 6 conjugacy classes whose 3-Sylow groups are not normal.

3.3. Groups of order $5n$

In this subsection, we shall consider the subgroups of $GL(4, \mathbb{Q})$ of order $5n$. There are 11 conjugacy classes (see Chapter 4). Among them, 6 classes contain $\langle c_5\rangle$ as normal subgroups, and 5 classes are non-solvable. It is already known that linear Noether's problem is affirmative for the latter 5 non-solvable classes. (Yamasaki [25]).

The former 6 classes are as follows.

$$\langle c_5\rangle \simeq C_5, \langle c_5, c_2c'_2\rangle \simeq C_{10}, \langle c_5, mc_4m'c'_4\gamma\rangle \simeq D_5,$$
$$\langle c_5, c_2c'_2, mc_4m'c'_4\gamma\rangle \simeq D_{10}, \langle c_5, \sigma_4\rangle \simeq C_4 \ltimes C_5, \langle c_5, \sigma_4, c_2c'_2\rangle \simeq C_4 \ltimes C_{10}.$$

Here c_5 acts as $x_1 \mapsto x_2 \mapsto x_3 \mapsto x_4 \mapsto -(x_1 + x_2 + x_3 + x_4) \mapsto x_1$ and σ_4 acts as $x_1 \mapsto x_2 \mapsto x_4 \mapsto x_3 \mapsto x_1$.

Theorem 3.5. *If G is one of the above six groups, then $\mathbb{Q}(x_1, x_2, x_3, x_4)^G$ is rational over $\mathbb{Q}$.*

Proof. First we shall determine $\mathbb{Q}(x_1, x_2, x_3, x_4)^{\langle c_5\rangle}$. Extend the coefficient field from $\mathbb{Q}$ to $\mathbb{Q}(\zeta)$, where ζ is a primitive fifth root of unity. Then we have $\mathbb{Q}(\zeta)(x_1, x_2, x_3, x_4) = \mathbb{Q}(\zeta)(y_1, y_2, y_3, y_4)$ where

$$y_j = x_1 + \zeta^j x_2 + \zeta^{2j} x_3 + \zeta^{3j} x_4 - \zeta^{4j}(x_1 + x_2 + x_3 + x_4).$$

Since c_5 acts as $y_j \mapsto \zeta^{-j} y_j$, $\mathbb{Q}(\zeta)(y_1, y_2, y_3, y_4)^{\langle c_5\rangle}$ is generated by $\{y_1^{n_1} y_2^{n_2} y_3^{n_3} y_4^{n_4} | (n_1, n_2, n_3, n_4) \in L\}$ where $L = \{(n_j) \in \mathbb{Z}^4 | \sum j n_j \equiv 0 \mod 5\}$. L is generated by $\{(-1, 1, 0, 1), (0, -1, 1, 1), (1, 1, -1, 0), (1, 0, 1, -1)\}$, so that $\mathbb{Q}(\zeta)(x_1, x_2, x_3, x_4)^{\langle c_5\rangle} = \mathbb{Q}(\zeta)(z_1, z_2, z_3, z_4)$ where $z_1 = \frac{y_2 y_4}{y_1}$, $z_2 = \frac{y_3 y_4}{y_2}$, $z_3 = \frac{y_1 y_2}{y_3}$, $z_4 = \frac{y_1 y_3}{y_4}$.

Let τ be the automorphism of $\mathbb{Q}(\zeta)/\mathbb{Q}$ defined by $\zeta \mapsto \zeta^2$. Then, τ maps y_j to y_{2j} (identifying y_{2j} with $y_{j'}$ such that $2j \equiv j' \mod 5$), so that τ acts as $z_1 \mapsto z_2 \mapsto z_4 \mapsto z_3 \mapsto z_1$ therefore $\{z_j\}$ are mutually conjugate in $\mathbb{Q}(\zeta)(x_1, x_2, x_3, x_4)/\mathbb{Q}(x_1, x_2, x_3, x_4)$.

Now from Theorem 2.5, we see that if $z_1 = m_1\zeta + m_2\zeta^2 + m_3\zeta^3 + m_4\zeta^4$, $m_j \in \mathbb{Q}(x_1, x_2, x_3, x_4)^{\langle c_5\rangle}$, then $\mathbb{Q}(x_1, x_2, x_3, x_4)^{\langle c_5\rangle} = \mathbb{Q}(m_1, m_2, m_3, m_4)$. We shall show that all of $c_2c_2', mc_4m'c_4'\gamma$ and σ_4 act linearly on (m_1, m_2, m_3, m_4).

c_2c_2' acts as $y_j \mapsto -y_j$, so that $z_j \mapsto -z_j$, therefore $m_j \mapsto -m_j$. $mc_4m'c_4'\gamma$ acts as $y_j \mapsto \zeta^{3j} y_{4j}$ so that $z_1 \leftrightarrow z_4, z_2 \leftrightarrow z_3$, therefore $m_1 \leftrightarrow m_4, m_2 \leftrightarrow m_3$. σ_4 acts as $y_j \mapsto \zeta^{2j} y_{3j}$, so that $z_1 \mapsto z_3 \mapsto z_4 \mapsto z_2 \mapsto z_1$ therefore $m_1 \mapsto m_2 \mapsto m_4 \mapsto m_3 \mapsto m_1$.

This shows that G-action on (m_1, m_2, m_3, m_4) is linear and thus $\mathbb{Q}(x_1, x_2, x_3, x_4)^G$ is rational over $\mathbb{Q}$. □

3.4. Groups of Crystal system 33

In this subsection, we shall consider the remaining 16 conjugacy classes which belong to the crystal system 33. The actions of these groups are much more complicated than ones of other crystal systems, so hard calculations are required for the study of them.

The largest group in question is $\widetilde{M_0}$, the group of all orthogonal matrices whose entries are half-integers. $\#(\widetilde{M_0}) = 1152$, and any other group is conjugate to a subgroup of $\widetilde{M_0}$

15 classes out of 16 ones contain $Q = \langle \boldsymbol{i}, \boldsymbol{j}\rangle$, which is isomorphic to the quaternion group, where $\boldsymbol{i} = c_4c_4', \boldsymbol{j} = mm'c_2'\gamma$, $\boldsymbol{i}^2 = \boldsymbol{j}^2 = -1$ $(= c_2c_2')$, $\boldsymbol{ij} = -\boldsymbol{ji}$ (= put $\boldsymbol{k}$). The conjugation by m' maps Q to $Q_1 = \langle \boldsymbol{i}_1, \boldsymbol{j}_1\rangle$, where $\boldsymbol{i}_1 = c_4c_4'c_2'$, $\boldsymbol{j}_1 = c_2'\gamma$. We see that $Q \cap Q_1 = \{\pm 1\}$, and Q and Q_1 commute with each other.

An element of order 3 is given by $\alpha = -\frac{1}{2}(1+\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k})$, and the conjugaction by α maps $\boldsymbol{i} \mapsto \boldsymbol{j} \mapsto \boldsymbol{k} \mapsto \boldsymbol{i}$. Similarly $\alpha_1 = -\frac{1}{2}(1 + \boldsymbol{i}_1 + \boldsymbol{j}_1 + \boldsymbol{k}_1)$, where $\boldsymbol{k_1} = \boldsymbol{i_1 j_1}$, is also of order 3, and acts on Q_1, just in the same way as α on Q.

$\alpha_0 = \frac{1}{2}(\boldsymbol{i}+\boldsymbol{j})(\boldsymbol{i}_1+\boldsymbol{j}_1)$ is of order 2, and the conjugation by α_0 maps $\boldsymbol{i} \leftrightarrow \boldsymbol{j}$, $\boldsymbol{k} \leftrightarrow -\boldsymbol{k}$, $\boldsymbol{i}_1 \leftrightarrow \boldsymbol{j}_1$, $\boldsymbol{k}_1 \leftrightarrow -\boldsymbol{k}_1$, so that the conjugation by $\boldsymbol{k}\alpha_0$ maps α to α^2.

