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On the characteristic classes of a submanifold

By

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In this paper, we first give some remarks on the differential forms introduced in our previous papers[1)2)], and we show secondly that the forms can also represent the characteristic cohomology classes of tangent and normal bundles over a submanifold imbedded in a Riemannian manifold $\boldsymbol{R}^n$, by integrating over suitable chains in the tangent frame bundles over $\boldsymbol{R}^n$.

§ 1. The formulas of obstruction cocycles and deformation cochains

Consider a Riemannian manifold $\boldsymbol{R}^n$ of dimension n which we shall suppose, as in previous papers, to be compact connected orientable and of class $\geqq 4$. The group of the tangent sphere bundle $\mathfrak{B}^{n-1}$ over $\boldsymbol{R}^n$ may be the proper orthogonal group, and so any element of the associated principal bundle $\mathfrak{B}^0$ of $\mathfrak{B}^{n-1}$ can be expressed by an n-frame $\boldsymbol{Pe_1e_2\cdots e_n}$ which determines one of the orientations of $\boldsymbol{R}^n$.

Take an even permutation σ of n figures $(1, 2, \cdots, n)$ and set $\sigma(A)=A'$ $(A=1, 2, \cdots, n)$. Let us denote any element of the tangent $(n-q)$-frame bundle $\mathfrak{B}^q$ associated with $\mathfrak{B}^{n-1}$ by

$$\boldsymbol{Pe}_{q+1)'}\boldsymbol{e}_{(q+2)'}\cdots\boldsymbol{e}_{n'}\in\mathfrak{B}^q.$$

And the natural projection $\mu:\mathfrak{B}^q\to\mathfrak{B}^{q+1}$ is defined by

$$\mu\boldsymbol{Pe}_{(q+1)'}\boldsymbol{e}_{(q+2)'}\cdots\boldsymbol{e}_{n'}=\boldsymbol{Pe}_{(q+2)'}\cdots\boldsymbol{e}_{n'}\in\mathfrak{B}^{q+1}.$$

Then, for any cross-section F into the bundle $\mathfrak{B}^{r-1}$ defined on the

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1) S. Takizawa: *On the Stiefel characteristic classes of a Riemannian manifold,* these Memoirs, Vol. **28**, No. 1 (1953).

2) S. Takizawa: *On the primary difference of two frame functions in a Riemannian manifold,* Ibid,

$(r-1)$-dimensional skeleton K^{r-1} of a cellular decomposition of R^n, there exists an extension G of the cross-section $\mu F: K^{r-1} \to \mathfrak{B}^r$ over the r-dimensional skeleton K^r. From the linear differential formes ω_A, ω_{AB} $(=-\omega_{BA})$ $(A, B=1, 2, \cdots, n)$ which define the Riemannian connexion and from its curvature forms Ω_{AB}, we construct the form Π^{r-1} $(1 \leqq r \leqq n)$ as follows:

$$(1) \quad \Pi^{r-1} = \frac{(-1)^r}{2^r \pi^{\frac{1}{2}(r-1)}} \sum_{\lambda=0}^{[\frac{1}{2}(r-1)]} (-1)^\lambda \frac{1}{\lambda!\, \Gamma(\frac{1}{2}(r-2\lambda+1))} \Pi_\lambda^{r-1},$$

where

$$(2) \quad \Pi_\lambda^{r-1} = \sum_{(A)} \epsilon_{A_1 A_2 \cdots A_{r-1} r' (r+1)' \cdots n'} \Omega^{(r)}_{A_1 A_2} \cdots \Omega^{(r)}_{A_{2\lambda-1} A_{2\lambda}} \omega_{A_{2\lambda+1} r'} \cdots \omega_{A_{r-1} r'},$$

and

$$(3) \qquad \Omega^{(r)}_{AB} = \Omega_{AB} + \sum_{Q=(r+1)'}^{n'} \omega_{AQ} \omega_{QB}.$$

Here, we altered the notations in the previous papers: namely the present form Π^{r-1} and Π_λ^{r-1} were respectively denoted by Π^r and Φ_λ^r in the previous ones.

