MEMOIRS OF THE COLLEGE OF SCIENCE, UNIVERSITY OF KYOTO, SERIES, A
Vol. XXVII, Mathematics No. 2, 1952.

Note on the Integral Representation of Mathieu Functions.

By

Toshizô MATSUMOTO

(Received Mach 31. 1952)

We have read McLachlan's treatise on the theory and application of Mathieu functions which has quite recently reached us for the first time. After Whittaker and Watsons' treatise on the modern analysis, Mathieu functions satisfy homogeneous integral equations. Until to-day certain special functions are known as the kernels of the integral equations. McLachlan found many other kernels although they are yet special functions. On the other hand Mr. Bunshiro Hiromoto, attending to my seminary of the first semester of two years ago, found the general form of the kernels which I intend to report here briefly.

After a long study of Mathieu functions, Hiromoto's research has begun from Whittaker's proposition that the odd Mathieu functions satisfy the integral equation

$$G(\eta)=\lambda\int_{-\pi}^{\pi}\sinh\ (k\sin\eta\sin\theta)\,G(\theta)d\theta$$

is not complete. It is satisfied by $se_{2n+1}(x,q)$ but not by $se_{2n}(x,q)$, since Hiromoto proved that the kernel of the integral equation satisfied by Mathieu functions belonging to one of four types is orthogonal to Mathieu functions of the remaining types in $(-\pi,\pi)$. There is another incomplete proposition.

Now Hiromoto reassured that if $K(x,\theta)$, twice continuously differentiable in $-\pi\leq x\leq\pi$, $-\pi\leq\theta\leq\pi$ and periodic with period 2π for both variables, satisfy

$$\frac{\partial^2 K}{\partial x^2}-\frac{\partial^2 K}{\partial\theta^2}=k^2(\cos^2 x-\cos^2\theta)K, \tag{1}$$

then the periodic solutions with period 2π say $G(x)$ of the Mathieu differential equation

$$\frac{d^2y}{dx^2}+(a-16q\cos 2x)y=0 \tag{2}$$

or

$$\frac{d^2y}{dx^2} + (A - k^2 \cos^2 x) y = 0 \tag{2'}$$

satisfy the intgral equation

$$G(x) = \lambda \int_{-\pi}^{\pi} K(x, \theta) G(\theta) d\theta. \tag{3}$$

McLachlan transformed (1) into an equation of the elliptic type bv some imaginary transformation whose several particular integrals are known from which many particular kernels are found. Without knowing it, Hiromoto solved directly the equation (1) by Riemann's method. By the transformation

$$\xi = k/2 \cos(\theta + x), \quad \eta = k/2 \cos(\theta - x), \tag{4}$$

(1) becomes

$$\frac{\partial^2 K}{\partial \xi \partial \eta} + K = 0. \tag{5}$$

Changing the notations,

$$\frac{\partial^2 K}{\partial x \partial y} + K = 0 \tag{5'}$$

has for its adjoined equation,

$$\frac{\partial^2 v}{\partial x \partial y} + v = 0. \tag{6}$$

Riemann's function $v(x, y; \xi, \eta)$ satisfies (6) and takes the value 1 on the characteristics $x=\xi$ and $y=\eta$ of (5′). Putting

$$X = 2\sqrt{(x-\xi)(y-\eta)},$$

such function satisfies

$$X \frac{d^2 v}{dX^2} + \frac{dv}{dX} + Xv = 0;$$

therefore we have

$$v(x, y; \xi, \eta) = J_0(2\sqrt{(x-\xi)(y-\eta)}). \tag{7}$$

If we will find the solution of (5) under the boundary conditions that for $y=0$, we have $K=\varphi(x)$ and for $x=0$, we have $K=\psi(y)$, we have by the Riemann's formula

$$K(\xi,\eta)=\varphi(0)J_0(2\sqrt{\xi\eta})+\int_0^\xi J_0\{2\sqrt{\eta(\xi-x)}\}\varphi'(x)dx$$
$$+\int_0^\eta J_0\{2\sqrt{\xi(\eta-y)}\}\psi'(y)dy \qquad (8)$$

provided $\varphi(0)=\psi(0)$. φ, ψ being arbitrary functions, (8) is the general integral of (5) from which the general solution of (1) can be found by the inverse of the transformation (4).

For an example let $\varphi(x)=\cos x, \psi(y)=\cos y$, then we have

$$K(\xi,\eta)=J_0(2\sqrt{\xi\eta})-\int_0^\xi J_0\{2\sqrt{\eta(\xi-x)}\}\sin x dx$$
$$-\int_0^\eta J_0\{2\sqrt{\xi(\eta-y)}\}\sin y dy.$$

Expanding J_0 into the power-series, we have by some calculations,

$$K(\xi,\eta)=\cos(\xi+\eta)$$

which becomes by the inverse transformation (4)

$$K(x,\theta)=\cos(k\cos x\cos\theta).$$

This is the wellknown kernel for ce_{2n}. We may obtain the remaining ones in the same way.