The only class which does not contain Q is $G=\langle\alpha, \boldsymbol{k}\boldsymbol{i}_1\alpha_0\rangle$. $\langle\alpha\rangle$ is normal in G and $(\boldsymbol{k}\boldsymbol{i}_1\alpha_0)^2=\boldsymbol{k}_1$ so that $\boldsymbol{k}\boldsymbol{i}_1\alpha_0$ is of order 8, thus $G\simeq C_8\ltimes C_3$. But the linear Noether's problem is negative for this group by the following reason. If $\mathbb{Q}(x_1,x_2,x_3,x_4)^G$ is rational over $\mathbb{Q}$, then Proposition 5.1.7 and Theorem 5.2.5 in [11] lead to the existence of a generic polynomial for $(\mathbb{Q},C_8)$, which contradicts to the well-known result of [20]. For detail, see Chapter 5 of [11].

The other 15 classes contain Q. The normalizer of Q in $\widetilde{M}_0$ is $\widetilde{M}_1=\langle Q,Q_1,\alpha,\alpha_1,\alpha_0\rangle$, and we have $\widetilde{M}_0=\langle\widetilde{M}_1,m'\rangle$. First we shall find suitable generators of $\mathbb{Q}(x_1,x_2,x_3,x_4)_0^Q$ on which $\widetilde{M}_1$ acts in a simple way.

$\widetilde{M}_0$ acts linearly on P_2, the space of all homogeneous quadratic polynomials. P_2 is ten dimensional, and spanned by $\{x_ix_j|1\le i,j\le 4\}$. Since $\langle Q,Q_1\rangle/\{\pm1\}$ is commutative, $\boldsymbol{i},\boldsymbol{j},\boldsymbol{i}_1$ and $\boldsymbol{j}_1$ are simultaneously diagonalized, and P_2 is decomposed into the direct sum of one dimensional spaces as follows.

$w_0=x_1^2+x_2^2+x_3^2+x_4^2,$	invariant by all of $\boldsymbol{i},\boldsymbol{j},\boldsymbol{i}_1,\boldsymbol{j}_1$
$y_{11}=\frac{1}{2}(x_1^2+x_2^2-x_3^2-x_4^2),$	invariant by $\boldsymbol{i}$ and $\boldsymbol{i}_1$
$y_{12}=x_1x_4-x_2x_3,$	invariant by $\boldsymbol{i}$ and $\boldsymbol{j}_1$
$y_{13}=x_1x_3+x_2x_4,$	invariant by $\boldsymbol{i}$ and $\boldsymbol{k}_1$
$y_{21}=x_1x_4+x_2x_3,$	invariant by $\boldsymbol{j}$ and $\boldsymbol{i}_1$
$y_{22}=\frac{1}{2}(x_1^2+x_3^2-x_2^2-x_4^2),$	invariant by $\boldsymbol{j}$ and $\boldsymbol{j}_1$
$y_{23}=x_1x_2-x_3x_4,$	invariant by $\boldsymbol{j}$ and $\boldsymbol{k}_1$
$y_{31}=x_1x_3-x_2x_4,$	invariant by $\boldsymbol{k}$ and $\boldsymbol{i}_1$
$y_{32}=x_1x_2+x_3x_4,$	invariant by $\boldsymbol{k}$ and $\boldsymbol{j}_1$
$y_{33}=\frac{1}{2}(x_1^2+x_4^2-x_2^2-x_3^2),$	invariant by $\boldsymbol{k}$ and $\boldsymbol{k}_1$

From this and the previous observations on conjugations, we have

α: $y_{11}\mapsto y_{21}\mapsto y_{31}\mapsto y_{11}$, $y_{12}\mapsto -y_{22}\mapsto y_{32}\mapsto y_{12}$, $y_{13}\mapsto -y_{23}\mapsto -y_{33}\mapsto y_{13}$,
α_1: $y_{11}\mapsto -y_{12}\mapsto y_{13}\mapsto y_{11}$, $y_{21}\mapsto y_{22}\mapsto -y_{23}\mapsto y_{21}$, $y_{31}\mapsto -y_{32}\mapsto -y_{33}\mapsto y_{31}$,
α_0: $y_{11}\leftrightarrow y_{22}$, $y_{33}\mapsto y_{33}$, $y_{12}\leftrightarrow -y_{21}$, $y_{13}\leftrightarrow y_{23}$, $y_{31}\leftrightarrow y_{32}$
m': $y_{11}\mapsto y_{11}$, $y_{22}\mapsto y_{22}$, $y_{33}\mapsto y_{33}$, $y_{12}\leftrightarrow -y_{21}$, $y_{13}\leftrightarrow y_{31}$, $y_{23}\leftrightarrow y_{32}$
(the sign ±1 is calculated separately).

We shall determine $\mathbb{Q}(x_1,x_2,x_3,x_4)_0^Q$. First note that $\mathbb{Q}(x_1,x_2,x_3,x_4)_0=\mathbb{Q}(z_1,z_2,z_3)$ where $z_1=\frac{x_2}{x_1}$, $z_2=\frac{x_4}{x_3}$, $z_3=\frac{x_3}{x_1}$. Put $u_1=y_{13}/y_{12}$, then $u_1\in\mathbb{Q}(x_1,x_2,x_3,x_4)_0^Q$ and $u_1=(1+z_1z_2)/(z_2-z_1)$ implies $\mathbb{Q}(z_1,z_2)=\mathbb{Q}(z_1,u_1)$. $u_2=y_{23}/y_{22}$ also belongs to $\mathbb{Q}(x_1,x_2,x_3,x_4)_0^Q$ and $u_2=(z_1-z_3^2z_2)/\frac{1}{2}(1+z_3^2-z_1^2-z_3^2z_2^2)$ implies $\mathbb{Q}(z_1,z_2,z_3^2)=\mathbb{Q}(z_1,z_2,u_2)=\mathbb{Q}(z_1,u_1,u_2)$. Since $\mathbb{Q}(z_1,z_2,z_3^2)=\mathbb{Q}(z_1,z_2,z_3)^{\langle c_2'\rangle}$ and Q acts on it, $\mathbb{Q}(z_1,z_2,z_3)^{\langle Q,c_2'\rangle}=\mathbb{Q}(z_1,u_1,u_2)^Q=\big(\mathbb{Q}(u_1,u_2)(z_1)\big)^Q$, so that by Lüroth's theorem, we get $\mathbb{Q}(z_1,z_2,z_3)^{\langle Q,c_2'\rangle}=\mathbb{Q}(u_1,u_2,a)$ where $a=\mathrm{Tr}_Q(z_1)=z_1-\frac{1}{z_1}-z_2+\frac{1}{z_2}$. ($\mathrm{Tr}_Q=\mathrm{Tr}_{\mathbb{Q}(z_1,z_2,z_3)/\mathbb{Q}(z_1,z_2,z_3)^Q}$). But a computer calculation shows that

$$a=\frac{-4(u_2+u_3)u_1}{u_2u_1^2+u_1(u_2u_3+1)-u_2}\ \text{for } u_1=\frac{y_{13}}{y_{12}},\ u_2=\frac{y_{23}}{y_{22}},\ u_3=\frac{y_{33}}{y_{32}},$$

thus we get $\mathbb{Q}(z_1,z_2,z_3)^{\langle Q,\boldsymbol{i}_1\rangle}=\mathbb{Q}(u_1,u_2,u_3)$. (Note that $\langle Q,c_2'\rangle=\langle Q,\boldsymbol{i}_1\rangle$).

This field has the index 2 in $\mathbb{Q}(z_1,z_2,z_3)^Q$, and the result of another computer calculation:

$$u_3=\frac{(u_1u_2+1)w^2+u_1^2}{u_1\{(u_1u_2+1)w^2+u_1u_2\}}\text{ for } w=\frac{y_{13}}{y_{11}}$$

shows that $\mathbb{Q}(u_1,u_2,u_3)=\mathbb{Q}(u_1,u_2,w^2)$, so we have $\mathbb{Q}(z_1,z_2,z_3)^Q=\mathbb{Q}(u_1, u_2,w)$.