It has been known that Π_λ^{r-1} is an $(r-1)$-form on $\mathfrak{B}^{r-1}$. If we set

$$(4) \qquad -d\Pi^{r-1} = \Omega^r,$$

Ω^r becomes an r-form on $\mathfrak{B}^r$ and satisfies the relations

$$(5) \qquad \begin{aligned} \Omega^r &= 0 && \text{if } r \text{ is odd}, \\ \Omega^r &= -2\Pi^r && \text{if } r \text{ is even and } r < n. \end{aligned}$$

Then the Kronecker product of an r-cell Δ^r of K^n and the obstruction cocycle $c(F)$ of a cross-section F: $K^{r-1} \to \mathfrak{B}^{r-1}$ is given by

$$(6) \qquad \begin{aligned} (-1)^r c(F) \cdot \Delta^r &= \int_{F\partial\Delta^n} \Pi^{n-1} + \int_{\Delta^n} \Omega^n && \text{if } r=n, \\ &= \int_{F\partial\Delta^r} \Pi^{r-1} && \text{if } r \text{ is odd and } < n, \\ &\equiv \int_{F\partial\Delta^r} \Pi^{r-1} + \int_{G\Delta^r} \Omega^r \pmod 2 && \text{if } r \text{ is even and } < n. \end{aligned}$$

Further, the deformation cochain $d(f_0, h, f_1)$ for two cross-sections $f_0, f_1 : K^r \to \mathfrak{B}^r$ and a homotopy $h : f_0|K^{r-1} \simeq f_1|K^{r-1}$ is expressed by

$$(7) \qquad \begin{aligned} &(-1)^r d(f_0, h, f_1) \cdot z^r \\ &= \int_{f_1 z^r} \Pi^r - \int_{f_0 z^r} \Pi^r && \text{if } r \text{ is even or } r=n-1, \end{aligned}$$

$$\equiv\int_{f_1 z^r}\Pi^r-\int_{f_0 z^r}\Pi^r-\int_{\psi(z^r\times I)}\Omega^{r+1}\quad (mod\ 2)\quad \text{if } r \text{ is odd and } < n-1,$$

where z^r is an r-cycle of K^n whose coefficients are integers.

The general theory of fibre bundles assures that the cochain $c(F)$ defined by (6) determines a unique cohomology class which does not depend on the choice of F. This result can be however easily proved from the formal relations (5) on the formes Π^{r-1} and Ω^r. It is obvious that $c(F)$ is a cocycle when r is odd or $r=n$. If r is even and $r<n$, for any $(r+1)$-cell Δ^{r+1}, taking an extension G of μF over K^r, we have

$$\delta c(F)\cdot\Delta^{r+1}=c(F)\cdot\partial\Delta^{r+1}\equiv\int_{G\partial\Delta^{r+1}}\Omega^r\quad (mod\ 2)$$

$$=-2\int_{G\partial\Delta^{r+1}}\Pi^r=2c(G)\cdot\Delta^{r+1}.$$

Since $c(G)\cdot\Delta^{r+1}$ is an integer, it follows that

$$\delta c(F)\cdot\Delta^{r+1}\equiv 0\quad (mod\ 2).$$

This shows that $c(F)$ is a cocycle. Secondly, let F and F' be two cross-section: $K^{r-1}\rightarrow\mathfrak{B}^{r-1}$, and let z^r be an r-cycle with integral coefficients. It is trivial that $c(F)\cdot z^r-c(F')\cdot z^r=0$, when r is odd or $r=n$. If r is even and $r<n$,

$$c(F)\cdot z^r-c(F')\cdot z^r\equiv\int_{Gz^r}\Omega^r-\int_{G'z^r}\Omega^r\quad (mod\ 2)$$

$$=-2\left\{\int_{Gz^r}\Pi^r-\int_{G'z^r}\Pi^r\right\}=2d(G,G')\cdot z^r,$$

where G' is an extension of $\mu F'$ over K^r. Since $d(G,G')\cdot z^r$ is an integer, it follows that

$$c(F)\cdot z^r-c(F')\cdot z^r\equiv 0\quad (mod\ 2).$$

The cohomology class of $c(F)$ is therefore independent on the choice of F. Thus, the Kronecker product of the r-th Stiefel class $C_r(R^n)$ and a homology class Z^r is given by

$$(8)\qquad (-1)^r C_r(R^n)\cdot Z^r \overset{*}{=} \int_{Gz^r}\Omega^r,$$

where z^r is a cycle chosen to represent Z^r and the equality "$\overset{*}{=}$" denotes "$=$" or "$\equiv(mod\ 2)$" according as the $(r-1)$-th homotopy group of the fibre is infinite cyclic or cyclic of order two.

