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On the surface integral and Gauss-Green's theorem

By

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We intend to give a general and direct definition of the surface integral (which appears in the Gauss-Green formura, i. e. $\iint f dxdy$).

It is defined ordinarily by means of the area of the surface where we suppose the existence of its tangent planes, while $\iint f dxdy$ depends only upon the area of the projection of the surface on xy-plane. In this case the area must be calculated, according to the multiplicity in positive and negative senses after the form of the surface. Theoretical difficulty lies in this respect. For the curvilinear integral it may be solved easily by the Stieltjets integral.

In this paper first we define the mapping of bounded variation of two dimensions[(2)] by using the theory of Brouwer's " Abbildungsgrad,"[(1)] and then two dimensional Stieltjes integral, after that we may extend immediately the surface integral.

This extension is not the Lebesgue-Stieltjes integral into two dimensions, but it is a direct extension of the original Stieltjes integral into two dimensions. We remark that the modern theory of the integral ignores this point.

In the next, I shall mention the validity of the Gauss-Green's theorem under the notion above mentioned of bounded variation for the closed (Jordan) surface without using any other condition.

This theorem, by its frequent application in mathematical physics, has hitherto been treated roughly until Kellogg who has proved it under certain mathematically rigorous conditions. Now our conditions are farther more general, and extremely simple (cf. (3) of the last remark). The present author believes that his reserch makes clear of the mathematical nature of this theorem, and he thinks it necessary to study the potential theory to which it is applied—in particular, several methods concerning to Dirichlet problem in space—from thie new stand point.

Further, the extention to the dimensions higher than two is not difficut, though such a one like Stokes's theorem (the relation between two dimensions and one dimension in three dimensions) is an important analogous problem, it is a question that it is discussed in the similar manner.

I. Mapping of the bounded variation. Let (u, v) and (x, y) be rectangular coordinates of respective planes, and $\varphi(u, v)$, $\psi(u, v)$ be continuous functions on a bounded closed region of uv-plane (the sum of open set and its boundary). We consider the mapping of $\bar{\varDelta}$ into xy-plane, namely

$$\mathfrak{A} : x=\varphi(u, v),\ y=\psi(u, v)$$

Briefly, we write it $(x, y)=\mathfrak{A}(u, v)$, also $\mathfrak{A}(\varDelta)$ the image of $\varDelta$ in xy-plane, and the like.

Let δ be an open subset of $\varDelta$. When a point $P(x, y)$ does not belong to the boundary of δ, $\mathfrak{A}(\bar{\delta}-\delta)$, we denote by $A_{\mathfrak{A}}[(x, y), \delta]$ or simply by $A(P, \delta)$, the " Abbildungsgrad " of δ at the point P.

We call the variation on δ of the mapping $\mathfrak{A}$, the following integral, (if integrable)

$\int A(P, \delta)dm(P)$ taken over the whole plane (or $\overset{(4)}{\mathfrak{A}}(\delta)-\mathfrak{A}(\bar{\delta}-\delta)$) and we denote it by $V(\delta)$. Especially, when the mapping functions φ, ψ are continuously differentiable in δ, and that $\iint\left|\frac{\partial(\varphi, \psi)}{\partial(u, v)}\right|dudv < +\infty$, we see easily from the definition of " Abbildungsgrad " that

$V(\delta)=\iint\left|\frac{\partial(\varphi, \psi)}{\partial(u, v)}\right| dudv$ (especially by the Nagumo's theory).

If the following conditions B_1 and B_2 about the mapping $\mathfrak{A}$ are fufilled, we call $\mathfrak{A}$ the mapping of the bounded variation:

B_1. We may divide $\bar{\varDelta}$ into the sum of a finite number of closed regions $\delta_i (i=1, 2, \ldots, n)$ (i. e. $\bar{\varDelta}=\overset{n}{\underset{i=1}{V}}\bar{\delta}_i$, when $i\neq j$, $\delta_i\cdot\delta_j=0$) however small it may be, such that $m[\overset{n}{\underset{i=1}{V}}\mathfrak{A}(\bar{\delta}-\delta)]=0$ is possible; we denote such a subdivision by (σ),

B_2. For every such subdivision (σ), $V(\delta_i)$ is determinate, and $\sum_{i=1}^{n}|V(\delta_i)|$ is bounded.

We call the upper bound $\sup_{(\sigma)}\sum_{i=1}^{n}|V(\delta_i)|$, the total variation on $\varDelta$ of the mapping $\mathfrak{A}$.