As seen here, the action of α on $\mathbb{Q}(z_1,z_2,z_3)^{\langle Q,\boldsymbol{i}_1\rangle}$ is much simpler than the action on $\mathbb{Q}(z_1,z_2,z_3)^Q$, so we shall assume $\boldsymbol{i}_1\in G$.

Theorem 3.6. *Let* $G_0=\langle Q,\alpha,\boldsymbol{i}_1\rangle$ *and* $\widetilde{M}_2=\langle G_0,\boldsymbol{j}_1,\alpha_0\alpha_1\rangle$. *If* $G_0\le G\le\widetilde{M}_2$, *then* $\mathbb{Q}(x_1,x_2,x_3,x_4)^G$ *is rational over* $\mathbb{Q}$. (*This theorem covers* 5 *conjugacy classes*).

Proof. $\widetilde{M}_2$ acts on $\mathbb{Q}(x_1,x_2,x_3,x_4)_0^{\langle Q,\boldsymbol{i}_1\rangle}=\mathbb{Q}(u_1,u_2,u_3)$ where $u_1=\frac{y_{13}}{y_{12}}$, $u_2=\frac{y_{23}}{y_{22}}$, $u_3=\frac{y_{33}}{y_{32}}$.

α maps as $u_1\mapsto u_2\mapsto -u_3\mapsto u_1$, $\boldsymbol{j}_1$ maps as $u_1\mapsto -u_1, u_2\mapsto -u_2, u_3\mapsto -u_3$, and $\alpha_0\alpha_1$ maps as $u_1\leftrightarrow -\frac{1}{u_2}, u_3\mapsto -\frac{1}{u_3}$. Since $\langle G_0,\boldsymbol{j}_1\rangle$ acts linearly on (u_1,u_2,u_3), the theorem is true for $G=G_0$ and $G=\langle G_0,\boldsymbol{j}_1\rangle$.

$\boldsymbol{j}_1\alpha_0\alpha_1$ acts as $u_1\leftrightarrow\frac{1}{u_2}$, $u_3\mapsto\frac{1}{u_3}$, so that $\langle G_0,\boldsymbol{j}_1\alpha_0\alpha_1\rangle$ acts purely monomially on $(u_1,u_2,-u_3)$, hence the theorem is true for $G=\langle G_0,\boldsymbol{j}_1\alpha_0\alpha_1\rangle$.

From the action of $\boldsymbol{j}_1$ above, $\mathbb{Q}(u_1,u_2,u_3)^{\langle \boldsymbol{j}_1\rangle}=\mathbb{Q}(v_1,v_2,v_3)$ where $v_1=u_2u_3$, $v_2=u_1u_3$, $v_3=u_1u_2$. Then both of α and $\alpha_0\alpha_1$ act purely monomially on $(v_1,v_2,-v_3)$ as follows. $\alpha: v_1\mapsto v_2\mapsto -v_3\mapsto v_1$ and $\alpha_0\alpha_1: v_1\leftrightarrow\frac{1}{v_2}, v_3\mapsto\frac{1}{v_3}$. This implies that the theorem holds for $G=\widetilde{M}_2$ by applying Theorem2.3 (2).

The proof for the fifth group $G=\langle G_0,\alpha_0\alpha_1\rangle$ is rather complicated. Let $\sigma_1,\sigma_2,\sigma_3$ be the fundamental symmetric polynomials of $u_1,u_2,-u_3$, namely $\sigma_1=u_1+u_2-u_3$, $\sigma_2=u_1u_2-u_1u_3-u_2u_3$, $\sigma_3=-u_1u_2u_3$. Since α acts as a cyclic permutation of $(u_1,u_2,-u_3)$, $\mathbb{Q}(u_1,u_2,u_3)^{\langle\alpha\rangle}=\mathbb{Q}(t_1,t_2,t_3)$ where $t_1=\sigma_1$, $t_2=\frac{\sigma_3'}{\sigma_2'}$, $t_3=\frac{\Delta}{\sigma_2'}$, $\sigma_2'=\sigma_2-\frac{1}{3}\sigma_1^2$, $\sigma_3'=\sigma_3-\frac{1}{3}\sigma_1\sigma_2+\frac{2}{27}\sigma_1^3$, $\Delta=(u_1-u_2)(u_1+u_3)(u_2+u_3)$, and the relation $\sigma_2'=-\frac{1}{4}t_3^2-\frac{27}{4}t_2^2$ holds.

We shall check the action of $\alpha_0\alpha_1$ on $\mathbb{Q}(t_1,t_2,t_3)$.

$\alpha_0\alpha_1$ acts as $\sigma_1\mapsto-\frac{\sigma_2}{\sigma_3}$, $\sigma_2\mapsto\frac{\sigma_1}{\sigma_3}$, $\sigma_3\mapsto-\frac{1}{\sigma_3}$ and $\Delta\mapsto-\frac{\Delta}{\sigma_3^2}$. Since $\sigma_3'+\frac{2}{9}\sigma_1\sigma_2'=\sigma_3-\frac{1}{9}\sigma_1\sigma_2$, the action of $\alpha_0\alpha_1$ maps $\sigma_3'+\frac{2}{9}\sigma_1\sigma_2'$ to $-\frac{1}{\sigma_3^2}(\sigma_3'+\frac{2}{9}\sigma_1\sigma_2')$ hence $w_3:=\frac{t_3}{t_2+\frac{2}{9}t_1}$ is invariant by $\alpha_0\alpha_1$.

Put $w_2=t_2+\frac{2}{9}t_1$ then we have $\mathbb{Q}(t_1,t_2,t_3)=\mathbb{Q}(t_1,w_2,w_3)$. We shall study the action on w_2. $w_2=(\sigma_3-\frac{1}{9}\sigma_1\sigma_2)/\sigma_2'=(\sigma_3-\frac{1}{9}\sigma_1\sigma_2)/(\sigma_2-\frac{1}{3}\sigma_1^2)$ is mapped to $-(\sigma_3-\frac{1}{9}\sigma_1\sigma_2)/(\sigma_3\sigma_1-\frac{1}{3}\sigma_2^2)$, which is equal to

$$-\left(t_2+\frac{2}{9}t_1\right)\Big/\left(t_2t_1+\frac{1}{12}t_3^2+\frac{9}{4}t_2^2+\frac{1}{9}t_1^2\right)=-\frac{4}{9}\frac{1}{w_2}\frac{1}{1+\frac{1}{27}w_3^2}.$$

Put $w_1=\sigma_3$, then $\mathbb{Q}(t_1,t_2,t_3)=\mathbb{Q}(w_1,w_2,w_3)$ because $\sigma_3=\sigma_3'+\frac{1}{3}\sigma_1\sigma_2'+\frac{1}{27}\sigma_1^3=(-\frac{1}{4}t_3^2-\frac{27}{4}t_2^2)(t_2+\frac{1}{3}t_1)+\frac{1}{27}t_1^3=\{-\frac{1}{4}w_2^2w_3^2-\frac{27}{4}(w_2-\frac{2}{9}t_1)^2\}(w_2+\frac{1}{9}t_1)+\frac{1}{27}t_1^3=\frac{9}{4}w_2^2(1-\frac{1}{81}w_3^2)t_1-\frac{27}{4}w_2^3(1+\frac{1}{27}w_3^2)$.

