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Note on the existence theorem of a periodic solution of the non-linear differential equation

By

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In the foregoing paper[1] we have obtained a *boundedness theorem* from which we have deduced an *existence theorem* of a periodic solution of the non-linear differential equation. Since there we have made use of the property of the ultimate boundedness, such existence theorem for a periodic solution is no more applicable to the case where the solutions are not ultimately bounded. Therefore in this paper we will search an existence theorem applicable to this new case. The principle is to establish that each solution starting from $t=0$ is bounded for $0 \leqq t < \infty$. Then since each solution is continuable and moreover there exists a bounded solution, we can obtain an existence theorem by aid of Massera's theorem.[2]

At first, as a sufficient condition for the boundedness of solutions, generalizing the problem for the general system of differential equations, we will prove the following theorem.

Theorem 1. *Consider a system of differential equations,*

$$(1) \qquad \frac{dx_i}{dt} = f_i(t, x_1, x_2, \cdots, x_n) \quad (i=1, 2, \cdots, n),$$

where $f_i(t, x_1, x_2, \cdots, x_n)$ *are continuous functions of* $(t, x_1, x_2, \cdots, x_n)$ *in the domain*

$$D_1: \quad 0 \leqq t < \infty, \quad -\infty < x_i < +\infty \quad (i=1,2,\cdots,n).$$

Now let R_0 *be a positive constant which may be sufficiently great and* D_2 *be the domain such as*

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1) These Memoirs, Series A, Mathematics, Vol. 28, pp. 133-141.

2) Wendel; Ann. Math. Stud. no. 20 (Princeton, 1950), p. 266 or Massera; "The existence of periodic solutions of systems of differential equations", Duke Math. Journal. Vol. 17 (1950), pp. 457-475.

$$0 \leqq t < \infty, \quad x_1^2 + x_2^2 + \cdots + x_n^2 \geqq R_0^2.$$

Suppose that there exists a continuous function $\Phi(t, x_1, x_2, \cdots, x_n)$ *of* $(t, x_1, x_2, \cdots, x_n)$ *satisfying the following conditions in this domain* D_2*; namely*

1° $\Phi(t, x_1, x_2, \cdots, x_n)$ *tends to zero uniformly as* $x_1^2 + x_2^2 + \cdots + x_n^2 \rightarrow \infty$,

2° *for any positive number* $R(>R_0)$, *there exists a positive constant* $G(R)$ *such that* $\Phi(t, x_1, x_2, \cdots, x_n) \geqq G(R) > 0$ *when* $x_1^2 + x_2^2 + \cdots + x_n^2 = R^2$,

3° $\Phi(t, x_1, x_2, \cdots, x_n)$ *satisfies locally the Lipschitz condition with regard to* $(x_1, x_2, \cdots, x_n)$ *and in the interior of this domain* D_2 *we have*

$$\varliminf_{h \to 0} \frac{1}{h}\{\Phi(t+h, x_1 + hf_1, x_2 + hf_2, \cdots, x_n + hf_n) - \Phi(t, x_1, x_2, \cdots, x_n)\} \geqq 0. \tag{2}$$

Then for any solution of (1) $x_1 = x_1(t), x_2 = x_2(t), \cdots, x_n = x_n(t)$, *being given any positive number* α, *if we have at an arbitrary* $t = t_0 (\geqq 0)$

$$x_1(t_0)^2 + x_2(t_0)^2 + \cdots + x_n(t_0)^2 \leqq \alpha^2, \tag{3}$$

then we can find another positive number β $(>\alpha)$ *such that for* $t \geqq t_0$

$$x_1(t)^2 + x_2(t)^2 + \cdots + x_n(t)^2 < \beta^2. \tag{4}$$

Proof. Let us assume that $\alpha > R_0$, for this case alone is worth to consider. By the conditions 1° and 2°, we can choose β, independent of t, so large that

$$G(\alpha) > \sup_{\substack{0 \leqq t < \infty \\ x_1^2 + \cdots + x_n^2 = \beta^2}} \Phi(t, x_1, \cdots, x_n) . \tag{5}$$

Now it is easy to prove that this β is the required number.