With this bounded variation, especially with the condition B_2, we will prove the following theorem, also necessary to prove Gauss-Green's theorem: For the mapping of the bounded variation, let (σ) be a subdivision satisfying the condition B_1; then to the point P of xy-plane, if $P \bar{\in} \overset{n}{\underset{i=1}{V}} \mathfrak{A}(\bar{\delta}_i - \delta_i)$, put $a_\sigma(P) = \sum_{i=1}^{n} |A(P, \delta_i)|$ otherwise $a_\sigma(P) = 0$, and let us denote the upper bound by $a(P)$ for all possible divisions (σ) i.e. $a(P) = \sup_{(\sigma)} a_\sigma(P)$. Then the intègral $\int a(p)dm$, the field of integration being the whole plane or $\mathfrak{A}(\Delta)$[6], is equal to the total variation of the mapping $\mathfrak{A}$; of course $\int a(p)dm < \infty$.

Proof. For a subdivision (σ), we have

$$(1) \quad \sum_{i=1}^{n} |V(\delta_i)| \leqq \sum_{i=1}^{n} \int |A(p, \delta_i)| dm = \int a_\sigma(p) dm.$$

$A(P, \delta)$ is constant in the neighbourhood of points which do not belong to $\mathfrak{A}(\bar{\delta} - \delta)$ (by continuity), and it is additive, i.e. $A(P, \sum_i \delta_i) = \sum_i A(P, \delta_i)$. Therefore divide δ_i into the following three parts:

$$\delta_i^{\pm} = E\{(u, v); \mathfrak{A}(u, v) \bar{\in} \mathfrak{A}(\bar{\delta}_i - \delta_i),\ A[\mathfrak{A}(u, v)\ \delta_i] \gtrless 0\}$$

$$\delta_i^0 = \text{The interior points of } E\{(u, v); \mathfrak{A}(u, v) \in \mathfrak{A}(\bar{\delta}_i - \delta_i)\},$$

then, they have no common points to one another, and δ_i^0, δ_i^+, δ_i^- are all open sets (by the continuity of the "Abbildungsgrad"), and we have $\bar{\delta}_i = \overline{\delta_i^+ + \delta_i^- + \delta_i^0}$; moreover, if $P \bar{\in} \mathfrak{A}(\bar{\delta}_i - \delta_i)$, then $A(P, \delta_i) = A(P, \delta_i^+) + A(P, \delta_i^-)$, where the latter two terms are respectively the positive and the negative parts of the first members (by the additivity and the property of the zero of mapping).

Following further the subdivision (σ), by dividing all δ_i successively, (1) will becomes an equality. Whence cleary

$$(2) \quad \sup_{(\sigma)} \sum_{i=1}^{n} |V(\delta_i)| = \sup_{(\sigma)} \sum_{i=1}^{n} \int |A(P, \delta_i)| dm = \sup_{(\sigma)} \int a_\sigma(P) dm.$$

we take a succession of subdivision (σ_ν) $(\nu = 1, 2, \ldots)$ mentioned in B_1; the divided parts tend to zero for $\nu \to \infty$. We may suppose that each (σ_ν) is the subdivision of the preceding $(\sigma_{\nu-1})$; For this, it suffices to designate anew by σ_ν the superposition of $\sigma_1, \sigma_2, \ldots, \sigma_{\nu-1}, \sigma_\nu$, the condition B_1 being yet fulfilled. In this case, summed up for all the image of boundary of division $V_i \mathfrak{A}(\bar{\delta}_i - \delta_i)$ for all σ_ν is of measure zero, and for the other points P, $a_{\sigma\nu}(P)$ increases monotonously with ν, since (σ_ν) is the subdivision of $(\sigma_{\nu-1})$ and the "Abbildungsgrad" is additive.

Hence, $\lim_{\nu\to\infty} a_{\sigma_\nu}(P)$ is deteminate. On the other hand for any subdivision (σ), if $P\bar{\epsilon}\, V_i\, \mathfrak{A}(\bar{\delta}_i - \delta_i)$, P has a positive distance from the closed set of the second member; therefore for sufficiently large ν the diameters of $\mathfrak{A}(\delta_i)$ for (σ_ν) will become smaller than the above distance. Therefore, by the additivity of the "Abbildungs-grad" and the property of the zero order, we have easily $a_\sigma(P) \leqq a_{\sigma_\nu}(P)$. Hence, $\lim_{\nu\to\infty} a_{\sigma_\nu}(P) \geqq a_\sigma(P)$. Since (σ) is arbitrary, taking the upper bound of the second member, we have $\lim_{\nu\to\infty} a_{\sigma_\nu}(P) \geqq a(P)$. This is true for all points of xy-plane except a set of measure zero, and of course, the inequality does not occur. thus $a(P)$, being measurable function, we have by the property of the integral

