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Note on the non-increasing solutions of $y''=f(x,y,y')$.

By

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The non-increasing solutions of the ordinary differential equation of the second order,

$$\frac{d^2y}{dx^2}=f\left(x, y, \frac{dy}{dx}\right), \tag{1}$$

have been discussed by A. Kneser, A. Mambriani, G. Scorza Dragoni and recently by P. Hartman and A. Wintner etc.. In the following we will give some comments on the same topics.

At first on the increasing solutions of (1), the

Theorem 1.[1] Let $f(x,y,z)$ be defined and continuous in the domain $0\leqq x<\infty$, $0\leqq y<\infty$, $-\infty<z<\infty$. Suppose that, for every $C>0$, there exists a continuous function $\Phi(x,y,z)=\Phi_C(x,y,z)$ for the domain

$$\Delta c : 0\leqq x\leqq C,\quad 0\leqq y\leqq C,\quad -k\leqq z\leqq 0$$

(constant $k>0$ may be arbitrarily small) with the continuous first partial derivatives in the interior of Δc, and that $\Phi(x,y,z)>0$ for $z\neq 0$, $\Phi(x,y,0)=0$, moreover that, in the interior of Δc, we have

$$\frac{\partial\Phi}{\partial x}+\frac{\partial\Phi}{\partial y}\cdot z+\frac{\partial\Phi}{\partial z}\cdot f(x,y,z)\leqq 0. \tag{2}$$

Then let $y=y(x)$ be a solution of (1) no $(0\leqq)a\leqq x\leqq b$ satisfying the initial conditions

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1) P. Hartman and A. Wintner have proved a theorem like this in American Journal of Mathematics, Vol. 73 (1951). Since their conditions are $f(x,y,0)\gtrless 0$ and $f(x,y,z)-f(x,y,0)\geqq Mz$, if we take $\Phi=e^{-2Mx}\cdot z^2$, then it satisfies evidently (2) and (5), and so their theorem becomes a special case of ours. Hence their condition $f(x,0,0)\equiv 0$ for $0\leqq x<\infty$ is not necessary. Also their Lemma 1 (p. 391) can be deduced from this theorem.

(3) $$y(a) \geqq 0 \quad \text{and} \quad y'(a) \geqq 0,$$

then we have

(4) $$y'(x) \geqq 0 \quad \text{for} \quad a \leqq x \leqq b.$$

Also, let Δc be the domain $[0 \leqq x \leqq C,\ 0 \leqq y \leqq C,\ k \geqq z \geqq 0]$ and if, instead of (2), $\Phi(x, y, z)$ satisfies

(5) $$\frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial y} \cdot z + \frac{\partial \Phi}{\partial z} \cdot f(x, y, z) \geqq 0$$

and $y'(a) > 0$, then

(6) $$y'(x) > 0 \quad \text{for} \quad a < x \leqq b.$$

Proof. Now we consider the case (3). By the continuity of $y(x)$ and the definition of $f(x, y, z)$, there exists a positive number L such as $0 \leqq y(x) \leqq L$ for $a \leqq x \leqq b$ except the trivial case $y \equiv 0$. Take $C = \max(b, L)$ and consider the function $\Phi(x, y, z)$ corresponding to this C. Now suppose that $y'(x)$ is negative at a value of x. Then, by the continuity of $y'(x)$, there exist two values of x, say $x = x_1$, and $x = x_2$ such that $y'(x_1) = 0$, $y'(x_2) > -k$ and $-k < y'(x) < 0$ for $x_1 < x < x_2$. In this interval, $\Phi(x, y(x), y'(x)) \equiv 0$ since $\Phi(x, y(x), y'(x))$ is non-increasing with respect to x and yet $\Phi(x_1, y(x_1), y'(x_1)) = \Phi(x_1, y(x_1), 0) = 0$. This contradicts with $\Phi(x_2, y(x_2), y'(x_2)) > 0$, for $y'(x_2) \neq 0$. Therefore we have (4).

In the case where $y'(a) > 0$, also we may conclude that there exist no points such as $y'(x) = 0$ by (5) in the same way, and so (6) is proved.

