MEMOIRS OF THE COLLEGE OF SCIENCE, UNIVERSITY OF KYOTO, SERIES, A
Vol. XXVIII, Mathematics No. 1, 1953.

On the Solutions of the System of Ordinary Differential Equations.[1]

By

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(Received April 9, 1953)

§ 1. Introduction.

In this paper we will treat the system of ordinary differential equations

$$\frac{dy_i}{dx}=f_i(x, y_1, y_2, \cdots, y_m) \quad (i=1, 2, \cdots, m). \tag{1}$$

Concerning the system, whose second members are continuous, many authors[2],[3] have investigated several fundamental theorems. Here we will extend some of them into the case of the system which has discontinuous second members.

In the following the integrals are of Lebesgue sense and y represents the vector in the space of m dimensions: namely $y=(y_1, y_2, \cdots, y_m)$, and $|y|=\sqrt{y_1^2+y_2^2+\cdots+y_m^2}$. Therefore (1) may be represented by

$$\frac{dy}{dx}=f(x, y). \tag{2}$$

And, we assume that $f(x, y)$ is defined in a domain G: $0\leqq x-x_0\leqq a$, $|y-y_0|\leqq b$, having the properties as follows:

a) $f(x, y)$ is measurable with regard to x, and continuous function of y,

b) $|f(x,y)|\leqq M(x)$, where $M(x)$ is summable, i. e. integrable in the sense of Lebesgue, for $0\leqq x-x_0\leqq a$.

For the differential equation (2) we call a curve $y=\varphi(x)$, the solution passing through the point $P(x_P, y_P)\in G$ provided

c) $\varphi(x)$ is defined in an interval I containing x_P, $\varphi(x_P)=y_P$ and $(x, \varphi(x))\in G(x\in I)$,

d) $\varphi(x)=y_P+\int_{x_P}^{x} f(x, \varphi(x))dx \quad (x\in I)$.

§ 2. Lemmas.

Lemma 1. *If* $\varphi(x)$ *is measurable in* $0 \leqq x - x_0 \leqq a$ *and* $(x, \varphi(x)) \in G$, *then* $f(x, \varphi(x))$ *is summable in* $0 \leqq x - x_0 \leqq a$.

For the proof we refer to carathéodory.[4]

Now consider a sequence of functions $\varphi_n(x)\ (n=1, 2, 3, \cdots)$, which are so defined in $x_0 \leqq x \leqq X (\leqq x_0 + a)$ that

$$(3)\quad \varphi_n(x) = \begin{cases} y_0 & (x_0 \leqq x < x_{n1}), \\ y_0 + \int_{x_0}^{x_{n1}} \{f(x, \varphi_n(x)) + \alpha_n(x)\}\, dx & (x_{n1} \leqq x < x_{n2}), \\ y_0 + \int_{x_0}^{x_{n2}} \{f(x, \varphi_n(x)) + \alpha_n(x)\}\, dx & (x_{n2} \leqq x < x_{n3}), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ y_0 + \int_{x_0}^{x_{nn-1}} \{f(x, \varphi_n(x)) + \alpha_n(x)\}\, dx & (x_{nn-1} \leqq x \leqq x_{nn} = X), \end{cases}$$

where $x_{nj} = x_0 + \dfrac{j(X - x_0)}{n}\ (j = 1, 2, \cdots, n)$, and $\alpha_n(x)\ (n = 1, 2, 3, \cdots)$ are given measurable functions and we suppose that there exists such a function $N(x)$, summable in $x_0 \leqq x \leqq X$, that $|\alpha_n(x)| \leqq N(x)$ and $\int_{x_0}^{X} \{M(x) + N(x)\}\, dx \leqq b$.