$$\lim_{\nu\to\infty} \int a_{\sigma_\nu}(P)\,dm = \int a(P)\,dm.$$

Since for all ν, we have

$$\int a_\sigma(P)\,dm \leqq \int a(P)\,dm,$$

we have

$$\sup_{(\sigma)} \int a_{\sigma'}(P)\,dm = \int a(P)\,dm.$$

Combined with (2), the proof is completed.

II. Stieltjes integral of two dimensions. Surface integral.

Now it is easy to define the integral. Let the mapping $\mathfrak{A}[x=\varphi(u,v),\ y=\psi(u,v),\ (u,v)\in\bar{\Delta}]$ be of the bounded vriation as in the preceding section, and $f(u,v)$ a continuous function in $\bar{\Delta}$, for a subdivision (σ) stated in B_1 $(\bar{\Delta} = \overset{n}{\underset{i=1}{V}} \bar{\delta}_i)$; for that the sum $\sum_{i=1}^{n} f(u_i, v_i)\, V(\delta_i)$, where $(u_i, v_i) \in \delta_i (i=1, 2, \cdots, n)$, converges in the limit of tending the subdivision (σ) to infinity. We call it two dimensional Stieltjes integral, and denote by

$$\int_\Delta f(u,v)\,d\mathfrak{A}(u,v) \quad \text{or} \quad \int_\Delta f(u,v)\,d\varphi * \psi(u,v),$$

where $d\mathfrak{A} = d\varphi * \psi$ means the infinitesimal dV of the variation V explained in the preceding section, i.e. the element of area considered with its sign.

To prove the existence of the limit we may go as in the case of classical integral of tne continuous function. We remark the following to the proof: By the condition B_1 of the bounded variation, two subdivisions may be superposed as to make a third subdivision (σ) by the additivity of the variation V we have $V(\sum_i \delta_i) = \sum_i V(\delta_i)$, moreover remark the condition B_2 and the uniform continuity of the function f.

It is clear that this gives immediately a definition of the surface integral.

Now let the surface S be represented by means of parameter (u, v) by the eqations.

$$S;\quad x=\varphi(u,v),\quad y=\psi(u,v),\quad z=\chi(u,v)$$

where φ, ψ, χ are supposed to be continuous in the closed domain $\bar{\varDelta}$ of uv-plane. Among them, the first two $x=\varphi(u,v)$, $y=\psi(u,v)$ give the mapping of the bounded variation. Given a continuous function $f(M)=f(u,v)$ of the point M on the surface, the Stieltjes integral $\int_\Delta f(u,v)d\varphi * \psi(u,v)$ is invariable by any topological transformation of the parameter (u, v); but by the change of the side, it will change only the sign. Now we denote it by

$$\int_S f(M)dx*y(M) \quad \text{or by} \quad \iint_S fdxdy,$$

and call it the surface integral. Of course, the sides of the surface are assigned which we denote by S^+ and S^- respectively. So we have

$$\int_{S^-} fdx*y = -\int_{S^+} fdx*y.$$

Obviously we have

$$\int_S fdy*x = -\int_S fdx*y.$$

Concluding these the two dimensional Stieltjes integral is a surface integral where the surface S overlaps several times over the xy-plane.

III. Gauss-Green's Theorem

Theorem. Let (x, y, z) be the rectangular of positive configuration, the function $F(x, y, z)$ be continuous in the closed domain $\bar{D}=D+S$ which consists of a jordan surface S and its interior D. We suppose that $\frac{\partial F}{\partial z}$ is continuous with respect to z in the open intervals contained in the section (closed set) of $\bar{D}$ cut by a straight line parallel to z-axis, and that $\frac{\partial F}{\partial z}$ is summable $\left(\iiint\left|\frac{\partial F}{\partial z}\right| dxdydz<\infty\right)$ on the set $D_1 (D\subset D_1 \subset \bar{D})$ on which it is defined.

Moreover we suppose that the surface S admits the mapping of $x{*}y$-bounded variation; then we have

$$(3)\quad \iiint_{D_1} \frac{\partial F}{\partial z} dxdydz = \int_S F dx{*}y \left(= \iint_S F dxdy \right)$$

where the outward of D is the positive side of S. This formula holds also for the domain enclosed by a finite number of Jordan surfaces.