Moreover, it is possible to prove in the same manner, which we shall omit here, that, if f_0 and f_1 be extendable over K^{r+1}, the cochain $d(f_0, h, f_1)$ is a cocycle whose cohomology class is independent on the choice of the homotopy h.

§ 2. Frame bundles over a submanifold

Let R^m ($m \leqq n-2$) be an m-dimensional closed orientable submanifold of class $\geqq 3$ imbedded in R^n. The groups of the tangent sphere bundle $\mathfrak{T}^{m-1}$ and the normal sphere bundle $\mathfrak{N}^{p-1}$ over R^m are proper orthogonal groups. Throughout this paper we shall set

$$m+p=n\,.$$

The elements of the associated principal bundle $\mathfrak{T}^0$ and $\mathfrak{N}^0$ shall be denoted by

$$Pe_1e_2\cdots e_m \in \mathfrak{T}^0 \quad \text{and} \quad Pe_{m+1}e_{m+2}\cdots e_n \in \mathfrak{N}^0.$$

Designating an orientation of R^m we can assume that the m-frame $Pe_1e_2\cdots e_m$ and the composite n-frame $Pe_1\cdots e_m e_{m+1}\cdots e_n$ determine the given orientations of R^m and R^n respectively. Let $\mathfrak{T}^s$ and $\mathfrak{N}^s$ be the $(m-s)$- and $(p-s)$-frame bundles associated with $\mathfrak{T}^{m-1}$ and $\mathfrak{N}^{p-1}$ respectively, and we shall denote their elements by

$$Pe_{s+1}e_{s+2}\cdots e_m \in \mathfrak{T}^s \quad \text{and} \quad Pe_{m+1}e_{m+2}\cdots e_{n-s} \in \mathfrak{N}^s.$$

For a family of tangent $(m-s)$-frames $\mathfrak{M} \subset \mathfrak{T}^s$, we define families $N(\mathfrak{M}) \subset \mathfrak{V}^{s+p-1}$ and $N_0(\mathfrak{M}) \subset \mathfrak{V}^s$ as follows:

$$Pe_{s+1}\cdots e_m e_{m+1} \in N(\mathfrak{M}),$$

if and only if $Pe_{s+1}\cdots e_m \in \mathfrak{M}$ and $Pe_{m+1} \in \mathfrak{N}^{p-1}$; and

$$Pe_{s+1}\cdots e_m e_{m+1}\cdots e_n \in N_0(\mathfrak{M}),$$

if and only if $Pe_{s+1}\cdots e_m \in \mathfrak{M}$ and $Pe_{m+1}\cdots e_n \in \mathfrak{N}^0$. When the family $\mathfrak{M}$ depends on k parameters, $N(\mathfrak{M})$ depends on $k+p-1$ parameters. Similarly, for a family $\mathfrak{M} \subset \mathfrak{N}^s$, we define families $T(\mathfrak{M}) \subset \mathfrak{V}^{s+m-1}$ and $T_0(\mathfrak{M}) \subset \mathfrak{V}^s$ as follows:

$$Pe_m e_{m+1}\cdots e_{n-s} \in T(\mathfrak{M}),$$

if and only if $Pe_{m+1}\cdots e_{n-s} \in \mathfrak{M}$ and $Pe_m \in \mathfrak{T}^{m-1}$; and

$$Pe_1\cdots e_m e_{m+1}\cdots e_{n-s} \in T_0(\mathfrak{M})$$

if and only if $Pe_{m+1}\cdots e_{n-s} \in \mathfrak{M}$ and $Pe_1\cdots e_m \in \mathfrak{T}^0$. When $\mathfrak{M}$ depends on k parameters, $T(\mathfrak{M})$ depends on $k+m-1$ parameters. Setting

$$\mathfrak{F}=N_0(\mathfrak{T}^0)=T_0(\mathfrak{N}^0),$$

we shall consider only on $\mathfrak{F}$ the forms ω_A, ω_{AB} and Ω_{AB} given on the principal bundle $\mathfrak{B}^0$.