Moreover we consider a sequence of functions $\Psi_n(x)\ (n= 1, 2, 3, \cdots)$:

$$(4)\qquad \Psi_n(x) = y_0 + \int_{x_0}^{x} \{f(x, \varphi_n(x)) + \alpha_n(x)\}\, dx.$$

Then,

$$|\Psi_n(x) - y_0| \leqq \int_{x_0}^{x} \{M(x) + N(x)\}\, dx \leqq b,$$

$$|\Psi_n(x) - \Psi_n(x')| \leqq \int_{x}^{x'} \{M(x) + N(x)\}\, dx \quad (x < x'),$$

and therefore, the sequence of functions $\{\Psi_n(x)\}$ is equicontinuous. Hence we can select a uniformly convergent subsequence. The limiting function $\Psi(x)$ is continuous in $x_0 \leqq x \leqq X$:

$$\Psi(x) = \lim_{n' \to \infty} \Psi_{n'}(x).$$

Since, at each point in $x_{nj} \leqq x < x_{nj+1}$,

$$|\Psi_n(x)-\varphi_n(x)|\leqq\int_{x_{nj}}^{x}|f(x,\varphi_n(x))+\alpha_n(x)|\,dx$$

$$\leqq\int_{x_{nj}}^{x}\{M(x)+N(x)\}\,dx\to 0\quad(n\to\infty),$$

we have

$$\lim_{n'\to\infty}\varphi_{n'}(x)=\lim_{n'\to\infty}\Psi_{n'}(x)=\Psi(x).$$

Therefore we have

$$\lim_{n'\to\infty}\int_{x_0}^{x}f(x,\varphi_{n'}(x))\,dx=\int_{x_0}^{x}f(x,\Psi(x))\,dx;$$

and hence there exists the limit $\alpha(x)=\lim\limits_{n'\to\infty}\int_{x_0}^{x}\alpha_{n'}(x)dx$ which is a continuous function of x. Consequently, we have

$$\Psi(x)=y_0+\int_{x_0}^{x}f(x,\Psi(x))\,dx+\alpha(x). \tag{5}$$

Especially put $\alpha(x)\equiv 0$, then $y=\Psi(x)$ is a solution of (2) passing through the point $P_0(x_0,y_0)$. If $\alpha_n(x)\equiv 0$ $(n=1,2,3,\ldots)$, we have the following

Lemma 2. (Carathéodory's existence theorem[1]). *If for a point X in $x_0<X\leqq x_0+a$,*

$$\int_{x_0}^{X}M(x)\,dx\leqq b,$$

then passing through the point P_0 there exists a solution of (2) which is defined in $x_0\leqq x\leqq X$.

§ 3. Theorems.

Theorem 1. *The set S of all points, which are on any solutions of (2) passing through the point P_0, is a closed set. And therefore the intersection S_ξ with a hyperplane $x=\xi$ $(x_0\leqq\xi\leqq x_0+a)$ is also a closed set.*

The proof is omitted since it is not difficult because of the equicontinuity of solutions of (2).

Theorem 2. (Kneser's theorem[2]). *When $x_0\leqq\xi\leqq x_0+a$ and $\int_{x_0}^{X}M(x)dx\leqq b$, S_ξ is a continuum.*

Proof. Suppose that, on the contrary, S_ξ consists of two closed sets Q and R and the distance between them is positive. And consider $F(P)=\overline{PR}-\overline{PQ}$ as a function of point on the solution[F] $y=\bar{y}_{nj}(x)$ of (2) which arrives at P_{nj} from the left and never enter into the interior of S. Then there exists a boundary point of S on the segment $\overline{P'_{nj-1}P''_{nj-1}}$. We represent this point by $P_{nj-1}(x_{nj-1}, \eta_{nj-1})$. If P'_{nj-1} is not a boundary point of S_{nj-1}, then we write

$$\rho_{nj}=\frac{\eta_{nj-1}-y_{nj-1}}{\bar{y}_{nj-1}-y_{nj-1}}.$$

Evidently $0<\rho_{nj}\leqq 1$. If P'_{nj-1} is a boundary point of S, we can choose P'_{nj-1} as P_{nj-1}. And we define $\rho_{nj}=0$. Consider $Y_{nj}(x)$ such that

$$\begin{aligned} Y_{nj}(x) &= y_{nj}(x)+\rho_{nj}\{\bar{y}_{nj}(x)-y_{nj}(x)\} \\ &= \eta_{nj}+\int_{x_{nj}}^{x}[f(x, y_{nj}(x))+\rho_{nj}\{f(x, \bar{y}_{nj}(x))-f(x, y_{nj}(x))\}]\,dx \\ &\qquad\qquad \text{for } x_{nj-1}\leqq x\leqq x_{nj}, \end{aligned}$$