For the next theorem, we use

Lemma 1. (Okamura's existence theorem[2]). Let $f(x, y, z)$ be continuous in the domain

$$\mathfrak{L}^* : \quad a \leqq x \leqq b,\ \underline{\omega}(x) \leqq y \leqq \bar{\omega}(x),\ -\infty < z < \infty,$$

where, for $a \leqq x \leqq b$, $\underline{\omega}(x)$ and $\bar{\omega}(x)$ are twice differentiable, $\underline{\omega}(x) \leqq \bar{\omega}(x)$ and satisfy

(7) $$\underline{\omega}''(x) \geqq f(x, \underline{\omega}(x), \underline{\omega}'(x))$$

and

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2) Okamura, "On $y'' = f(x, y, y')$ (II)". Functional Equations (in Japanese), Vol. 30 (1941), pp. 14-19.

$$\bar{\omega}''(x) \leqq f(x, \bar{\omega}(x), \bar{\omega}'(x)) \tag{8}$$

respectively. And let $\psi(x)$ be a function defined on $a \leqq x \leqq b$, satisfying $\omega(x) \leqq \psi(x) \leqq \bar{\omega}(x)$ $(a \leqq x \leqq b)$ and

$$|\psi(x') - \psi(x'')| \leqq L \cdot |x' - x''| \quad (a \leqq x', x'' \leqq b), \tag{9}$$

where L is a constant.

Moreover suppose that there exist four functions $\Phi_i(x, y, z)$ and $\Psi_i(x, y, z)$ $(i=1, 2)$ such as follows; $\Phi_1(x, y, z)$ is continuous in $[a \leqq x \leqq b, \omega(x) \leqq y \leqq \psi(x), z \geqq K]$ (constant $K>0$ may be arbitrarily great) and has continuous first partial derivatives in the interior of this domain. Ψ_1, Φ_2 and Ψ_2 are the same respectively in $[a \leqq x \leqq b, \psi(x) \leqq y \leqq \bar{\omega}(x), z \geqq K]$, $[a \leqq x \leqq b, \psi(x) \leqq y \leqq \bar{\omega}(x), z \leqq -K]$ and $[a \leqq x \leqq b, \omega(x) \leqq y \leqq \psi(x), z \leqq -K]$. These functions are supposed positive in their definition domains, while in their domains they converge uniformly to zero for $z \rightarrow \pm\infty$ respectively; and finally in the interior of their respective domains, the inequalities

$$\frac{\partial \Phi_i}{\partial x} + \frac{\partial \Phi_i}{\partial y} \cdot z + \frac{\partial \Phi_i}{\partial z} \cdot f(x, y, z) \geqq 0 \tag{10}$$

and

$$\frac{\partial \Psi_i}{\partial x} + \frac{\partial \Psi_i}{\partial y} \cdot z + \frac{\partial \Psi_i}{\partial z} \cdot f(x, y, z) \leqq 0 \tag{11}$$

hold.

Then, in $\mathfrak{L}^*$, there exists at least a solution of (1), $y=y(x)$, satisfying $y(a)=\psi(a)$ and $y(b)=\psi(b)$.

This lemma is proved by the following idea; We can suppose that $K>L$, $K>|\bar{\omega}'(x)|$ and $K>|\omega'(x)|$, for K may be great. Let M be greater than K and satisfy for $a \leqq x \leqq b$, $\omega(x) \leqq y \leqq \psi(x)$,

$$\min \Phi_1(x, y, K) > \max \Phi_1(x, y, M)$$

and for $a \leqq x \leqq b$, $\psi(x) \leqq y \leqq \bar{\omega}(x)$,

$$\min \Phi_2(x, y, -K) > \max \Phi_2(x, y, -M);$$

the same with Ψ_1 and Ψ_2, where we take for M a common value. Now we define a bounded continuous function $g(x, y z)$ as follows. Namely when $\omega(x) \leqq y \leqq \bar{\omega}(x)$,

$$g(x,y,z)=\begin{cases} f(x,y,M) & \text{for } z>M \\ f(x,y,z) & \text{for } -M\leqq z\leqq M \\ f(x,y,-M) & \text{for } z<-M, \end{cases}$$