If Φ is bounded for t and the right hand side of the inequality (2) is replaced by $\geqq \varepsilon (>0)$ when x_i are bounded, the solutions are *ultimately bounded.* Moreover if $\Phi(t, x_1, x_2, \cdots, x_n)$ be *totally differentiable,* (2) becomes

$$\frac{\partial \Phi}{\partial t} + \sum_{k=1}^{n} \frac{\partial \Phi}{\partial x_k} f_k \geqq 0 .$$

Here we may add a notice that as a special case of this theorem, a simple criterion due to A. Wintner[3] may be given. Let $r=\sqrt{x_1^2+\cdots+x_n^2}$ and suppose that $f_i(t, x_1, x_2, \cdots, x_n)$ satisfy

$$|f_i(t, x_1, \cdots, x_n)| \leqq \lambda(t)\ \varphi(r),$$

when $\lambda(u)$ and $\varphi(u)$ are positive when $0 \leqq u < \infty$, continuous when $0 < u < \infty$, and are subjected to

$$\int^{\infty} \lambda(t)\,dt < \infty$$

and

$$\int^{\infty} dr/\varphi(r) = \infty.$$

Then we may put

$$\Phi(t, x_1, x_2, \cdots, x_n) = e^{-\int_{r_0}^{r} \frac{dr}{\varphi(r)} + n\int_0^t \lambda(t)dt}.$$

But there exist $\lim\limits_{t\to\infty} x_i(t)$ *in this case.*

Now with a system of differential equations of two unknowns

$$\text{(6)} \qquad \begin{cases} \dfrac{dx}{dt} = f(t, x, y) \\[2mm] \dfrac{dy}{dt} = g(t, x, y) \end{cases},$$

we can proceed in the following way. Namely let us assume that $f(t, x, y)$ and $g(t, x, y)$ are *continuous* in the domain

$$\Delta_1:\ 0 \leqq t < \infty, \quad -\infty < x < +\infty, \quad -\infty < y < +\infty,$$

and we shall prove the following theorem.

Theorem 2. *Let Δ_2 be the domain such as*

$$0 \leqq t < \infty, \quad (x, y) \in \text{complement of } \Delta_3,$$

where Δ_3 is

$$|x| < K_1, \quad |y| < K_2$$

(positive constants K_1 and K_2 may be arbitrarily great). Suppose that there exists a continuous and always positive function $\Phi(t, x, y)$ in the domain Δ_2 which satisfies the following conditions; *namely*

3) Wintner; "An Abelian lemma concerning asymptotic equilibria" American Journal of Mathematics, Vol. 68 (1946), pp. 451–454.

1° $\Phi(t, x, y)$ *tends to zero uniformly for* (t, x) *as* $y \to \pm\infty$,

2° *for every pair of positive constants* N_1 *and* N_2 $(N_1 \geqq K_1, N_2 \geqq K_2)$, *there exists a positive number* $G(N_1, N_2)$ *such as*

$$\Phi(t, x, y) \geqq G(N_1, N_2) > 0$$

for $0 \leqq t < \infty$, $|x| = N_1$, $|y| = N_2$,

3° $\Phi(t, x, y)$ *satisfies locally the Lipschitz condition with regard to* (x, y) *and we have for all points in the interior of this domain*

$$\lim_{h \to 0} \frac{1}{h} \{\Phi(t+h, x+hf(t, x, y), y+hg(t, x, y)) - \Phi(t, x, y)\} \geqq 0. \tag{7}$$

Now for every $M > 0$ (*M may be sufficiently great*), *let* $\Delta_4(M)$ *and* $\Delta_5(M)$ *be the domains such as*

$$0 \leqq t < \infty, \quad x \geqq L, \quad |y| \leqq M$$

and

$$0 \leqq t < \infty, \quad x \leqq -L, \quad |y| \leqq M$$

respectively, where the constant $L > 0$ *may be arbitrarily great and may depend on* M.

Moreover suppose that there exist two continuous functions of (t, x, y), $\Psi_1(t, x, y)$ *and* $\Psi_2(t, x, y)$, *defined in* $\Delta_4(M)$ *and* $\Delta_5(M)$ *respectively such that they satisfy the following conditions; namely*

4° *they are always positive and for every* $L' (> L)$ *there exists a positive number* $H(L')$ *such as*

$$\Psi_i(t, x, y) \geqq H(L') > 0 \qquad (i = 1, 2)$$

when $x = L'$ *and* $x = -L'$ *respectively,*

5° *they tend to zero uniformly for* (t, y) *when* $x \to \pm\infty$ *respectively,*

6° *they satisfy locally the Lipschitz condition with regard to* (x, y) *and we have in the interior of their respective domains*

$$\lim_{h \to 0} \frac{1}{h} \{\Psi_i(t+h, x+hf, y+hg) - \Psi_i(t, x, y)\} \geqq 0 \quad (i = 1, 2). \tag{8}$$

Then for any solution of (6) *starting from* $t = 0$, *there exist two* *positive constants A and B such that for* $0 \leqq t < \infty$

$$|x(t)| < A, \quad |y(t)| < B,$$

where of course A and B vary from solutions to solutions.