Let us agree with the following ranges of indices:

$$A, B \cdots =1, 2, \cdots, n;$$

$$i, j \cdots =1, 2, \cdots, m;$$

$$\alpha, \beta \cdots =m+1, m+2, \cdots, m+p=n.$$

Over a coordinate neighborhood $U\subset R^m$, we take local cross-sections $\tau: U\to\mathfrak{T}^0$ and $\nu: U\to\mathfrak{N}^0$; and set

$$\tau P=Pe_1^0e_2^0\cdots e_m^0 \quad \text{and} \quad \nu P=Pe_{m+1}^0e_{m+2}^0\cdots e_n^0 \quad \text{for} \quad P\in U.$$

We have then a *repère mobile* $Pe_1^0\cdots e_m^0e_{m+1}^0\cdots e_n^0$ on $U\subset R^m$, and any frame $Pe_1e_2\cdots e_n\in\mathfrak{F}$ over U is given by

$$e_i=\sum_j u_{ij}e_j^0, \quad e_\alpha=\sum_\beta v_{\alpha\beta}e_\beta^0 \quad \text{and} \quad P=P\in U,$$

where (u_{ij}) and $(v_{\alpha\beta})$ are proper orthogonal matrices. There exist natural homeomorphisms of $T_0(\nu U)$ and of $N_0(\tau U)$ respectively onto the portions of bundles $\mathfrak{T}^0$ and $\mathfrak{N}^0$ over U. Let ι^* and κ^* denote respectively the dual maps of the inclusion maps

$$\iota:\ T_0(\nu U)\to\mathfrak{F} \quad \text{and} \quad \kappa:\ N_0(\tau U)\to\mathfrak{F},$$

and we set

$$\text{(9)} \qquad \begin{aligned} &\theta_A=\iota^*\omega_A, \quad \theta_{AB}=\iota^*\omega_{AB},\\ &\varphi_A=\kappa^*\omega_A, \quad \varphi_{AB}=\kappa^*\omega_{AB}. \end{aligned}$$

Then the forms ω_A, ω_{AB} on $\mathfrak{F}$ over U are written

$$\text{(10)} \qquad \begin{aligned} &\omega_i=\theta_i=\sum_j u_{ij}\varphi_j, \qquad \omega_\alpha=\sum_\beta v_{\alpha\beta}\theta_\beta=\varphi_\alpha=0\\ &\omega_{ij}=\theta_{ij}=\sum_k du_{ik}u_{jk}+\sum_{k,h}u_{ik}u_{jh}\varphi_{kh}\\ &\omega_{i\alpha}=\sum_\beta v_{\alpha\beta}\theta_{i\beta}=\sum_j u_{ij}\varphi_{j\alpha}, \qquad \omega_{\alpha i}=-\omega_{i\alpha},\\ &\omega_{\alpha\beta}=\sum_\gamma dv_{\alpha\gamma}v_{\beta\gamma}+\sum_{\gamma,\delta}v_{\alpha\gamma}v_{\beta\delta}\theta_{\gamma\delta}=\varphi_{\alpha\beta}. \end{aligned}$$

Moreover we have

$$\text{(11)} \qquad \begin{aligned} &\Omega_{ij}=\iota^*\Omega_{ij}=\sum_{k,h}u_{ik}u_{jh}\kappa^*\Omega_{kh},\\ &\Omega_{i\alpha}=\sum_\beta v_{\alpha\beta}\iota^*\Omega_{i\beta}=\sum_j u_{ij}\kappa^*\Omega_{j\alpha}, \qquad \Omega_{\alpha i}=-\Omega_{i\alpha}, \end{aligned}$$

$$\Omega_{\alpha\beta}=\sum_{\gamma,\delta} v_{\alpha\gamma} v_{\beta\delta} \iota^* \Omega_{\gamma\delta}=\kappa^* \Omega_{\alpha\beta}\,,$$
$$\iota^*(\sum_{\alpha} \omega_{i\alpha}\omega_{\alpha j})=\sum_{\alpha} \theta_{i\alpha}\theta_{\alpha j}$$
$$\kappa^*(\sum_{i} \omega_{\alpha i}\omega_{i\beta})=\sum_{i} \varphi_{\alpha i}\varphi_{i\beta}\,.$$