$g(x,y,z)>g(x,\bar{\omega}(x),z)$ for $y>\bar{\omega}(x)$ and $g(x,y,z)<g(x,\underline{\omega}(x),z)$ for $y<\underline{\omega}(x)$. Then $g(x,y,z)$ coincides with $f(x,y,z)$ in $\Delta\,[a\leqq x\leqq b$, $\underline{\omega}(x)\leqq y\leqq\bar{\omega}(x)$, $-M\leqq z\leqq M]$ and becomes bounded[3] continuous in $[a\leqq x\leqq b$, $-\infty<y<\infty$, $-\infty<z<\infty]$. Hence $y''=g(x,y,y')$ has at least a solution $y=y(x)\,(a\leqq x\leqq b)$ such as $y(a)=\psi(a)$ and $y(b)=\psi(b)$, and we see that this solution satisfies $\underline{\omega}(x)\leqq y(x)\leqq\bar{\omega}(x)$. Moreover considering the functions $\Phi_i(x,y(x),y'(x))$ and $\Psi_i(x,y(x),y'(x))$, it will follow that there exist no points satisfying $y'(x)\geqq M$ or $y'(x)\leqq -M$, by the properties of M and inequalities (10) and (11). Therefore $|y'(x)|<M$ and so $y=y(x)$ lying in Δ is a solution of $y''=g=f$.

Using this lemma and Theorem 1, we obtain the following theorem.

Theorem 2. Let $f(x,y,z)$ be a continuous function in the domain

$$(12)\qquad R:\ 0\leqq x<\infty,\ 0\leqq y<\infty,\ -\infty<z\leqq 0$$

and suppose that

$$(13)\qquad f(x,0,0)\equiv 0 \text{ for } 0\leqq x<\infty$$

and that

$$(14)\qquad f(x,y,0)\geqq 0 \text{ for } 0\leqq x<\infty,\ 0\leqq y<\infty.$$

Moreover suppose that for every $C>0$ there exist two functions $\Phi(x,y,z)=\Phi_C(x,y,z)$ and $\Psi(x,y,z)=\Psi_C(x,y,z)$ as follows; namely Φ and Ψ are positive continuous and converge uniformly to zero for $z\to-\infty$ in

$$(15)\qquad R_C:\ 0\leqq x\leqq C,\ 0\leqq y\leqq C,\ z\leqq -K$$

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3) E. g., for $y>\bar{\omega}(x)$

$$g(x,y,z)=g(x,\bar{\omega}(x),z)+\frac{y-\bar{\omega}(x)}{1+y-\bar{\omega}(x)},$$

and for $y<\underline{\omega}(x)$

$$g(x,y,z)=g(x,\underline{\omega}(x),z)-\frac{\underline{\omega}(x)-y}{1+\underline{\omega}(x)-y}.$$

(constant $K>0$ may be arbitrarily great) and, in the interior of R_C, they have continuous first partial derivatives which satisfy

$$\frac{\partial \Phi}{\partial x}+\frac{\partial \Phi}{\partial y}\cdot z+\frac{\partial \Phi}{\partial z}\cdot f(x, y, z) \geqq 0 \tag{16}$$

and

$$\frac{\partial \Psi}{\partial x}+\frac{\partial \Psi}{\partial y}\cdot z+\frac{\partial \Psi}{\partial z}\cdot f(x, y, z) \leqq 0 \tag{17}$$

respectively.

Then for $x_0>0$ and $y_0>0$ (1) has at least a solution $y=y(x)$ on $0 \leqq x \leqq x_0$, satisfying

$$y(0)=y_0 \text{ and } y(x_0)=0. \tag{18}$$

Proof. Let $\underline{\omega}(x) \equiv 0$, $\bar{\omega}(x) \equiv y_0$ and

$$f^*(x, y, z)=\begin{cases} f(x, y, z) & (-\infty<z \leqq 0) \\ f(x, y, 0) & (0<z<\infty). \end{cases}$$

Then $f^*(x, y, z)$ becomes continuons in $[0 \leqq x \leqq x_0,\ 0 \leqq y \leqq y_0,\ -\infty<z<+\infty]$, and $\underline{\omega}(x)$, $\bar{\omega}(x)$ satisfy (7) and (8) respectively, for $\underline{\omega}''(x)=0=f^*(x, \underline{\omega}(x), \underline{\omega}'(x))$ and $\bar{\omega}''(x)=0 \leqq f^*(x, y_0, 0)=f^*(x, \bar{\omega}(x), \bar{\omega}'(x))$. Let $\psi(x)=y_0\left(1-\frac{x}{x_0}\right)$ and then $\psi(x)$ satisfies (9) clearly. As Φ_2 and Ψ_2 in Lemma 1, we use Φ_C and Ψ_C corresponding to C respectively, where $C=\max(x_0, y_0)$. For $z>0$, there is a positive number M such that, for $0 \leqq x \leqq x_0$, $0 \leqq y \leqq y_0$, $0 \leqq f(x, y, 0) \leqq \frac{M}{2}$, hence we have

$$0 \leqq f(x, y, 0)=f^*(x, y, z) \leqq \frac{M}{2}(1+z^2) \quad (z>0). \tag{19}$$