Since $\sigma_3 \mapsto -\frac{1}{\sigma_3}$, the action of $\alpha_0\alpha_1$ is monomial on $\mathbb{Q}(w_3)(w_1, w_2)$, the rational function field of two variables w_1, w_2 over $\mathbb{Q}(w_3)$. This implies that $\mathbb{Q}(w_1, w_2, w_3)^{\langle \alpha_0\alpha_1\rangle} = \mathbb{Q}(x_1, x_2, x_3, x_4)_0^{\langle G_0, \alpha_0\alpha_1\rangle}$ is rational over $\mathbb{Q}$ in virtue of Theorem 2.3(1). □

Theorem 3.7. *Let $\widetilde{M_3} = \langle G_0, \boldsymbol{j}_1, \alpha_1\rangle$. If $\widetilde{M_3} \le G$, then $\mathbb{Q}(x_1, x_2, x_3, x_4)^G$ is rational over $\mathbb{Q}$. (This theorem covers 4 conjugacy classes).*

Proof. As shown in the proof of Theorem 3.6, we have $\mathbb{Q}(x_1, x_2, x_3, x_4)_0^{\langle Q, \boldsymbol{i}_1, \boldsymbol{j}_1\rangle} = \mathbb{Q}(v_1, v_2, v_3)$ where $v_1 = u_2u_3, v_2 = u_1u_3, v_3 = u_1u_2$, and α acts as $v_1 \mapsto v_2 \mapsto -v_3 \mapsto v_1$ while $\alpha_0\alpha_1$ acts as $v_1 \leftrightarrow \frac{1}{v_2}, v_3 \mapsto \frac{1}{v_3}$.

Due to the relation mentioned before Theorem 3.6,

$$u_3 = \frac{(u_1u_2+1)w^2 + u_1^2}{u_1\{(u_1u_2+1)w^2 + u_1u_2\}} \text{ for } w = \frac{y_{13}}{y_{11}},$$

we have $\left(\frac{y_{13}}{y_{11}}\right)^2 = \frac{u_1^2(1-u_2u_3)}{(u_1u_3-1)(u_1u_2+1)}$. Multiplying with the image by α, we get $\left(\frac{y_{13}}{y_{11}}\frac{y_{23}}{y_{21}}\right)^2 = \frac{u_1^2u_2^2}{(u_1u_2+1)^2}$, hence $\frac{y_{13}}{y_{11}}\frac{y_{23}}{y_{21}} = \frac{u_1u_2}{u_1u_2+1}$ (The sign ± 1 is calculated separately). Therefore, the action of α_1 is calculated as follows:

$$v_3 = u_1u_2 = \frac{y_{13}y_{23}}{y_{12}y_{22}} \mapsto -\frac{y_{11}y_{21}}{y_{13}y_{23}} = -\frac{v_3+1}{v_3}.$$

Since α_1 commutes with α, we get $v_1 \mapsto \frac{v_1-1}{v_1}$, $v_2 \mapsto \frac{v_2-1}{v_2}$.

To make the action of α_1 simpler, we shall extend the coefficient field from $\mathbb{Q}$ to $\mathbb{Q}(\sqrt{-3})$, and put

$$a_1 = \frac{v_1+\omega^2}{v_1+\omega}, a_2 = \frac{v_2+\omega^2}{v_2+\omega}, a_3 = \frac{-v_3+\omega^2}{-v_3+\omega}, \omega = \frac{-1+\sqrt{-3}}{2}.$$

Then α acts as $a_1 \mapsto a_2 \mapsto a_3 \mapsto a_1$ and α_1 acts as $a_i \mapsto \omega^2 a_i\,(i = 1, 2, 3)$ while $\alpha_0\alpha_1$ acts as $a_1 \mapsto \frac{\omega}{a_2}, a_2 \mapsto \frac{\omega}{a_1}, a_3 \mapsto \frac{\omega}{a_3}$. Denote the conjugation in $\mathbb{Q}(\sqrt{-3})/\mathbb{Q}$ with $^-$, then $\overline{a}_i = \frac{1}{a_i}\,(i = 1, 2, 3)$.

Since α acts as a cyclic permutation of (a_1, a_2, a_3), $\mathbb{Q}(\sqrt{-3})(a_1, a_2, a_3)^{\langle\alpha\rangle} = \mathbb{Q}(\sqrt{-3})(\tau_1, \tau_2, \tau_3)$ where τ_1, τ_2, τ_3 are obtained from t_1, t_2, t_3 in the proof of Theorem 3.6 replacing $u_1, u_2, -u_3$ by a_1, a_2, a_3.

Just as in the proof of Theorem 3.6, we have $\mathbb{Q}(\sqrt{-3})(\tau_1, \tau_2, \tau_3) = \mathbb{Q}(\sqrt{-3})(\xi_1, \xi_2, \xi_3)$ where ξ_1, ξ_2, ξ_3 are obtained from w_1, w_2, w_3 replacing t_1, t_2, t_3 by τ_1, τ_2, τ_3. Then, actions on (ξ_1, ξ_2, ξ_3) are as follows.

α_1: $\xi_1 \mapsto \xi_1, \xi_2 \mapsto \omega^2\xi_2, \xi_3 \mapsto \xi_3$

$\alpha_0\alpha_1$: $\xi_1 \mapsto \frac{1}{\xi_1}, \xi_2 \mapsto \frac{4}{9}\frac{\omega^2}{\xi_2}\frac{1}{1+\frac{1}{27}\xi_3^2}, \xi_3 \mapsto \xi_3$

$^-$: $\xi_1 \mapsto \frac{1}{\xi_1}, \xi_2 \mapsto \frac{4}{9}\frac{1}{\xi_2}\frac{1}{1+\frac{1}{27}\xi_3^2}, \xi_3 \mapsto -\xi_3$

Since $1+\frac{1}{27}\xi_3^2 = (1+\frac{\sqrt{-3}}{9}\xi_3)(1-\frac{\sqrt{-3}}{9}\xi_3)$, putting $\eta_1 = \xi_1$, $\eta_2 = \frac{3}{2}(1-\frac{\sqrt{-3}}{9}\xi_3)\xi_2$, $\eta_3 = (1-\frac{\sqrt{-3}}{9}\xi_3)/(1+\frac{\sqrt{-3}}{9}\xi_3)$, we have

α_1: $\eta_1 \mapsto \eta_1, \eta_2 \mapsto \omega^2\eta_2, \eta_3 \mapsto \eta_3$

$\alpha_0\alpha_1$: $\eta_1 \mapsto \frac{1}{\eta_1}, \eta_2 \mapsto \frac{\omega^2\eta_3}{\eta_2}, \eta_3 \mapsto \eta_3$

$^-$: $\eta_1 \mapsto \frac{1}{\eta_1}, \eta_2 \mapsto \frac{\eta_3}{\eta_2}, \eta_3 \mapsto \eta_3$

Evidently $\mathbb{Q}(\sqrt{-3})(\eta_1,\eta_2,\eta_3)^{\langle\alpha_1\rangle} = \mathbb{Q}(\sqrt{-3})(\eta_1,\eta_2',\eta_3)$ where $\eta_2' = \frac{\eta_2^3}{\eta_3}$, and η_2' is mapped to $\frac{\eta_3}{\eta_2'}$ by both $\alpha_0\alpha_1$ and $\bar{\ }$. From this, we have $\mathbb{Q}(v_1,v_2,v_3)^{\langle\alpha,\alpha_1\rangle} = \mathbb{Q}(\sqrt{-3},\eta_1,\eta_2,\eta_3)^{\langle\alpha_1,\bar{\ }\rangle} = \mathbb{Q}(\sqrt{-3},\eta_1,\eta_2',\eta_3)^{\langle\bar{\ }\rangle} = \mathbb{Q}\big(\sqrt{-3}\frac{\eta_1-1}{\eta_1+1},\eta_2'+\frac{\eta_3}{\eta_2'},\sqrt{-3}(\eta_2'-\frac{\eta_3}{\eta_2'})\big)$. Since $\mathbb{Q}(v_1,v_2,v_3)^{\langle\alpha,\alpha_1\rangle} = \mathbb{Q}(x_1,x_2,x_3,x_4)_0^{\widetilde{M}_3}$, the theorem has been proved for $G = \widetilde{M}_3$.