Now we set

$$(12)\qquad \begin{aligned}\Theta_{ij}&=\iota^*(\Omega_{ij}+\sum_{\alpha}\omega_{i\alpha}\omega_{\alpha j}),\\ \Phi_{\alpha\beta}&=\kappa^*(\Omega_{\alpha\beta}+\sum_{i}\omega_{\alpha i}\omega_{i\beta}).\end{aligned}$$

According to the above relations, it is clear that the forms θ_i, θ_{ij}, Θ_{ij} are invariable under the change of local cross-section ν, and so they can be regarded as the forms on $\mathfrak{T}^0$ which are so-called the differential forms of induced Riemannian connexion and its curvature forms. Similarly $\varphi_{\alpha\beta}$ and $\Phi_{\alpha\beta}$ can be regarded as forms on $\mathfrak{N}^0$.

Decompose $\boldsymbol{R}^m$ into a finite cell complex L^m of class $\geqq 3$ so fine that each cell σ^s may be included in a coordinate neighborhood of $\boldsymbol{R}^m$, and let L^s be its s-dimensional skeleton. For any cross-section $f: L^{s-1}\to\mathfrak{T}^{s-1}$, there exists an extension $g: L^s\to\mathfrak{T}^s$ of $\mu' f$, where $\mu':\mathfrak{T}^{s-1}\to\mathfrak{T}^s$ denotes the natural projection defined by

$$\mu' P e_s e_{s+1}\cdots e_m = P e_{s+1}\cdots e_m\,.$$

In the same way as we have constructed the $(r-1)$-form Π^{r-1} on $\mathfrak{B}^{r-1}$, from θ_{ij} and Θ_{ij} we can construct the $(s-1)$-form Λ^{s-1} on $\mathfrak{T}^{s-1}$. If we set $-d\Lambda^{s-1}=\Theta^s$, then Θ^s becomes a form on $\mathfrak{T}^s$, and the formula (6) for tangent bundles over $\boldsymbol{R}^m$ are written

$$(13)\qquad (-1)^s c(f)\cdot\sigma^s \overset{*}{=} \int_{f\partial\sigma^s}\Lambda^{s-1}+\int_{g\sigma^s}\Theta^s.$$

Similarly, from $\varphi_{\alpha\beta}$ and $\Phi_{\alpha\beta}$, we can construct the form Ψ^{s-1} on $\mathfrak{N}^{s-1}$. If we set $-d\Psi^{s-1}=\Phi^s$, then Φ^s becomes a form on $\mathfrak{N}^s$, and the obstruction cocycle $c(f)$ of a cross-section $f: L^{s-1}\to\mathfrak{N}^{s-1}$ is given by the formula

$$(14)\qquad (-1)^s c(f)\cdot\sigma^s \overset{*}{=} \int_{f\partial\sigma^s}\Psi^{s-1}+\int_{g\sigma^s}\Phi^s,$$

which was published by Yagyu[3].

Hence by taking arbitrary cross-sections $g: L^s\to\mathfrak{T}^s$ and $\bar{g}: L^s$

3) T. Yagyu: *On the Whitney characteristic classes of the normal bundles*, these Memoirs, Vol. **28**, No. 1 (1953).