Hence putting

$$\Phi_1(x, y, z)=e^{y-\frac{1}{M}\log(1+z^2)} \quad (z \geqq 0),$$

and

$$\Psi_1(x, y, z)=e^{-z} \quad (z \geqq 0);$$

clearly Φ_1 and Ψ_1 satisfy the conditions of Lemma 1. Therefore

$$\frac{d^2y}{dx^2}=f^*\left(x, y, \frac{dy}{dx}\right). \tag{20}$$

has at least a solution $y=y(x)$ passing through $(x=0, y=y_0)$ and $(x=x_0, y=0)$.

Now considering the function $\Phi(x, y, z)=z^2$, it is continuous in the domain Δc of Theorem 1, $\Phi>0$ for $z \neq 0$ and $\Phi=0$ for $z=0$, and finally,

$$\frac{\partial\Phi}{\partial x}+\frac{\partial\Phi}{\partial y}\cdot z+\frac{\partial\Phi}{\partial z}f^*(x, y, z)$$
$$=2z\cdot f^*(x, y, z)\geqq 0,$$

provided $z>0$. Therefore if $y'(x)$ is positive at a value of x say $x=\xi$, then by Theorem 1, $y'(x)>0$ for $\xi<x\leqq x_0$; but this contradicts with $y(x_0)=0$ since $y(\xi)\geqq 0$. Hence $y'(x)\leqq 0$ for $0\leqq x\leqq x_0$ and so Theorem 2 is proved.

From this proof, we see that (13) and (14) are not essential conditions. It is the same with Theorem 3 mentioned below.[4)]

For Theorem 3, the following Lemmas are needed.

Lemma 2.[5)] Let $\mathfrak{L}$ be a closed bounded domain in (x, y) plane and $\mathfrak{L}^*$ be the three-dimensional domain of (x, y, z) such that $(x, y)\in\mathfrak{L}$ and $|z|<\infty$. Let $f(x, y, z)$ be a continuous function in $\mathfrak{L}^*$. Moreover we suppose that there exist two functions $\Phi_i(x, y, z)$ $(i=1, 2)$ as follows; $\Phi_1(x, y, z)$ is continuous in the domain $[(x, y)\in\mathfrak{L}, z\geqq K]$ (K: constant) and has contiouous first partial derivatives in the interior of the domain, and $\Phi_2(x, y, z)$ is the same in $[(x, y)\in\mathfrak{L}, z\leqq -K]$. These functions are always positive and converge uniformly to zero for $(x, y)\in\mathfrak{L}$ and $z\to\pm\infty$ and finally, in the interior of their respective domains, the following inequalities

$$\frac{\partial\Phi_i}{\partial x}+\frac{\partial\Phi_i}{\partial y}\cdot z+\frac{\partial\Phi_i}{\partial z}\cdot f(x, y, z)\geqq 0 \quad (i=1, 2)$$

hold.

Then for an arbitrary positive number α, there is another positive number $\beta(\alpha)$ such that for any solution $y=y(x)$ of (1) with the initial conditions $y(x_0)=y_0$ and $|y'(x_0)|\leqq\alpha$, where (x_0, y_0) is an arbitrary point in $\mathfrak{L}$, the inequality $|y'(x)|<\beta(\alpha)$ always

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4) We can modify the Theorems 1, 2, 3, 4 for more general regions of y and y'.

5) Okamura, "On $y''=f(x, y, y')$", Functional Equations (in Japanese), Vol. 27 (1941), pp. 29-30.

holds, as long as $y=y(x)$ lies in the interior of $\mathcal{L}$ for $x_0 \leqq x$.