Proof. Now let $(x(t), y(t))$ be any solution of (6) starting from $t=0$. At first we wiil show that there exists a positive number M such as $|y(t)| < M$. Let us assume that

$$|x(o)| < \bar{A} \text{ and } |y(o)| < \bar{B},$$

where $\bar{A} > K_1$ and $\bar{B} > K_2$. And we choose a positive constant M such that

$$G(\bar{A}, \bar{B}) > \sup_{\substack{o \leqq t < \infty \\ |x| < \infty}} \Phi(t, x, \pm M). \tag{9}$$

By the assumptions it is clear that we can choose such an M. Now if we suppose that at some t we have $|y(t)| = M$, there arises a contradiction by consideration of the function $\Phi(t, x(t), y(t))$. Therefore we have $|y(t)| < M$.

Then we consider $\Psi_1(t, x, y)$ for the above M. Thus for $\bar{A'}$ such as $\bar{A'} \geqq \bar{A}$ and $\bar{A'} \geqq L$, choosing another positive constant M' so that

$$H(\bar{A'}) > \sup_{\substack{o \leqq t < \infty \\ |y| \leqq M}} \Psi_1(t, M', y),$$

we can see that $x(t) < M'$ in the same way; so also there exists M'' such as $x(t) > -M''$. Hence it is shown that there exist A and B such that

$$|x(t)| < A, \quad |y(t)| < B.$$

Remark 1. We may reason, interchanging x and y.

Remark 2. If Φ and Ψ do not include t, there is no need to show the existence of G and H and their existence is a natural consequence.

Therefore according to this theorem, each solution is continuable and bounded. Hence we obtain the following theorem by aid of Massera's theorem.

Theorem 3. *Suppose that the same assumptions as those in Theorem 2 and the condition for the uniqueness of solutions in the Cauchy-problem hold good. Moreover suppose that*

$$f(t+\omega, x, y)=f(t, x, y)$$

and

$$g(t+\omega, x, y)=g(t, x, y).$$

Then (6) *has at least a periodic solution of period* ω.

Remark. The conditions in Theorem 2 may be for $t_0 \leqq t < \infty$ (t_0 may be great).

Example. Mizohata and Yamaguti have remarkably proved the following existence theorem of a periodic solution in *these Memoirs, Series A, Vol. 27.*

The differential equation,

$$\ddot{x}+a(x)\dot{x}+\varphi(x)=p(t), \tag{10}$$

possesses at least a periodic solution of period ω, when the following conditions are fulfilled:

a) $A(x)=\int_0^x a(x)dx \to \pm\infty$, for $x \to \pm\infty$ respectively,

b) sgn $x \cdot \varphi(x) \geqq 0$, for $|x|>q$,

where $a(x)$, $\varphi(x)$, $\varphi'(x)$ and $p(t)$ are continuous functions and $p(t)$ is periodic of period ω and $\int_0^\omega p(t)dt=0$ and q is a positive number.

For our part, we consider the equations

$$\begin{cases} \dot{x}=y-A(x)+P(t) \\ \dot{y}=-\varphi(x) \end{cases},$$

where $P(t)=\int_0^t p(t)dt$. And choosing positive numbers a and b suitably great, we may define the function Φ in Theorem 2 as follows; namely

$$\Phi(t, x, y)=\begin{cases} e^{-u(x,y)} & (x \geqq a,\ |y|<\infty) \\ e^{-u(x,y)+x-a} & (|x| \leqq a,\ y \geqq b) \\ e^{-u(x,y)-2a} & (x \leqq -a,\ y \geqq b) \\ e^{-u(x,y)-\frac{2a}{b}y} & (x \leqq -a,\ |y| \leqq b) \\ e^{-u(x,y)+2a} & (x \leqq -a,\ y \leqq -b) \\ e^{-u(x,y)-x+a} & (|x| \leqq a,\ y \leqq -b), \end{cases}$$

where $u(x,y)=\tilde{\Phi}(x)+\dfrac{y^2}{2}$ and $\tilde{\Phi}(x)=\int_0^x \varphi(x)dx$.

As $\Psi_i(t, x, y)$, choosing $c > 0$ suitably, we may put

$$\Psi_1 = e^{-x}, \quad \Psi_2 = e^x$$

for $x \geqq c$ and $x \leqq -c$ respectively. Then by Theorem 2, we can see that the differential equation (10) possesses at least a periodic solution of period ω.