$\to \mathfrak{N}^s$, and by choosing an arbitrary cycle z^s which represents a homology class Z^s of R^m, the s-th Stiefel class C_s of R^m and the s-th Whitney class W_s of the normal sphere bundle over R^m are expressed by the formulas

(15) $$(-1)^s C_s \cdot Z^s \overset{*}{=} \int_{gz^s} \Theta^s ,$$

(16) $$(-1)^s W_s \cdot Z^s \overset{*}{=} \int_{\bar{g}z^s} \Phi^s .$$

Writing the form Θ^s and Φ^s in detail, we get

(17)
$$\Theta^s = \begin{cases} (-1)^{\frac{s}{2}} \dfrac{1}{2^s \pi^{\frac{s}{2}} \left(\dfrac{s}{2}\right)!} \sum\limits_{(i)} \epsilon_{i_1 i_2 \cdots i_s s+1 \cdots n} \Theta^{(s)}_{i_1 i_2} \Theta^{(s)}_{i_3 i_4} \cdots \Theta^{(s)}_{i_{s-1} i_s} & \textit{if } s \textit{ is even}, \\ 0 & \textit{if } s \textit{ is odd}, \end{cases}$$

(18)
$$\Phi^s = \begin{cases} (-1)^{\frac{s}{2}} \dfrac{1}{2^s \pi^{\frac{s}{2}} \left(\dfrac{s}{2}\right)!} \sum\limits_{(\alpha)} \epsilon_{\alpha_1 \alpha_2 \cdots \alpha_s 12 \cdots n-s} \Phi^{(s)}_{\alpha_1 \alpha_2} \Phi^{(s)}_{\alpha_3 \alpha_4} \cdots \Phi^{(s)}_{\alpha_{s-1} \alpha_s} & \textit{if } s \textit{ is even}, \\ 0 & \textit{if } s \textit{ is odd}. \end{cases}$$

where

(19) $$\Theta^{(s)}_{ij} = \theta_{ij} + \sum_{h=s+1}^{m} \theta_{ih} \theta_{hj} ,$$

(20) $$\Phi^{(s)}_{\alpha\beta} = \Phi_{\alpha\beta} + \sum_{\sigma=m+1}^{n-s} \varphi_{\alpha\sigma} \varphi_{\sigma\beta} .$$

It is however possible to show that the characteristic classes of the bundles over R^m may be expressed, without using the forms on tangent or normal bundles, but in terms of the forms Π^{r-1} on $\mathfrak{B}^{r-1}$ by integrating over the suitable cycles in $\mathfrak{B}^{r-1}$. In the following sections we shall make clear the relations between the classes of the bundles over R^m and the forms Π^{r-1} on $\mathfrak{B}^{r-1}$.

§ 3. The characteristic classes of the normal bundles

Denoting by $Pe_m e_{m+1} \cdots e_{n-s}$ any element of the $(p-s+1)$-frame bundle $\mathfrak{B}^{s+m-1}$ over R^n and defining the natural projection $\mu : \mathfrak{B}^{s+m-1} \to \mathfrak{B}^{s+m}$ by

$$\mu P e_m e_{m+1} \cdots e_{n-s} = P e_{m+1} \cdots e_{n-s} \in \mathfrak{B}^{s+m},$$

the form Π^{s+m-1} on $\mathfrak{B}^{s+m-1}$ is now given by

(21)
$$\Pi^{s+m-1}=\frac{(-1)^{s+m}}{2^{s+m}\pi^{\frac{1}{2}(s+m-1)}}\sum_{\lambda=0}^{[\frac{1}{2}(s+m-1)]}(-1)^{\lambda}\frac{1}{\lambda!\,\Gamma(\frac{1}{2}(s+m-2\lambda+1))}\Pi_{\lambda}^{s+m-1},$$
where

(22)
$$\Pi_{\lambda}^{s+m-1}=\sum_{(A)}\epsilon_{A_1A_2\cdots A_{s+m-1}\,m\,m+1\cdots n-s}$$
$$\times\,\Omega^{(s+m)}_{A_1A_2}\cdots\Omega^{(s+m)}_{A_{2\lambda-1}A_{2\lambda}}\,\omega_{A_{2\lambda+1}m}\cdots\omega_{A_{s+m-1}m}$$
and

(23)
$$\Omega^{(s+m)}_{AB}=\Omega_{AB}+\sum_{\sigma=m+1}^{n-s}\omega_{A\sigma}\,\omega_{\sigma B}\,.$$