On the other hand, $\mathbb{Q}(\sqrt{-3})(\eta_1,\eta_2',\eta_3)^{\langle\alpha_0\alpha_1\rangle} = \mathbb{Q}(\sqrt{-3})(\zeta_1,\zeta_2,\zeta_3)$ where $\zeta_1 = (\eta_1 - 1/\eta_1)/(\eta_1\eta_2' - \eta_3/\eta_1\eta_2')$, $\zeta_2 = (\eta_2' - \eta_3/\eta_2')/(\eta_1\eta_2' - \eta_3/\eta_1\eta_2')$, $\zeta_3 = \eta_3$. Since ζ_1,ζ_2 and ζ_3 are invariant by $\bar{\ }$ also, we have $\mathbb{Q}(x_1,x_2,x_3,x_4)_0^{\langle\widetilde{M}_3,\alpha_0\alpha_1\rangle} = \mathbb{Q}(\zeta_1,\zeta_2,\zeta_3)$, so that the theorem is true for $G = \langle\widetilde{M}_3,\alpha_0\alpha_1\rangle$.

The remaining two groups are $\widetilde{M}_0$ and $\langle\widetilde{M}_3,m'\rangle$. $\widetilde{M}_0$ is a reflection group, so that $\mathbb{Q}(x_1,x_2,x_3,x_4)^{\widetilde{M}_0}$ is rational over $\mathbb{Q}$. $\mathbb{Q}(x_1,x_2,x_3,x_4)^{\langle\widetilde{M}_3,m'\rangle}$ is its quadratic extension, and also can be proved to be rational over $\mathbb{Q}$. See Yamasaki [25]. □

When $\boldsymbol{i}_1 \notin G$, the linear Noether's problem is more difficult. As mentioned before, we have $\mathbb{Q}(x_1,x_2,x_3,x_4)_0^Q = \mathbb{Q}(\frac{y_{13}}{y_{12}},\frac{y_{13}}{y_{11}},\frac{y_{23}}{y_{22}})$, but the action of G is very complicated for this generator system.

Theorem 3.8. *The linear Noether's problem is affirmative for* $G = \langle Q,\alpha_1\rangle$ *and* $G = \langle Q,\alpha_1,\boldsymbol{k}_1\alpha_0\rangle$.

Proof. Let $u_1 = \frac{y_{13}}{y_{12}}, u_2 = \frac{y_{13}}{y_{11}}, u_3 = \frac{y_{23}}{y_{22}}$. (The notations are independent of the ones in the proof of Theorem 3.6 and 3.7). α_1 acts as $u_1 = \frac{y_{13}}{y_{12}} \mapsto -\frac{y_{11}}{y_{13}} = -\frac{1}{u_2}$, $u_2 = \frac{y_{13}}{y_{11}} \mapsto -\frac{y_{11}}{y_{12}} = -\frac{u_1}{u_2}$, and $u_3 = \frac{y_{23}}{y_{22}} \mapsto \frac{y_{21}}{y_{23}}$, but a computer calculation shows that $\frac{y_{21}}{y_{23}} = \frac{u_2(1+u_1u_3)}{u_1u_3}$. Therefore $t = u_1u_3$ is mapped to $-\frac{1+t}{t}$.

Put $s = \frac{c_1t+c_2}{c_3t+c_4}$ with $c_1,c_2,c_3,c_4 \in \mathbb{Q}(u_1,u_2)$. Denoting the image by α_1 with $'$, we have

$$s' = \frac{-c_1'(1+t)+c_2't}{-c_3'(1+t)+c_4't}.$$

Let us choose c_1,c_2,c_3,c_4 properly to get $s' = s$. From $c_1 = c_2' - c_1'$ and $c_2 = -c_1'$, we get $c_1 + c_1' + c_1'' = 0$. Similar equations hold for c_3 and c_4. Therefore, for any $c_1 \in \mathbb{Q}(u_1,u_2)$ such that $c_1 + c_1' + c_1'' = 0$, it is enough to put $c_2 = -c_1', c_3 = -c_1'', c_4 = c_1$.

Now we get $\mathbb{Q}(u_1,u_2,t)^{\langle\alpha_1\rangle} = \mathbb{Q}(s)(u_1,u_2)^{\langle\alpha_1\rangle}$.

Let $v_1 = \frac{1}{1-u_1-\frac{u_1}{u_2}} - \frac{1}{3}$, $v_2 = \frac{-u_1}{1-u_1-\frac{u_1}{u_2}} - \frac{1}{3}$, then $\mathbb{Q}(u_1,u_2) = \mathbb{Q}(v_1,v_2)$ and α_1 acts as $v_1 \mapsto v_2 \mapsto -(v_1+v_2) \mapsto v_1$, so that $\mathbb{Q}(v_1,v_2)^{\langle\alpha_1\rangle} = \mathbb{Q}(\xi_1,\xi_2)$ where

$$\xi_1 = \frac{1}{v_1} + \frac{1}{v_2} - \frac{1}{v_1+v_2},\ \xi_2 = \frac{(v_1-v_2)(2v_1+v_2)(v_1+2v_2)}{v_1v_2(v_1+v_2)}.$$

This shows the rationality of $\mathbb{Q}(x_1,x_2,x_3,x_4)_0^{\langle Q,\alpha_1\rangle}$, thus the theorem has been proved for $G = \langle Q,\alpha_1\rangle$.

As for the group $G = \langle Q, \alpha_1, \boldsymbol{k}_1\alpha_0\rangle$, since the conjugation by α keeps $\langle Q, \alpha_1\rangle$ invariant, we shall consider $\widetilde{\alpha}_0 = \alpha\alpha_0\alpha^2$ instead of α_0. Then $\boldsymbol{k}_1\widetilde{\alpha}_0$ acts as $y_{11} \leftrightarrow -y_{12}$, $y_{13} \mapsto y_{13}$, $y_{23} \leftrightarrow -y_{33}$, $y_{22} \leftrightarrow -y_{31}$ so that $u_1 = \frac{y_{13}}{y_{12}} \leftrightarrow -\frac{y_{13}}{y_{11}} = -u_2$, $u_3 = \frac{y_{23}}{y_{22}} \leftrightarrow \frac{y_{33}}{y_{31}} = \frac{u_1^2+u_2^2+u_1u_2^2u_3}{u_1u_2(u_1-u_3)}$ (by a computer calculation), hence $t \mapsto \frac{u_2^2t+u_1^2+u_2^2}{t-u_1^2}$.

This implies $v_1 \leftrightarrow -(v_1+v_2), v_2 \mapsto v_2$ so that $\xi_1 \mapsto \xi_1$, $\xi_2 \mapsto -\xi_2$. The action on s is complicated. Let $s = \frac{ct-c'}{-c''t+c}$ with $c \in \mathbb{Q}(u_1, u_2)$ such that $c + c' + c'' = 0$. Denoting the image by $\boldsymbol{k}_1\widetilde{\alpha}_0$ with $*$, we have

$$s^* = \frac{c^*t^* - c'^*}{-c''^*t^* + c^*},\ t^* = \frac{u_2^2t + u_1^2 + u_2^2}{t - u_1^2},\ t = \frac{cs + c'}{c''s + c}.$$

From these three equations, we obtain $s^* = \frac{d_1s+d_2}{d_3s+d_4}$, where

$$\begin{aligned}
d_1 &= c^*cu_2^2 + c^*c''(u_1^2 + u_2^2) - c'^*c + c'^*c''u_1^2\\
d_2 &= c^*c'u_2^2 + c^*c(u_1^2 + u_2^2) - c'^*c' + c'^*cu_1^2\\
d_3 &= -c''^*cu_2^2 - c''^*c''(u_1^2 + u_2^2) + c^*c - c^*c''u_1^2\\
d_4 &= -c''^*c'u_2^2 - c''^*c(u_1^2 + u_2^2) + c^*c' - c^*cu_1^2
\end{aligned}$$

We see that $d_2^* = d_2, d_3^* = d_3, d_1^* + d_4 = 0$. Put $\xi_3 = s - \frac{d_1^*}{d_3}$, then

$$\xi_3^* = s^* - \frac{d_1}{d_3} = \frac{d_2 - \frac{d_1d_4}{d_3}}{d_3s + d_4} = \frac{d_2 - \frac{d_1d_4}{d_3}}{d_3\xi_3 + d_1^* + d_4} = \frac{1}{\xi_3}\frac{d_2d_3 - d_1d_4}{d_3^2}.$$