If in place of Φ_i, there exist similar functions $\Psi_i(x, y, z)$ $(i=1, 2)$ satisfying

$$\frac{\partial \Psi_i}{\partial x} + \frac{\partial \Psi_i}{\partial y} \cdot z + \frac{\partial \Psi_i}{\partial z} \cdot f(x, y, z) \leqq 0 \quad (i=1, 2),$$

then the above stated inequality holds for $x_0 \geqq x$.

For consider $\Phi_i(x, y(x), y'(x))$ which is positive at $x=x_0$ and non-decreasing with x, then the assumption that $\beta(a)$ does not exist contradicts with $\Phi_i(x, y, y') \to 0$ for $|y'| \to \infty$ and uniformly for $(x, y) \in \mathcal{L}$.

Lemma 3.[6] Let $\omega(x)$ and $\bar{\omega}(x)$ be twice differentiable for $a \leqq x \leqq b$ and $\omega(a) > \bar{\omega}(a)$. Suppose that the function $f(x, y, z)$ satisfies the same conditions as in Lemma 2 and that there exist $\Phi_i(x, y, z)$ and $\Psi_i(x, y, z)$ $(i=1, 2)$ as in Lemma 2.

Then there exists a positive number γ having the following properties; Every solution curve of (1) with the initial conditions $\omega(a) \leqq y(a) < \bar{\omega}(a)$ and $y'(a) \geqq \gamma$ meets the curve $y=\bar{\omega}(x)$ for a value of x in $a < x < b$, and every solution curve with the initial conditions $\omega(a) < y(a) \leqq \bar{\omega}(a)$ and $y'(a) \leqq -\gamma$ meets the curve $y=\omega(x)$ for a value of x in $a < x < b$.

Under the above preparations, we shall prove the following theorem which is an extension of Hartman and Wintner's theorem.

Theorem 3. Suppose that $f(x, y, z)$ satisfies the same conditions as in Theorem 2. Then for every $y_0 > 0$, there exists at least a solution of (1) for $0 \leqq x < \infty$, satisfying the initial condition

$$y(0) = y_0 \tag{21}$$

and the inequalities

$$y(x) \geqq 0 \quad \text{and} \quad y'(x) \leqq 0. \tag{22}$$

Proof. Let $y_0 > 0$ be fixed and $y=y(x, x_0)$ be a solution of (1), for $0 \leqq x \leqq x_0$, satisfying (18). Its existence is evident from Theorem 2. We can suppose that $y(x, x_0) > 0$ for $0 \leqq x \leqq x_0$, for if it be not so, the assertion is clear. Let $x_0 = n$ $(n=1, 2, \ldots)$, then it follows

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6) Okamura, loc. cit., p. 30. C. f. Nagumo, "Über die Differentialgleichung $y''=f(x, y, y')$," Proc. of Physico-Math. Soc. of Japan, ser 3, vol. 19 (1937), pp. 863-864.

from Lemma 2 and 3 that, for any integer $k>0$, there exists a constant $M>0$ corresponding to k such that

$$0 \geqq y'(x, n) \geqq -M \text{ for } 0 \leqq x \leqq k \text{ and } n=k+1, k+2, \dots .$$

Hence by equicontinuity, $y(x, n_1), y(x, n_2), \dots$ tends to a desired solution of (1), if $n_1, n_2, \dots$ are chosen suitably.

Remark. Hartman and Wintner's theorem[7] is a special case of the above; for we may put

$$\Phi(x, y, z) = e^{-y-\int_0^z \frac{zdz}{\phi(z)}} \quad (z \leqq 0)$$

$$\Psi(x, y, z) = e^{y-\int_0^z \frac{zdz}{\phi(z)}} \quad (z \leqq 0),$$

e.g., Φ satisfies the conditions clearly, because

$$\frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial y} \cdot z + \frac{\partial \Phi}{\partial z} \cdot f(x, y, z)$$

$$= e^{-y-\int_0^z \frac{zdz}{\phi(z)}} \left\{ -z - \frac{z}{\phi(z)} \cdot f(x, y, z) \right\}$$

$$\geqq e^{-y-\int_0^z \frac{zdz}{\phi(z)}} \{ -z+z \} = 0.$$

Then concerning the solutions of (1) which satisfy (22) for $0 \leqq x < \infty$, there arises a question, whether the limit relation

$$\lim_{x \to \infty} y(x) = 0 \tag{23}$$

is true or not. About it we have

Theorem 4. Let $f(x, y, z)$ be a continuous function in R. Suppose that for every pair of constants $0 < c < C$, there exists a positive continuous function $\Phi(y, z)$ in R^* $[c \leqq y \leqq C, z \leqq -K]$ (K: constant) with continuous first partial derivatives in the interior of R^*, converging uniformly to zero for $c \leqq y \leqq C$ when $z \to -\infty$ and satisfying, in the interior of $[0 \leqq x < \infty, c \leqq y \leqq C, z \leqq -K]$, the inequality