Then, in view of (12) and (20) it holds in $\mathfrak{R}^n$ that

(24)
$$\Phi^{(s)}_{\alpha\beta}=\Omega^{(s+m)}_{\alpha\beta}-\sum_{i=1}^{m}\varphi_{\alpha i}\,\varphi_{\beta i}\,.$$

For a cross-section $\bar{g}: L^s\rightarrow\mathfrak{R}^s$ and an oriented s-cell σ^s of L^m, we have an oriented cell $\bar{g}\sigma^s$ in $\mathfrak{R}^s$. Then $T(\bar{g}\sigma^s)$, which is homeomorphic to the topological product of σ^s and the $(m-1)$-sphere S^{m-1}, may be regarded as an $(s+m-1)$-chain in $\mathfrak{B}^{s+m-1}$. Therefore, to an s-chain γ^s of L^m with integral coefficients, an $(s+m-1)$-chain $T(\bar{g}\gamma^s)$ in $\mathfrak{B}^{s+m-1}$ corresponds, by considering that the correspondence T is linear with respect to the cells of L^m.

Since $T(\bar{g}\sigma^s)$ depends on s local parameters of $\mathfrak{R}^s$ and $m-1$ parameters u_{mi}, the terms in the expressions of the forms Π_{λ}^{s+m-1} vanish in $T(\bar{g}\sigma^s)$ except ones being $2\lambda\leqq s$ and involving the factor
$$A^{m-1}=\omega_{m1}\,\omega_{m2}\cdots\omega_{m,m-1}\,.$$
Hence, according to (10) we have in $T(\bar{g}\sigma^s)$

(25)
$$\Pi_{\lambda}^{s+m-1}=(-1)^{m-1}\frac{(s+m-2\lambda-1)!}{(s-2\lambda)!}\sum_{(\alpha)}\epsilon_{\alpha_1\alpha_2\cdots\alpha_s\,12\cdots n-s}$$
$$\times\,\Omega^{(s+m)}_{\alpha_1\alpha_2}\cdots\Omega^{(s+m)}_{\alpha_{2\lambda-1}\alpha_{2\lambda}}\Big(\sum_i u_i\varphi_{\alpha_{2\lambda+1}i}\Big)\cdots\Big(\sum_i u_i\varphi_{\alpha_s i}\Big)\,A^{m-1},$$

where $u_i=u_{mi}$. On the exterior product of linear differential forms ψ_{AB} possessing two indices, we shall introduce the following symbols:
$$\psi_{A_1Q}\,\psi_{A_2Q}\cdots\psi_{A_jQ}=\psi_{A_1A_2\cdots A_j;Q(j)}$$
and
$$\psi_{A_1A_2\cdots A_j;Q(j)}\,\psi_{B_1B_2\cdots B_k;R(k)}\cdots\psi_{C_1C_2\cdots C_l;S(l)}$$
$$=\psi_{A_1A_2\cdots A_jB_1B_2\cdots B_k\cdots C_1C_2\cdots C_l;Q(j)R(k)\cdots S(l)}\,.$$

The form $A^{m-1}=(-1)^{m-1}\omega_{12\cdots m-1;m(m-1)}$ gives the surface element of the tangent $(m-1)$-sphere S^{m-1} of R^n over a point $P\in R^m$ described by the unit vectors Pe_m. Expanding (25), we have

$$\Pi_\lambda^{s+m-1}=(-1)^{m-1}(s+m-2\lambda-1)!\sum_{(\alpha)}\sum_{l_1+l_2+\cdots+l_m=s-2\lambda}\frac{1}{l_1!l_2!\cdots l_m!}\epsilon_{\alpha_1\alpha_2\cdots\alpha_s 12\cdots n-s}$$
$$\times\Omega^{(s+m)}_{\alpha_1\alpha_2}\cdots\Omega^{(s+m)}_{\alpha_{2\lambda-1}\alpha_{2\lambda}}\varphi_{\alpha_{2\lambda+1}\cdots\alpha_s;1(l_1)2(l_2)\cdots m(l_m)}u_1^{l_1}u_2^{l_2}\cdots u_m^{l_m}A^{m-1}.$$