Replace each d_i by $\frac{d_i}{(1+u_2-u_1)^2}$ and use the same symbol for the replaced quantity, then we have $d_i' = d_i$, hence each $d_i \in \mathbb{Q}(\xi_1, \xi_2)$. With a special choice as $c = v_2 - v_1$, we get the following result from computer calculations again. [d_i's written below are divided by the common factor $\frac{\xi_2^2+27}{12\xi_1^2\left(4\xi_1^2-3(\xi_2^2+27)\right)^2}$.]

$$d_3 = 16\xi_1^4 + 288\xi_1^3 + 9(8\xi_1^2 + 24\xi_1 - 3\xi_2^2 + 27)(\xi_2^2 + 27),$$
$$d_2d_3 - d_1d_4 = 24\big(2\xi_1^2 + 3(\xi_2^2 + 27)\big)\big(4\xi_1^3 - 9(\xi_1 + 3)(\xi_2^2 + 27)\big)^2.$$

Therefore $\xi_3^* = \frac{1}{\xi_3}(A\xi_2^2 + B)C^2$ where $C \in \mathbb{Q}(\xi_1, \xi_2)$, $C^* = C$, $A = 18$, $B = 6(2\xi_1^2 + 81)$.

It has been proven in [17] that

$$K(x, y)^{\langle\sigma\rangle},\ \sigma : x \mapsto -x, y \mapsto \frac{ax^2 + b}{y}\ (a, b \in K)$$

is rational over K. So we see that $\mathbb{Q}(x_1, x_2, x_3, x_4)_0^{\langle Q,\alpha_1,\boldsymbol{k}_1\widetilde{\alpha}_0\rangle}$ is rational, namely the linear Noether's problem is affirmative for $G = \langle Q, \alpha_1, \boldsymbol{k}_1\widetilde{\alpha}_0\rangle$. □

There remain four groups for further analysis. They are $\langle Q, \alpha\rangle$, $\langle Q, \alpha, \alpha_0\rangle$, $\langle Q, \alpha, \alpha_1\rangle$ and $\langle Q, \alpha, \alpha_1, \boldsymbol{k}_1\alpha_0\rangle$. Up to the present, we have not succeeded to find suitable generators of $\mathbb{Q}(x_1, x_2, x_3, x_4)_0^Q$ for an effective analysis of these groups.

4. List of conjugacy classes of finite subgroups of $GL(4,\mathbb{Q})$

The following is a list of conjugacy classes of finite subgroups of $GL(4,\mathbb{Q})$. We refer to [1]. But the generators in [1] were changed by other ones so that we can apply theorems easily. We omit groups of order 2^n from the list because they all have been completed in Kitayama [17]. In the following table, the symbol "**A**" means affirmative answer and "**N**" means negative answer.

4.1. Groups of order 3×2^n

Number	Order	Structure	Generators	Result
(4,8,1)	3	C_3	c_3	**A** by Theorem3.1
(4,8,2)	6	C_6	c_3, c_2c_2'	**A** by Theorem3.1
(4,8,3)	6	D_3	c_3, mc_4	**A** by Theorem3.1
(4,8,4)	6	D_3	$c_3, c_2'mc_4$	**A** by Theorem3.1
(4,8,5)	12	D_6	c_3, c_2c_2', mc_4	**A** by Theorem3.1
(4,9,1)	6	C_6	c_3, c_2	**A** by Theorem3.1
(4,9,2)	6	C_6	c_3, c_2'	**A** by Theorem3.1
(4,9,3)	12	$C_6{\times}C_2$	c_3, c_2, c_2'	**A** by Theorem3.1
(4,9,4)	12	D_6	c_3, c_2, mc_4	**A** by Theorem3.1
(4,9,5)	12	D_6	$c_3, c_2, c_2'mc_4$	**A** by Theorem3.1
(4,9,6)	12	D_6	c_3, mc_4, c_2'	**A** by Theorem3.1
(4,9,7)	24	$D_6{\times}C_2$	c_3, c_2, c_2', mc_4	**A** by Theorem3.1
(4,11,1)	3	C_3	c_3c_3'	**A** by Theorem3.3
(4,11,2)	6	C_6	c_3c_3', c_2c_2'	**A** by Theorem3.3
(4,14,1)	6	C_6	$c_3, c_2m'c_2'$	**A** by Theorem3.1
(4,14,2)	6	C_6	$c_3, c_2'm'$	**A** by Theorem3.1
(4,14,3)	6	D_3	c_3, c_2mc_4m'	**A** by Theorem3.1
(4,14,4)	12	$C_6{\times}C_2$	$c_3, c_2c_2', c_2'm'$	**A** by Theorem3.1
(4,14,5)	12	D_6	$c_3, c_2c_2', mc_4c_2'm'$	**A** by Theorem3.1
(4,14,6)	12	D_6	$c_3, mc_4, c_2c_2'm'$	**A** by Theorem3.1
(4,14,7)	12	D_6	$c_3, mc_4c_2', c_2c_2'm'$	**A** by Theorem3.1
(4,14,8)	12	D_6	$c_3, mc_4, c_2'm'$	**A** by Theorem3.1
(4,14,9)	12	D_6	$c_3, mc_4c_2', c_2'm'$	**A** by Theorem3.1
(4,14,10)	24	$D_6{\times}C_2$	$c_3, mc_4, c_2'm', c_2c_2'$	**A** by Theorem3.1
(4,15,1)	12	$C_6{\times}C_2$	$c_3, c_2, c_2'm'$	**A** by Theorem3.1
(4,15,2)	12	$C_6{\times}C_2$	c_3, c_2', m'	**A** by Theorem3.1
(4,15,3)	12	$C_6{\times}C_2$	$c_3, m'c_2, c_2'$	**A** by Theorem3.1
(4,15,4)	12	D_6	c_3, c_2, mc_4m'	**A** by Theorem3.1
(4,15,5)	12	D_6	c_3, c_2', c_2mc_4m'	**A** by Theorem3.1
(4,15,6)	24	$D_6{\times}C_2$	$c_3, c_2, mc_4, c_2'm'$	**A** by Theorem3.1
(4,15,7)	24	$D_6{\times}C_2$	$c_3, c_2', mc_4, c_2'm'$	**A** by Theorem3.1
(4,15,8)	24	$C_6{\times}C_2{\times}C_2$	c_3, c_2, c_2', m'	**A** by Theorem3.1
(4,15,9)	24	$D_6{\times}C_2$	$c_3, c_2, c_2'm', mc_4c_2'$	**A** by Theorem3.1
(4,15,10)	24	$D_6{\times}C_2$	c_3, c_2m', mc_4, c_2'	**A** by Theorem3.1
(4,15,11)	24	$D_6{\times}C_2$	c_3, c_2, c_2', mc_4m'	**A** by Theorem3.1
(4,15,12)	48	$D_6{\times}C_2{\times}C_2$	c_3, c_2, c_2', m', mc_4	**A** by Theorem3.1
(4,17,1)	6	D_3	$c_3c_3', \gamma mc_4m'c_4'$	**A** by Theorem3.3
(4,17,2)	12	D_6	$c_3c_3', c_2c_2', \gamma mc_4m'c_4'$	**A** by Theorem3.3
(4,20,1)	12	C_{12}	c_3, c_4'	**A** by Theorem3.1