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7) P. Hartman and A. Wintner, "On the non-increasing solutions of $y''=f(x, y, y')$", American Journal of Math., Vol. 73 (1951), p. 391.

$$\frac{\partial \Phi}{\partial y}\cdot z+\frac{\partial \Phi}{\partial z}\cdot f(x,y,z)\geqq 0. \tag{24}$$

Moreover suppose that, for every pair of constants $0<c'<C'$, there exists a positive continuous function $\Psi(x,y,z)$ in $\bar{R}\,[\xi\leqq x<\infty,\ c'\leqq y\leqq C',\ -C'\leqq z\leqq 0]$ (ξ: constant) such that it converges uniformly to zero for $c'\leqq y\leqq C'$, $-C'\leqq z\leqq 0$ when $x\to\infty$ and that, in the interior of $\bar{R}$ (including $z=0$), has continuous first partial derivatives and satisfies

$$\frac{\partial \Psi}{\partial x}+\frac{\partial \Psi}{\partial y}\cdot z+\frac{\partial \Psi}{\partial z}\cdot f(x,y,z)\geqq 0. \tag{25}$$

Then for any solution $y=y(x)$ of (1) on $0\leqq x<\infty$ satisfying (22), the relation (23) is true.

Proof. For if otherwise, then $\lim_{x\to\infty} y(x)=c>0$ would hold for a certain solution $y=y(x)$. Hence if $y(0)\leqq C$, $c\leqq y(x)\leqq C$ for $0\leqq x<\infty$. Now we may consider $y'(0)>-K$, since K may be great. Choose $N(>K>0)$ such as

$$\min_{c\leqq y\leqq C}\Phi(y,-K)>\max_{c\leqq y\leqq C}\Phi(y,-N),$$

and then we shall have $-N<y'(x)\leqq 0$ for $0\leqq x<\infty$. Because if for a value of x, say $x=\xi_1$, $y'(\xi_1)\leqq -N$ holds, by the continuity of $y'(x)$, there are ξ_2 and ξ_3 such that $y'(\xi_2)=-K$, $y'(\xi_3)=-N$ and $-K>y'(x)>-N$ for $\xi_2<x<\xi_3$, and since we have $\Phi(y(\xi_2),y'(\xi_2))=\Phi(y(\xi_2),-K)>0$ and $\Phi(y(x),y'(x))$ is non-decreasing with x by (24), it follows that $\Phi(y(\xi_2),-K)\leqq\Phi(y(\xi_2),-N)$ which contradicts with the choice of N.

Now choose $0<c'<c$ and $C'>\max(y(0),N)$ and $\Psi(x,y,z)$ corresponding to c' and C'. As $\Psi(x,y(x),y'(x))$ is non-decreasing with x by (25) and $\Psi(\xi,y(\xi),y'(\xi))>0$, we have

$$\lim_{x\to\infty}\Psi(x,y(x),y'(x))\geqq\Psi(\xi,y(\xi),y'(\xi))>0.$$

This contradiction proves the theorem.

Remark 1. $\Psi(x,y,z)$ can be replaced by $\Psi^*(x,y,z)$ which is positive, tends to infinity ($x\to\infty$) and satisfies, instead of (25),

$$\frac{\partial \Psi^*}{\partial x}+\frac{\partial \Psi^*}{\partial y}\cdot z+\frac{\partial \Psi^*}{\partial z}\cdot f(x,y,z)\leqq 0.$$

Remark 2. Rutting

$$\Phi(y, z) = e^{-z^2} \quad (z \leqq -K)$$

and

$$\Psi(x, y, z) = e^{-\int_0^x x\sigma(x)dx - y + xz} \quad (0 \leqq x < \infty,\ c' \leqq y \leqq C',\ -C' \leqq z \leqq 0),$$

we obtain Hartman and Wintner's theorem.[8)]

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8) Hartman and Wintner, loc. cit., p. 399.