Remark. 1. Let $r_i (i=1,2,\dots)$ be all the rational numbers, since D_1 is closed, the set $e_i = E\{(x,y) \mid (x,y,r_i) \in \bar{D}\}$ are also closed in the plane and hence measurable in the plane. Further the set

$$D_1 = D \underset{j,k}{V} E\{(x,y,z) \mid (x,y) \in \underset{i\{r_j < r_i < r_k\}}{e_i}, r_j < z < r_k\}$$

is also measurable. If we continue the function F to be continuous up to $\bar{D}$, its partial derivative with respect to z is equal to $\frac{\partial F}{\partial z}$ in D_1.

This shows that $\frac{\partial F}{\partial z}$ is a measurable function.

2. Explaining in detail, the Jordan surface S is a topological image of a sphere, therefore it cannot be expressed at the same time by a single function of the parameter (u, v) which appears in the preceding section. But it can be expressed locally (i.e. for a part $\bar{\delta}_i$ of the surface S), and the surface will be covered by a finite number of such part, in this case we may suppose that the variation is bounded in each of such parts of the surface; hence particularly the images on the xy-plane of the boundaries of these parts $x{*}y(\bar{\delta}-\delta)$ may be supposed to be sets of measure zero. Thus the surface S can be regarded as the union of a finite number of such parts which are not overlapping to one another, and it can easily be seen that the surface integral on S can be determined without contradiction as the sum of the surface integrals on each of such parts (independent of the mode of subdivision into such parts).

Proof of the theorem. By the assumption dividing S such that $S = \overset{n}{\underset{i=1}{V}} \bar{\delta}_i$, the sum $\sum_{i=1}^{n} F(M_i) V(\delta_i) [M_i \in \bar{\delta}_i, V(\delta_i) \doteq \int A_{x*y}(P, \delta_i) dm(P)]$, approaches the integral $\int_S F dx{*}y$ of (3) in the limit of the subdivisions. On the other hand the projection of the boundaries of the subdivided parts on xy-plane, $d_0 = \underset{i}{V}_{x*y}(\bar{\delta}_i - \delta_i)$ is a closed set and

$m(d_0)=0$. Then all the points on xy-plane are classified completely whether they belong to some of $x*y(\delta_i)$ or not; then such set being denoted by $d_j (j=1,2,\dots,\lambda)$, they are all open sets. Divide δ_i into $\lambda+1$ parts such that $\bar{\delta}_i=\overset{\lambda}{\underset{t=0}{V}}\bar{\delta}_{ij}$ where δ_{ij} are all open sets and $\delta_{ij}\cdot\delta_{ik}=0$, $(j\neq k)$ $x*y(\delta_{ij})\subset d_j$. Of course by that $d_j\cdot d_k=0$, $(j\neq k)$ and all d_j except d_0 of measure zero are all open, $S=\underset{i,j}{V}\bar{\delta}_{ij}$, is regarded as a subdivision (σ), next to the subdivision of $S=V_i\bar{\delta}_i$. We may continue the subdivision as far as we please. Noticing $m(d_0)=0$, hence $V(\delta_{i0})=0$, the sum $\sum_{i=1}^{n}\sum_{j=1}^{\lambda}F(M_{ij})V(\delta_{ij})$ tends to $\int_S F dx*y$ of (3).

On the other hand, for the triple integral of (3) we have after Fubini

$$\iiint_{D_1}\frac{\partial F}{\partial z}dxdydz=\iint_{x*y(S)}dxdy\int\frac{\partial F}{\partial z}dz=\sum_{j=1}^{\lambda}\int_{d_j}\left(\int\frac{\partial F}{\partial z}dz\right)dm,$$

where we apply the fact that $x*y(S)\text{-}\sum_{j=1}^{n}d_j(\subset d_0)$ is of measure zero. the difference between this and the former is

$$\sum_{j=1}^{\lambda}\left[\sum_{i=1}^{n}F(M_{ij})V(\delta_{ij})-\int_{d_j}\left(\int\frac{\partial F}{\partial z}dz\right)dm\right]$$

$$=\sum_j\int_{d_j}\left[\sum_i F(M_{ij})A(P,\delta_{ij})-\left(\int\frac{\partial F}{\partial z}dz\right)_P\right]dm(P)$$

by the definition of $V(\delta_{ij})$ of § 1, It converges to zero which will provod as follows.