Number	Order	Structure	Generators	Result
(4,20,2)	12	C_{12}	c_3, c_2c_4'	**A** by Theorem3.1
(4,20,3)	12	Q_{12}	c_3, c_2', mc_4c_4'	**A** by Theorem3.1
(4,20,4)	24	$D_3{\times}C_4$	$c_3, mc_4, c_2'c_4'$	**A** by Theorem3.1
(4,20,5)	24	$C_{12}{\times}C_2$	c_3, c_2, c_2', c_4'	**A** by Theorem3.1
(4,20,6)	24	$D_4{\times}C_3$	c_3, c_2', c_4', c_2m'	**A** by Theorem3.1
(4,20,7)	24	D_{12}	$c_3, c_2', c_4', c_2mc_4m'$	**A** by Theorem3.1
(4,20,8)	24	$D_4{\times}C_3$	$c_3, c_2'c_4', m'$	**A** by Theorem3.1
(4,20,9)	24	$D_3{\times}C_4$	c_3, c_2', mc_4, c_2c_4'	**A** by Theorem3.1
(4,20,10)	24	$D_4{\times}C_3$	$c_3, c_2', c_2m', c_2c_4'$	**A** by Theorem3.1
(4,20,11)	24	D_{12}	$c_3, c_2', c_2c_4', mc_4c_2'm'$	**A** by Theorem3.1
(4,20,12)	24		$c_3, m', c_2', c_2mc_4c_4'$	**A** by Theorem3.1
(4,20,13)	24		$c_3, c_2', c_2m', mc_4c_4'$	**A** by Theorem3.1
(4,20,14)	24	$Q_{12}{\times}C_2$	c_3, c_2, c_2', mc_4c_4'	**A** by Theorem3.1
(4,20,15)	48	$D_6{\times}C_4$	$c_3, c_2, c_2', c_4', mc_4$	**A** by Theorem3.1
(4,20,16)	48	$D_4{\times}D_3$	c_3, c_4', mc_4, c_2m'	**A** by Theorem3.1
(4,20,17)	48	$D_4{\times}D_3$	c_3, c_4', mc_4, m'	**A** by Theorem3.1
(4,20,18)	48	$D_4{\times}D_3$	c_3, c_2, c_2', m', c_4'	**A** by Theorem3.1
(4,20,19)	48	$D_{12}{\times}C_2$	$c_3, c_2, c_2', c_4', mc_4m'$	**A** by Theorem3.1
(4,20,20)	48	$D_4{\times}D_3$	$c_3, c_2', c_2m', mc_4, c_2c_4'$	**A** by Theorem3.1
(4,20,21)	48		$c_3, c_2, c_2', m', mc_4c_4'$	**A** by Theorem3.1
(4,20,22)	96	$D_6{\times}D_4$	c_3, mc_4, m', c_4', c_2	**A** by Theorem3.1
(4,21,1)	6	C_6	c_3c_3', c_2	**A** by Theorem3.3
(4,21,2)	12	$C_6{\times}C_2$	c_3c_3', c_2, c_2'	**A** by Theorem3.3
(4,21,3)	12	D_6	$c_3c_3', c_2, mc_4m'c_4'$	**A** by Theorem3.3
(4,21,4)	24	$D_6{\times}C_2$	$c_3c_3', c_2, c_2', mc_4m'c_4'$	**A** by Theorem3.3
(4,24,1)	12	A_4	$\mathfrak{A}_4 = \langle g_1, mm', c_2' \rangle$	**A** by Theorem3.2
(4,24,2)	24	$A_4{\times}C_2$	$\mathfrak{A}_4, c_2$	**A** by Theorem3.2
(4,24,3)	24	S_4	$\mathfrak{A}_4, m'c_4'$	**A** by Theorem3.2
(4,24,4)	24	S_4	$\mathfrak{A}_4, c_2m'c_4'$	**A** by Theorem3.2
(4,24,5)	48	$S_4{\times}C_2$	$\mathfrak{A}_4, c_2, m'c_4'$	**A** by Theorem3.2
(4,25,1)	24	$A_4{\times}C_2$	$\mathfrak{A}_4, m$	**A** by Theorem3.2
(4,25,2)	24	$A_4{\times}C_2$	$\mathfrak{A}_4, c_2m$	**A** by Theorem3.2
(4,25,3)	24	S_4	$\mathfrak{A}_4, mm'c_4'$	**A** by Theorem3.2
(4,25,4)	24	S_4	$\mathfrak{A}_4, c_2mm'c_4'$	**A** by Theorem3.2
(4,25,5)	48	$A_4{\times}C_2{\times}C_2$	$\mathfrak{A}_4, c_2, m$	**A** by Theorem3.2
(4,25,6)	48	$S_4{\times}C_2$	$\mathfrak{A}_4, c_2, mm'c_4'$	**A** by Theorem3.2
(4,25,7)	48	$S_4{\times}C_2$	$\mathfrak{A}_4, m, m'c_4'$	**A** by Theorem3.2
(4,25,8)	48	$S_4{\times}C_2$	$\mathfrak{A}_4, m, c_2m'c_4'$	**A** by Theorem3.2
(4,25,9)	48	$S_4{\times}C_2$	$\mathfrak{A}_4, c_2m, m'c_4'$	**A** by Theorem3.2
(4,25,10)	48	$S_4{\times}C_2$	$\mathfrak{A}_4, c_2m, mm'c_4'$	**A** by Theorem3.2
(4,25,11)	96	$S_4{\times}C_2{\times}C_2$	$M_1 = \langle \mathfrak{A}_4, c_2, m, m'c_4' \rangle$	**A** by Theorem3.2
(4,28,1)	12	C_{12}	$c_3c_3', c_2c_2', \gamma c_2$	**A** by Theorem3.3
(4,28,2)	24	D_{12}	$c_3c_3', \gamma c_2, mc_4m'c_4'$	**A** by Theorem3.3
(4,30,1)	12	Q_{12}	$c_3c_3', c_2c_2', \gamma mc_4m'c_4'c_2$	**A** by Theorem3.3
(4,30,2)	24	$D_3{\times}C_4$	$c_3c_3', mc_4m'c_4', c_2c_2', \gamma c_2$	**A** by Theorem3.3
(4,30,3)	24	$D_4{\times}C_3$	$c_3c_3', c_2, c_2', \gamma$	**A** by Theorem3.3
(4,30,4)	24		$c_3c_3', c_2, c_2', \gamma mc_4m'c_4'$	**A** by Theorem3.3
(4,30,6)	48	$D_4{\times}D_3$	$c_3c_3', c_2, c_2', \gamma, mc_4m'c_4'$	**A** by Theorem3.3
(4,32,5)	24	$SL(2,3)$	$g_1, c_4c_4', \gamma c_2mm'$	**A** by [22], [23]

Number	Order	Structure	Generators	Result
(4,32,11)	48	$GL(2,3)$	$g_1, c_4c_4', \gamma c_2mm', \gamma c_2c_4$	**A** by [22], [23]
(4,32,16)	96		$M_2 = \langle \mathfrak{A}_4, \gamma \rangle$	**A** by Theorem3.2
(4,32,18)	192		M_2, m	**A** by Theorem3.2
(4,32,19)	192		$M_2, m'c_4'$	**A** by Theorem3.2
(4,32,20)	192		$M_2, mm'c_4'$	**A** by Theorem3.2
(4,32,21)	384		$M_0 = \langle M_2, m, m'c_4' \rangle$	**A** by Theorem3.2
(4,33,1)	24	$Q_8{\times}C_3$	Q, α_1	**A** by Theorem3.8
(4,33,2)	24		$\alpha, \boldsymbol{k}\boldsymbol{i}_1\alpha_0$	**N** (p.p. 370)
(4,33,3)	24	$SL(2,3)$	Q, α	?
(4,33,4)	48		$Q, \alpha_1, \boldsymbol{k}_1\alpha_0$	**A** by Theorem3.8
(4,33,5)	48		$G_0 = \langle Q, \alpha, \boldsymbol{i}_1 \rangle$	**A** by Theorem3.6
(4,33,6)	48	$GL(2,3)$	Q, α, α_0	?
(4,33,8)	96		$G_0, \boldsymbol{j}_1$	**A** by Theorem3.6
(4,33,9)	96		$G_0, \boldsymbol{j}_1\alpha_0\alpha_1$	**A** by Theorem3.6
(4,33,10)	96		$G_0, \alpha_0\alpha_1$	**A** by Theorem3.6
(4,33,12)	192		$\widetilde{M_2} = \langle G_0, \boldsymbol{j}_1, \alpha_0\alpha_1 \rangle$	**A** by Theorem3.6