then we have

$$Y_{nj}(x_{nj})=\eta_{nj} \text{ and } Y_{nj}(x_{nj-1})=\eta_{nj-1}.$$

Put

$$\rho_n(x)=\begin{cases} \rho_{nj} & (x_{nj-1}< x\leqq x_{nj},\ j=2, 3, \ldots n), \\ \rho_{n1} & (x_0\leqq x\leqq x_{n1}), \end{cases}$$

then $0\leqq \rho_n(x)\leqq 1$ for $x_0\leqq x\leqq \xi$. Moreover consider following functions:

$$y_n(x)=\begin{cases} y_{nj}(x) & (x_{nj-1}< x\leqq x_{nj}, j=2, 3,\ldots, n), \\ y_{n1}(x) & (x_0\leqq x\leqq x_{n1}), \end{cases}$$

$$\bar{y}_n(x)=\begin{cases} \bar{y}_{nj}(x) & (x_{nj-1}< x\leqq x_{nj}, j=2, 3, \cdots, n), \\ \bar{y}_{n1}(x) & (x_0\leqq x\leqq x_{n1}), \end{cases}$$

and

$$Y_n(x)=\begin{cases} Y_{nj}(x) & (x_{nj-1}< x\leqq x_{nj}, j=2, 3, \ldots, n), \\ Y_{n1}(x) & (x_0\leqq x\leqq x_{n1}), \end{cases}$$

then we may write

$$(8)\quad Y_n(x)=\eta+\int_{\xi}^{x}[f(x, y_n(x))+\rho_n(x)\{f(x, \bar{y}_n(x))-f(x, y_n(x))\}]dx.$$

Since, evidently, the sequence of functions $\{Y_n(x)\}$ $(n=1, 2, 3,$

...) is uniformly bounded and equicontinuous, we can choose a uniformly convergent subsequence $\{Y_{n'}(x)\}$. The limiting function $Y(x)$ is a continuous function and $Y(x_0)=y_0$, $Y(\xi)=\eta$.

Since, in $x_{nj-1}<x\leqq x_{nj}$,

$$Y_n(x)-y_n(x)=\int_{x_{nj}}^{x}\rho_n(x)\{f(x,y_n(x))-f(x,y_n(x))\}dx,$$

$$|Y_n(x)-y_n(x)|\leqq 2\int_{x}^{x_{nj}}M(x)dx,$$

$$\bar{y}_n(x)-y_n(x)=\int_{x_{nj}}^{x}\{f(x,y_n(x))-f(x,y_n(x))\}dx$$

and

$$|\bar{y}_n(x)-y_n(x)|\leqq 2\int_{x}^{x_{nj}}M(x)dx,$$

and then we have

$$\lim_{n'\to\infty}y_{n'}(x)=\lim_{n'\to\infty}\bar{y}_{n'}(x)=\lim_{n'\to\infty}Y_{n'}(x)=Y(x). \tag{9}$$

Consequently we have in the limit

$$Y(x)=\eta+\int_{\xi}^{x}f(x,Y(x))dx=y_0+\int_{x_0}^{x}f(x,Y(x))dx. \tag{10}$$

Therefore we have obtained a solution $y=Y(x)$ of (2), passing through P_0 and P and consisting of boundary points of S.

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Notes.

(1) Read before the annual meeting of Japanese Mathematical Society of 1952 held in Tokyo.

(2) Kneser: S.-B. preuss. Akad. Wiss., 1923, p. 171.

(3) Fukuhara: Jap. J. Math., 5 (1928), p. 345; Nagumo-Fukuhara: Proc. Phys.-Math. Soc. Jap., (3) 12 (1930), P. 233; Kamke: Acta Math., 58 (1931), p. 57.

(4) Carathéodory: Vorlesungen über reelle Funktionen, 2te Anf., pp. 665–672.

(5) The existence of such a solution may be easily proved from the equicontinuity of solutions of the equation (2).