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Notes on Chow points of algebraic varieties.

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Let V be an algebraic variety embedded in a projective space. Then as is well known we can represent V by a point in a suitable projective space by the method of associated forms[1]. Henceforth we shall call it briefly the Chow point of V and denote it by $c(V)$. In this short note we shall prove two theorems, one concerning the Chow point of a variety, and the other concerning the Chow point of the divisors on a variety.

THEOROM 1. Let V be a variety embedded in a projective space and κ the prime field of characteristic p. Let M_λ $(\lambda=1,2,\cdots)$ be a sequence of independent generic points of V over some field of definition k for V, then for sufficiently large α we have $c(V) \subset \kappa(M_1,\cdots,M_\alpha)$.

PROOF. As is well known a projective model has the smallest field of definition $k_0=\kappa(c(V))$.[2] Let $\mathfrak{P}$ be the defining ideal of V in $k[X]$. Then we can select special basis $(P_1(X),\cdots,P_s(X))$ for $\mathfrak{P}$ having the following properties.

(1) k_0 is get by the adjunction of the coefficients of $P_j(X)$ to κ.

(2) Let $\mathfrak{M}_\lambda(X)$ be monomials in X with suitable ordering and J_i be the set of indices such that $P_i(X)$ is exactly the linear forms in $\mathfrak{M}_{\lambda_i}(X)$ with $\lambda_i \in J_i$. Then for any proper subset J_i' of J_i, the linear forms $\sum u_\beta \mathfrak{M}_\beta(X)$ with $\beta \in J_i'$ and $u_\beta \in k_0$ can not be contained in $\mathfrak{P}$. Such basis can be get by the procedure given in W-I,[3] lemma 2. Let

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1) Cf. B. L. van der Waerden, "Einführung in die algebraische Geometrie". Julius Springer in Berlin, 1939.

2) Cf. S. Nakano, "Note on gruop varietirs", Mem. Coll. Sci., Univ. of Kyoto, vol. XXVII, 1942.

3) This means the lemma 2 of Chap. I of "Foundations of algebraic geometry" written by A. Weil,

$$P_i(X) = \sum_{j=1}^{\alpha_i} a_{ij}\, \mathfrak{M}_{\lambda_j}(X)$$

Without loss of generalities we can suppose that $a_{i1}=1$. Then by the property (2) we see that $\mathfrak{M}_{\lambda_j}(Q)$ $(j=2,\cdots,\alpha_i)$ are linearly independent over k_0, where Q is a generic point of V over k_0. Since $k_0(Q)$ is regular over k_0, they are still linearly independent over $\bar{k}_0$. Hence by W-II, Prop. 19, there exist (α_i-1) generic points Q_ρ of V over k_0 such that det $|\mathfrak{M}_{\lambda_j}(Q_\rho)|$ $(j=2,\cdots,\alpha_i\,;\rho=1,\cdots,\alpha_i-1)$ is not zero. Hence for independent generic points $M_1,\cdots,M_{\alpha_{i-1}}$ of V over k_0, we have a fortiori det $|\mathfrak{M}_{\lambda_j}(M_\rho)|\neq 0$. Hence we can solve the linear equations

$$-\sum_{j=2}^{\alpha_i} a_{ij}\, \mathfrak{M}_{\lambda_j}(M_\rho) = \mathfrak{M}_{\lambda_1}(M_\rho)$$

$$(\rho=1,\cdots,\alpha_i-1)$$

in $a_{ij}(j=2,\cdots,\alpha_i)$, and we have $a_{ij}\in\kappa\,(M_1,\cdots,M_{\alpha_i-1})$, $(j=2,\cdots,\alpha_i)$. Now taking $\alpha=\max(\alpha_i)-1$, we see that all the coefficients of $P_i(X)$ are in $\kappa(M_1,\cdots,M_\alpha)$, i.e. $k_0=\kappa(c(V))$ is contained in $\kappa\,(M_1,\cdots,M_\alpha)$.

Let V^n be a projective model and $X=\sum a_i A_i-\sum b_j B_j$ a V-divisor, where A_i and B_j are simple subvarieties of dimension *n-1*, $c(X)$ the Chow point of X and k a field of definition for V. Then as is known[4] the field $k(c(X))$ is the smallest one containing k over which X is rational. Then we have

THEOREM 2. Using the same notations as above, $\dim_k(c(X))$ is equal to the maximal number of independent generic points of V over k lying on X.

PROOF. Let $P_1,\cdots,P_s$ be the independent generic points of V over k lying on X. Then since X is rational over $k(x)$, wher $x=c(X)$, each P_i has at most dimension *n-1* over $k(x)$. Hence we must have

$$ns=\dim_k(P_1,\cdots,P_s)\leq \dim_k(x)+\dim_{k(x)}(P_1,\cdots,P_s)$$
$$\leq \dim_k(x)+(n-1)s$$

i. e. $\qquad s\leq \dim_k(x)$

Then if we denote bv m the maximal number of independent generic

4) Cf. W. L. Chow, "On the defining field of a divisor in an algebraic variety", Proc. Amer. Math. Soc. vol. 1, no 6, 1950.

points of V over k lying on X, we must have $\dim_k(x) \geq m$.

We shall now say that $Q_1,\cdots,Q_s$ are independent generic points of X over $k(x)$ when we have the relation

$$\dim_{k(x)}(Q_1,\cdots,Q_s) = (n-1)s$$

Then by Th. 1 if we take sufficiently many independent generic points of X over $k(x)$ in a suitable manner we have $c(A_i)$, $c(B_j)$ are contained in $\varkappa(Q_1,\cdots,Q_t)$. Hence $k(x) \subset k(c(A_i),\ c(B_j))$ is contained in $k(Q_1,\cdots,Q_t)$. Then have

$$(1) \qquad \dim_k(x) + \dim_{k(x)}(Q_1,\cdots,Q_t) = \dim_k(Q_1,\cdots,Q_t)$$

But by the hypothesis there exist at most m independent generic points of V over k among $Q_1,\cdots,Q_t$, hence we must have

$$(2) \qquad \dim_k(Q_1,\cdots,Q_t) \leq nm + (n-1)(t-m) = t(n-1) + m$$

Combinig (1) and (2) we have

$$\dim_k(x) \leq m$$

Thus the proof is completed.