4.2. Groups of order 9×2^n

Number	Order	Structure	Generators	Result
(4,22,1)	9	$C_3{\times}C_3$	c_3, c_3'	**A** by Theorem3.4
(4,22,2)	18	$C_6{\times}C_3$	c_3, c_3', c_2c_2'	**A** by Theorem3.4
(4,22,3)	18	$D_3{\times}C_3$	c_3, c_3', c_2mc_4	**A** by Theorem3.4
(4,22,4)	18	$D_3{\times}C_3$	$c_3, c_3', c_2m'c_4'$	**A** by Theorem3.4
(4,22,5)	18		$c_3, c_3', mc_4m'c_4'$	**A** by Theorem3.4
(4,22,6)	36	$D_3{\times}C_6$	$c_3, c_3', c_2mc_4, c_2c_2'$	**A** by Theorem3.4
(4,22,7)	36		$c_3, c_3', c_2mc_4m'c_4', c_2c_2'$	**A** by Theorem3.4
(4,22,8)	36	$D_3{\times}D_3$	$c_3, c_3', c_2mc_4, c_2'm'c_4'$	**A** by Theorem3.4
(4,22,9)	36	$D_3{\times}D_3$	$c_3, c_3', c_2c_2'mc_4, c_2'm'c_4'$	**A** by Theorem3.4
(4,22,10)	36	$D_3{\times}D_3$	$c_3, c_3', c_2'mc_4, c_2m'c_4'$	**A** by Theorem3.4
(4,22,11)	72	$D_6{\times}D_3$	$c_3, c_3', c_2c_2', c_2mc_4, c_2'm'c_4'$	**A** by Theorem3.4
(4,23,1)	18	$C_6{\times}C_3$	c_3, c_3', c_2	**A** by Theorem3.4
(4,23,2)	36	$C_6{\times}C_6$	c_3, c_3', c_2, c_2'	**A** by Theorem3.4
(4,23,3)	36	$D_3{\times}C_6$	$c_3, c_3', c_2, c_2'mc_4$	**A** by Theorem3.4
(4,23,4)	36	$D_3{\times}C_6$	c_3, c_3', c_2, mc_4	**A** by Theorem3.4
(4,23,5)	36		$c_3, c_3', c_2, mc_4m'c_4'$	**A** by Theorem3.4
(4,23,6)	36	$D_3{\times}C_6$	$c_3, c_3', c_2, c_2'm'c_4'$	**A** by Theorem3.4
(4,23,7)	72	$D_6{\times}C_6$	$c_3, c_3', c_2, c_2', mc_4$	**A** by Theorem3.4
(4,23,8)	72		$c_3, c_3', c_2, c_2', mc_4m'c_4'$	**A** by Theorem3.4
(4,23,9)	72	$D_6{\times}D_3$	$c_3, c_3', c_2, mc_4c_2', m'c_4'$	**A** by Theorem3.4
(4,23,10)	72	$D_6{\times}D_3$	$c_3, c_3', c_2, mc_4, m'c_4'$	**A** by Theorem3.4
(4,23,11)	144	$D_6{\times}D_6$	$c_3, c_3', c_2, c_2', mc_4, m'c_4'$	**A** by Theorem3.4
(4,29,1)	18	$D_3{\times}C_3$	c_3, c_3', γ	**A** by Theorem3.4
(4,29,2)	36	$D_3{\times}C_6$	$c_3, c_3', c_2c_2', \gamma mc_4m'c_4'$	**A** by Theorem3.4
(4,29,3)	36	$D_3{\times}D_3$	$c_3, c_3', \gamma, mc_4m'c_4'$	**A** by Theorem3.4
(4,29,4)	36		$c_3, c_3', \gamma mc_4, mc_4m'c_4'$	**A** by Theorem3.4
(4,29,5)	72	$D_6{\times}D_3$	$c_3, c_3', c_2c_2', mc_4m'c_4', \gamma$	**A** by Theorem3.4
(4,29,6)	72		$c_3, c_3', c_2c_2', \gamma c_2mc_4, mc_4m'c_4'$	**A** by Theorem3.4
(4,29,7)	72		$c_3, c_3', c_2'm'c_4', \gamma c_2mc_4$	**A** by Theorem3.4

Number	Order	Structure	Generators	Result
(4,29,8)	72		$c_3, c_3', c_2m'c_4', \gamma c_2c_2'mc_4m'c_4'$	**A** by Theorem3.4
(4,29,9)	144		$c_3, c_3', c_2c_2', \gamma, m'c_4'$	**A** by Theorem3.4
(4,30,5)	36	$Q_{12} \times C_3$	$c_3, c_3', c_2c_2', \gamma c_2mc_4m'c_4'$	**A** by Theorem3.4
(4,30,7)	72		$c_3, c_3', \gamma, c_2, c_2'$	**A** by Theorem3.4
(4,30,8)	72		$c_3, c_3', c_2c_2', \gamma c_2, c_2mc_4m'c_4'$	**A** by Theorem3.4
(4,30,9)	72		$c_3, c_3', c_2c_2', mc_4m'c_4', \gamma c_2'$	**A** by Theorem3.4
(4,30,10)	144		$c_3, c_3', c_2, c_2', mc_4m'c_4', \gamma$	**A** by Theorem3.4
(4,30,11)	144		$c_3, c_3', c_2, c_2', \gamma mc_4$	**A** by Theorem3.4
(4,30,12)	144		$c_3, c_3', c_2c_2', c_2'm'c_4', \gamma mc_4$	**A** by Theorem3.4
(4,30,13)	288		$c_3, c_3', c_2, c_2', mc_4, m'c_4', \gamma$	**A** by Theorem3.4
(4,33,7)	72		Q, α, α_1	?
(4,33,11)	144		$Q, \alpha, \alpha_1, \boldsymbol{k}_1\alpha_0$	?
(4,33,13)	288		$\widetilde{M}_3 = \langle Q, \alpha, \boldsymbol{i}_1, \boldsymbol{j}_1, \alpha_1 \rangle$	**A** by Theorem3.7
(4,33,14)	576		$\widetilde{M}_3, \alpha_0\alpha_1$	**A** by Theorem3.7
(4,33,15)	576		$\widetilde{M}_3, m'$	**A** by Theorem3.7([25])
(4,33,16)	1152		$\widetilde{M}_0 = \langle \widetilde{M}_3, m', \alpha_0\alpha_1 \rangle$	**A** by Theorem3.7([25])

4.3. Groups of order 5^n

Number	Order	Structure	Generators	Result
(4,27,1)	5	C_5	c_5	**A** by Theorem3.5
(4,27,2)	10	C_{10}	c_5, c_2c_2'	**A** by Theorem3.5
(4,27,3)	10	D_5	$c_5, mc_4m'c_4'\gamma$	**A** by Theorem3.5
(4,27,4)	20	$D_5 \times C_2$	$c_5, c_2c_2', mc_4m'c_4'\gamma$	**A** by Theorem3.5
(4,31,1)	20	$C_5 \rtimes C_4$	c_5, σ_4	**A** by Theorem3.5
(4,31,2)	40	$C_2 \times (C_5 \rtimes C_4)$	c_5, σ_4, c_2c_2'	**A** by Theorem3.5
(4,31,3)	60	A_5	c_5, γ	**A** by Yamasaki[25]
(4,31,4)	120	S_5	c_5, mc_4	**A** by Yamasaki[25]
(4,31,5)	120	S_5	$c_5, c_2c_2'mc_4$	**A** by Yamasaki[25]
(4,31,6)	120	$A_5 \times C_2$	c_5, γ, c_2c_2'	**A** by Yamasaki[25]
(4,31,7)	240	$S_5 \times C_2$	c_5, mc_4, c_2c_2'	**A** by Yamasaki[25]

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