Remark. These considerations may be applied in any other case, but the partial differential equation satisfied by the kernel is always difficult to be integrated in a finite form. For Lamé's differential equation

$$\frac{d^2y}{dx^2}=\{n(n+1)k^2sn^2x+A\}y,$$

we have for the kernel $K(x,\theta)$,

$$\frac{\partial^2K}{\partial\xi\partial\eta}=\frac{n(n+1)}{(1-\xi\eta)^2}K,$$

where

$$\xi=k\,sn^2\frac{x+\theta}{2}, \quad \eta=k\,sn^2\frac{x-\theta}{2}.$$

For Whittaker's differential equation

$$\frac{d^2y}{dx^2}+\alpha\sin 2x\frac{dy}{dx}+(\beta-p\alpha\cos 2x)y=0,$$

we have for the kernel

$$2\frac{\partial^2 K}{\partial\xi\partial\eta}-\alpha\xi\frac{\partial K}{\partial\xi}-\alpha\eta\frac{\partial K}{\partial\eta}+p\alpha K=0$$

which may be transformed into

$$2\frac{\partial^2 V}{\partial\xi\partial\eta}+\left\{(p+1)\alpha-\frac{1}{2}\alpha^2\xi\eta\right\}V=0,\quad K=e^{\frac{\alpha}{2}\xi\eta}.$$

This is a particular case of that of Hill's equation.

Generally when the function under the integral sign does not need necessarily to be equal to the function which we want to represent by the integral formula. Now let

$$L_x(y)=a_0\frac{d^n y}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+\dots+a_n y=0,$$

$$\overset{\circ}{L}_x(y)=(-1)^n\frac{d^n(a_0 y)}{dx^n}+(-1)^{n-1}\frac{d^{n-1}(a_1 y)}{dx^{n-1}}+\dots+a_n y=0,$$

$$M_\theta(z)=b_0\frac{d^m z}{d\theta^m}+b_1\frac{d^{m-1}z}{d\theta^{m-1}}+\dots+b_m z=0,$$

$$\overset{\circ}{M}_\theta(z)=(-1)^m\frac{d^m(b_0 z)}{d\theta^m}+(-1)^{m-1}\frac{d^{m-1}(b_1 z)}{d\theta^{m-1}}+\dots+b_m z=0.$$

Let the kernel satisfy the partial differential equation

$$L_x\{K(x,\theta)\}=M_\theta\{K(x,\theta)\},$$

then in case of periodic integral, we have

$$y(x)=\lambda\int_{-\pi}^{\pi}K(x,\theta)z(\theta)d\theta$$

where y is the integral of $L_x(y)=0$; z that of $\overset{\circ}{M}_\theta(z)=0$.

For Example let

$$L_x(y)=x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+x^2y=0,$$

$$M_\theta(z)=-\frac{d^2z}{d\theta^2}=0,$$

then the partial differential equation satisfied by the kernel $K(x,\theta)$ becomes

$$x^2\frac{\partial^2 K}{\partial x^2}+x\frac{\partial K}{\partial x}+x^2K+\frac{\partial^2 K}{\partial\theta^2}=0, \tag{9}$$

and

$$y(x)=\lambda\int_{-\pi}^{\pi}K(x,\theta)z(\theta)d\theta.$$

Under the assumption of the periodicity, $z(\theta)$ must be constant, say 1. Putting $t=\cos\theta$, (9) becomes

$$x^2\frac{\partial^2 K}{\partial x^2}+(1-t^2)\frac{\partial^2 K}{\partial t^2}+x\frac{\partial K}{\partial x}-t\frac{\partial K}{\partial t}+x^2K=0.$$

If we put $x=e^{\xi}$, (9) becomes

$$\frac{\partial^2 K}{\partial \xi^2}+\frac{\partial^2 K}{\partial \theta^2}+e^{2\xi}K=0.$$

This is the equation well known under the vibrating membranes with several boundary conditions. Here we shall solve simply by the seperation of variables. Putting $K=X(x)\Theta(\theta)$, we have at once

$$x^2\frac{d^2X}{dx^2}+x\frac{dX}{dx}+(x^2-m^2)X=0,$$

$$\frac{d^2\Theta}{d\theta^2}+m^2\Theta=0,$$

where m^2 is a constant. As the general integral, we shall have in form

$$K=\sum_m(A_mJ_m(x)+B_mJ_{-m}(x))(a_m\cos m\theta+b_m\sin m\theta).$$

If we assign the arbitrary constants so as

$$K=J_0(x)-2J_2(x)\cos 2\theta+2J_4(x)\cos 4\theta-\dots$$

and $$=\cos(x\cos\theta),$$

then we have the well known formula *provided* $\lambda=\frac{1}{2\pi}$,

$$J_0(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(x\cos\theta)d\theta$$

By the differentiation under the sign of integral, we may have the general formula, $\Re\left(n+\frac{1}{2}\right)>0$,

$$x^{-n}J_n(x)=\frac{1}{2^{n+1}\sqrt{\pi}\,\Pi\left(n-\frac{1}{2}\right)}\int_{-\pi}^{\pi}\sin^{2n}\theta\cos(x\cos\theta)d\theta$$