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Notes on the F. K. Schmidt's "Quasidifferente" in Function-Fields.

By

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This note is concerned with the relations betwen the F. K. Schmidt's "Quasidifferente"[1] of an inseparable extention and that of its subfield. This relation is quite similar to the Dedekind's "Differenten-Kettensatz".[2] And as a direct consequence of this "Kettensatz", I have obtained the generalized Riemann-Hurwitz's formula.

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(I) Parastrophic matrix[3] and Schmidt's "Quasidifferente". Let L be an algebraic extention of a given rational function-field $K=k(x)$, and let M be an intermediate field of L and K. Let $\mathfrak{S}$, $\mathfrak{T}$ and $\mathfrak{R}=k[x]$ be integral domains of L, M and K respectively. Then we may assume

$$\mathfrak{T}=\mathfrak{R}[w_1, \ldots\ldots, w_m], \quad \mathfrak{S}=\mathfrak{R}[w_1, \ldots\ldots w_m, \ldots\ldots, w_{mn}]$$

and $$M=K[w_1, \ldots\ldots, w_m], \quad L=K[w_1, \ldots\ldots, w_m, \ldots\ldots, w_{mn}]$$

and $$w_i\, w_j=\textstyle\sum_k a_{ijk} w_k \qquad (i,\ j,\ k=1,\ 2, \ldots\ldots, m, \ldots\ldots mn.)$$

Then the parastrophic matrix formed by this basis (w_i) can be written

$$P_\lambda=(\textstyle\sum_k a_{ijk}\lambda_k)_{ij} \quad , \tag{1}$$

where $\lambda_k (k=1, \ldots\ldots, mn)$ are independent variables, and

$$a_{ijk}=0 \text{ for} \begin{cases} i,\ j=1, \ldots\ldots, m\ ;\ k=m+1, \ldots\ldots, mn \\ i=m+1, \ldots\ldots, mn\ ;\ j,\ k=1, \ldots\ldots, m \\ j=m+1, \ldots\ldots, mn\ ;\ i, k=1, \ldots\ldots,\ m, \end{cases}$$

and $a_{ijk}=a_{jik}$

We can easily verify that P_λ is regular[3], and that if

$$\alpha(w_i)=(w_i)A$$
$$(w_i)^T\alpha=A^T(w_i)^T$$

so we have

$$A^T=P_\lambda A P_\lambda^{-1}. \tag{2}$$

As L/K is a field, P_{λ_0} is regular for each $\lambda_i=\lambda^0, \lambda_i^0 \in K$, in which λ_i^0 is not all zero.[3] Let

$$(w'_1, \ldots\ldots, w'_{mn})=(w_1, \ldots\ldots, w_{mn})P_{\lambda_0}^{-1} \tag{3}$$

so we have easily

$$\alpha(w'_i)=(w'_i)A^T. \tag{4}$$

Consequently (w'_i) is a almost-complementary basis[4] of (w_i). Now let (λ^0) and (μ^0) be two systems of elements of K, for which P_{λ^0} and P_{μ^0} are regular, and let $(w''_i)=(w_i)P_{\mu^0}^{-1}$

then $\qquad P_{\lambda^0} A P_{\lambda^0}^{-1}=P_{\mu^0} A P_{\mu^0}^{-1}$,

therefore $\qquad P_{\mu^0}^{-1}P_{\lambda^0} A=A P_{\mu^0}^{-1}P_{\lambda^0}$

considering upon the non-commutative Galois theory of simple algebras, this shows that there exists such an element β of L that $\beta(w_i)=(w_i)B$ and $P_{\mu^0}^{-1}P_{\lambda^0}=B$ i. e. $P_{\mu^0}^{-1}=B P_{\lambda^0}^{-1}$, in other words two almost-complementary basis (w''_i) and (w'_i) are related by