Here, the field of integration of $\left(\int\frac{\partial F}{\partial z}dz\right)_P$ is the parts D_1 with the vertical line through $(x,y)=(x_P,y_P)$. The set of points P such that the parts said above contain any segment $MN\neq 0$ $(M\neq N)$ lying on the surface S has the measure zero, because if we take a sequence of subdivisions $\sigma:\sigma_1,\sigma_2,\sigma_3,\dots\dots\sigma_n\dots$ indefinitely subdividing, then the fact that M and N belong to distinct δ_i shows that the above points P belong finally to the image of the boundary $\underset{i}{V}x*y(\bar{\delta}_i-\delta_i)$. Therefore such points P are negligible. Then we may say $\alpha(P)\geqq 2\mu$ when the intersection by the vertical line is a sum of μ non-overlapping open intervals (this holds good even when $\mu=\infty$.) To prove it, it is enough to show $\alpha(P)\geqq 2\mu$ for μ finite and for the above open sets containing μ open intervals

$N_{2k-1}N_{2k}(k=1,.2,\ldots\ldots,\mu)$ which are either disjoint and, if contiguous, then separated by exterior points of $\bar{D}$. Now we assign such a exterior point to each of them, and appoint to each interval $N_{2k-1}N_{2k}$ an interior point, and denote by $O_l(l=1,2,\ldots\ldots,2\mu-1)$ these two kinds of points arranging by the increasing Z-coordinates. Consequently O_{2k-1} is an interior point and O_{2k} is an exterior point. In the next place we cut off by the cylinder of sufficient small diameter having this vertical line as the axis, then the parts of the surface S in this cylinder are divided into 2μ parts separated by the above points O_l, and we denote these 2μ parts by δ_l successively. Denoting the remaining parts of the surface S by $\delta_i(i>2\mu)$ appropriately, we set a subdivision σ. By the topological property of " Abbildungsgrad ", we have $A(P,\delta_l)=(-1)^l$.[(6)]

Therefore by the definition of section I, $\alpha_\sigma(P)=\sum_i|A(P,\delta_i)|=\sum_{i=1}^{2\mu}1=2\mu$, (by the property of " Abbildungsgrad " zero) and hence $\alpha(P)\geqq 2\mu$, (section I) from $\alpha(P)\geqq\alpha_\sigma(P)$. On the other hand, by the theorem of the last part of section I, $\int\alpha(P)dm$ is finite; consequently the set of point P such as $\mu=\infty$ is a set of measure zero in xy-plane; therefore these points are negligible by the integration in (4). Thus, we may consider only the case where the parts of the intersection of D_1 by a vertical line are the simple sum of a finite number of open intervals $N_{2k-1}N_{2k}$ contains an interior point. In this case, we have

$$\left(\int\frac{\partial F}{\partial z}dz\right)_P=\sum_{k=1}^{\mu}[F(N_{2k})-F(N_{2k-1})]=\sum_{l=1}^{2\mu}(-1)^lF(N_l).$$

Therefore, the integrand of (4) takes the following from:

$$\sum_i F(M_{ij})A(P,\delta_{ij})-\left(\int\frac{\partial F}{\partial z}dz\right)_P=\sum_i F(M_{ij})A(P,\delta_{ij})-\sum_{l=1}^{2\mu}(-1)^lF(N_l),$$

where $A(P,\delta_{ij})=\sum_{l\{N_l\in\delta_{ij}\}}(-1)^l$ is evident from the topological consideration.[(6)]

Therefore if we suppose $|F(N)-F(M_{ij})|<\varepsilon$ when $N\in\delta_{ij}$, the absolute value of the integrand of (4) does not surpass $2\mu\varepsilon$. As above mentioned $2\mu\leqq\alpha(P)$, therefore the absolute value of integral (4) does not surpass the following quantity

$$\sum_j\int_{d_j}\left|\sum_i F(M_{ij})A(P,\delta_{ij})-\left(\int\frac{\partial F}{\partial z}dz\right)_P\right|dm(P)\leqq\varepsilon\int\alpha(P)dm(P).$$

By the theorem in the last part of the section I, $\int \alpha(P)dm(P)$ is finite, and ε may be taken as small as pleased, by the uniform continuity of $F(M)$ on S, provided the subdivision is undergone infinitely. Thus the proof of the Gauss-Green's theorem is completed.