$$(w''_i)=(\beta w_i') \tag{5}$$

(II) "Quasidifferenten-kettensatz". We can write P_λ in the following form

$$P_\lambda=\begin{pmatrix} Q_\lambda & * \\ * & * \end{pmatrix},$$

where Q_λ is regarded as the parastrophic matrix of $(w_1, \ldots\ldots, w_m)$. For $(\lambda_1^0, \ldots\ldots \lambda_m^0) \fallingdotseq (0, 0, \ldots\ldots, 0)$ and $\lambda_{m+1}^0=\ldots\ldots=\lambda_{mn}^0=0$, we have

$$P_{\lambda^0}=\begin{pmatrix} Q_{\lambda^0} & 0 \\ 0 & P_{\lambda^0} \end{pmatrix}$$

with regular Q_{λ^0} and R_{λ^0}. Using this parastrophic matrix we construct an almost-complementary basis $(w'_i)=(w_i)P_{\lambda 0}^{-1}$, so we have also $(w'_1,\ldots\ldots, w'_m)=(w_1, \ldots\ldots, w_m)Q_{\lambda 0}^{-1}$. This shows that the ideals $\mathfrak{e}_{31}^0$ and $\mathfrak{e}_{21}^0$ generated by $(w'_1, \ldots\ldots, w'_m, \ldots\ldots, w'_{mn})$ and $(w'_1, \ldots\ldots, w'_m)$ are related by $\mathfrak{e}_{32}^0\mathfrak{e}_{21}^0=\mathfrak{e}_{31}^0$. As $\mathfrak{e}_{31}^0$ and $\mathfrak{e}_{21}^0$ are uniquely determined by L/K and M/K except principal ideals, that is so for $\mathfrak{e}_{32}^0$. Therefore "Quasidifferentes" $\mathfrak{d}_{31}^0=(\mathfrak{e}_{31}^0)^{-1}$ and $\mathfrak{d}_{21}^0=(\mathfrak{e}_{21}^0)^{-1}$ are related by $\mathfrak{d}_{32}^0\mathfrak{d}_{21}^0=\mathfrak{d}_{31}^0$, where $\mathfrak{d}_{32}^0=(\mathfrak{e}_{32}^0)^{-1}$.

Considering (4) we have the

THEOREM 1. *Two "Quasidiffererentes" $\mathfrak{d}_{31}$ and $\mathfrak{d}_{21}$ are related by*

$$\mathfrak{d}_{32}^0\mathfrak{d}_{21}=(\xi)\mathfrak{d}_{31}$$

where (ξ) denotes a principal ideal of L.

REMARK The proof adopted in the above includes the proof of the Dedekind's "Differenten-Kettensatz". For example, if M is the maximally separable subfield of L, we may regard $\mathfrak{d}_{21}$ as Dedekind's Differente. Clearly if $M=K$, it follows $\mathfrak{d}_{32}^0\sim\mathfrak{d}_{31}$.

(III) Riemann-Hurwitz's formula. Using the theorem 1., we have the Riemann-Hurwitz's formula of the general case. For let g_L and g_M be the genera of L and M respectively, then we have[1)]

$$2g_L-2=\deg.(\mathfrak{d}_{31})-2mn$$
$$2g_M-2=\deg.(\mathfrak{d}_{21})-2m$$

by the definition of the genus due to Schmidt. So we have

THEOREM 2. *Let L be an algebraic functionfield of one variable over k, and M its subfield, then we have*

$$2g_L-2=\deg.(\mathfrak{d}_{32}^0)+n(2g_M-2)$$

COROLLARY *It follows from $\mathfrak{d}_{32}^0\sim 1$, that $L=M$.*

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1) F. K. Schmidt : Zur arithmetischen Theorie der algebraischen Funktionen I. Math. Zeitschr. Bd. 41 (1936)

2) E. Hecke : Theorie der algebraischen Zahlen. (1923)

3) T. Nakayama : On Frobeniusean algebras. Ann. of Math. Vol. 41 (1940)
G. Frobenius : Theorie der hyperkomplexen Grössen. I, II. Bericht der Berl. Akad. Jahrgang 1903;

4) I have used "almost-complementary" instead of Schmidt's "fastkomplementär".