Remark. 1. If the surface is a set of measure zero (volume zero), we may replace the field of integration in the second member of (3) by D. Even when we suppose the surface is of bounded variation as above mentioned, we cannot say generally the surface S is of volume zero. It is difficult to give an example of the surface of volume positive. In this point differs from the case of curves.

2. Supposing that the surface is not only of $x*y$ bounded variation but also of $y*z$, $z*x$ bounded variation, and S is of volume zero, then, for any continuously differentiable vector (X, Y, Z) in $\bar{D}$, we obtain

$$\iiint_D \left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right)dxdydz=\int Xdy*z+Ydz*x+Zdx*y,$$

but generally we cannot prove this. Further, we connt say generally that this second member is a geometrical quantity (independent of the change of the axis of coordinates). In order to verifying this, we must calculate the transformation of the surface integral corresponding to the transformation of axis, for this purpose we have only to show that the Stieltjes integral $\int fd\varphi*\psi$ is linear type. But the relation which is easily proved in the case of one dimension

$$\int fd(\varphi_1+\varphi_2)*\psi=\int fd\varphi_1*\psi+\int fd\varphi_2*\psi$$

remains unproved. Considering these points, it seems that our conditions on variation boundedness is too weak, so that the appropriate restriction will be desirable, but in that case it will be difficult keep the analogy to one dimension—except the continuity (cf. footnote (2))—and, I remark that, if it is possible, we obtain, by developing into two dimensions the analogy of the definition of (continuous) curve by Jordan, a definition of the area of surface independent of the tangent plane.(7)

3. In particular, when the functions $\varphi(u, v)$, $\psi(u, v)$ are continuous in the bounded closed domain $\bar{\Delta}$, and moreover continuously differentiable in Δ, and that the image of boundary is of measure

zero ($m[\varphi * \psi(\bar{\Delta}-\Delta)]=0$), and moreover $\iint\left|\frac{\partial(\varphi, \psi)}{\partial(u, v)}\right| dudv < \infty$, then the mapping is of bounded variation, therefore the Stieltjes integral (Section I) takes the form

$$\int_{\Delta} f(u, v)\, d\varphi * \psi(u, v) = \iint f \frac{\partial(\varphi, \psi)}{\partial(u, v)} dudv.$$

this easily is proved from the fact $V(\delta) = \iint \frac{\partial(\varphi, \psi)}{\partial(u, v)} dudv$ (Section 1).

Further. when the surface S is given by the equation

$$S:\quad x=\varphi(u, v),\quad y=\psi(u, v),\quad z=\chi(u, v),$$

it is easily proved that the surface is of volume zero. Consequently, by Gauss-Green's theorem, when the surface S is such a surface (or the union of such surfaces), we conclude

$$\iiint_{\Delta} \frac{\partial F}{\partial z} dxdydz = \int_{S} Fdx*y = \iint_{\Delta} F \frac{\partial(\varphi, \psi)}{\partial(u, v)} dudv,$$

in this relation the sense of (u, v) must be taken properly (such that the positive side of S may be the exterior domain).

1) Brouwer; Math. Ann. 71 (1912), p. 97. It will be better to apply the recent, excellent Nagumo's theory: Nagumo, Syazôdo to Sonzaiteiri (Kawade Syobô Sûgaku Shûsyo).

2) It is restricted within the continuous mapping. In this point the analogy to one dimension is lost, but so long as we use the theory of " Abbildungsgrad" we cannot remove this restriction.

3) Kellogg, Foundations of Potential Theory (1929); Chap. IV. Except this there are some literature, for example: Tôzirô Ogasawara, Green no Teiri ni tsuite (Hiroshima Bunrika Daigaku Kiyô 12 kan p. 101, Shôwa 17 Nen)

4) the property of "Abbildungsgrad" zero: when $P \bar{\in} \mathfrak{A}(\delta)$, then $A(P, \delta)=0$.

5) by the property of "Abbildungsgrad" zero; we have $a(P)=0$ when $P \bar{\in} \mathfrak{A}(\Delta)$.

6) I have not yet the rigorous proof.

7) This subject was discussed by Banach by the paper; Sur les lignes rectifiables et les surfaces dont l'aire est finie. (Fundam. Math. 7(1925), p. 225). But the area defined by Banach in that paper is not verified to have the geometrical property (i. e. invariableness with respect to the choice of the xyz-coordinates axis in the space). I think that it is not a satisfactory definition, and by the same reason, we cannot succeed in the definition of the area of the surface.

(Redacted by T. Matsumoto and M. Yamaguchi and S. Mizohata)