It has been known that if $u_1, u_2, \cdots, u_m$ denote the components of unit vectors whose origins are a fixed point in an m-dimensional Euclidean space, the integral

$$I=\int_{S^{m-1}}u_1^{l_1}u_2^{l_2}\cdots u_m^{l_m}A^{m-1}$$

over the unit sphere S^{m-1} is zero unless all exponents $l_1, l_2, \cdots, l_m$ are even, and in the later case

$$I=\frac{\pi^{\frac{1}{2}m}}{2^{2k-1}\Gamma\left(\frac{m}{2}+k\right)}\cdot\frac{l_1!l_2!\cdots l_m!}{k_1!k_2!\cdots k_m!},$$

where $l_i=2k_i$ and $k=k_1+k_2+\cdots+k_m$. Integrating the form over the tangent sphere S^{m-1} and employing a relation on the Gamma function

$$\Gamma\left(\frac{x}{2}\right)\cdot\Gamma\left(\frac{x+1}{2}\right)=2^{-(x-1)}(x-1)!\pi^{\frac{1}{2}}$$

for a positive integer x, we obtain if s is even

$$\int_{S^{m-1}}\Pi^{s+m-1}=\frac{(-1)^{s-1}}{2^{s+m}\pi^{\frac{1}{2}(s+m-1)}}\sum_{\lambda=0}^{\frac{s}{2}}\frac{(-1)!}{\lambda!}\cdot\frac{(s+m-2\lambda-1)!}{\Gamma(\frac{1}{2}(s+m-2\lambda+1))} \tag{26}$$
$$\times\sum_{k_1+k_2+\cdots+k_m=\frac{s}{2}-\lambda}\frac{1}{(2k_1)!(2k_2)!\cdots(2k_m)!}\int_{S^{m-1}}u_1^{2k_1}u_2^{2k_2}\cdots u_m^{2k_m}A^{m-1}$$
$$\times\sum_{(\alpha)}\epsilon_{\alpha_1\alpha_2\cdots\alpha_s 12\cdots n-s}\Omega^{(s+m)}_{\alpha_1\alpha_2}\cdots\Omega^{(s+m)}_{\alpha_{2\lambda-1}\alpha_{2\lambda}}\varphi_{\alpha_{2\lambda+1}\cdots\alpha_s;1(2k_1)2(2k_2)\cdots m(2k_m)}$$
$$=\frac{-1}{2^s\pi^{\frac{s}{2}}}\sum_{\lambda=0}^{\frac{s}{2}}\frac{(-1)^\lambda}{\lambda!}\sum_{k_1+k_2+\cdots+k_m=\frac{s}{2}-\lambda}\frac{1}{k_1!k_2!\cdots k_m!}$$
$$\times\sum_{(\alpha)}\epsilon_{\alpha_1\alpha_2\cdots\alpha_s 12\cdots n-s}\Omega^{(s+m)}_{\alpha_1\alpha_2}\cdots\Omega^{(s+m)}_{\alpha_{2\lambda-1}\alpha_{2\lambda}}\varphi_{\alpha_{2\lambda+1}\cdots\alpha_s;1(2k_1)2(2k_2)\cdots m(2k_m)},$$

and if s is odd

$$\int_{S^{m-1}}\Pi^{s+m-1}=0. \tag{27}$$

On the other hand, expanding the expression of the form Φ^s into which the relations (24) are substituted, we can easily see that the right hand side of (26) coincides with $-\Phi^s$. Consequently it follows that

$$(28) \qquad -\int_{S^{m-1}} \Pi^{s+m-1} = \Phi^s.$$

Taking into account that $T(\bar{g}\sigma^s)$ is homeomorphic to $\bar{g}\sigma^s \times S^{m-1}$, we get from (16)

$$(29) \qquad (-1)^s W_s \cdot Z^s \overset{*}{=} \int_{\bar{g}z^s} \Phi^s = -\int_{T(\bar{g}z^s)} \Pi^{s+m-1}.$$

Thus we have obtained the formula which expresses the Whitney classes of the normal sphere bundle over R^m in terms of the forms Π^{r-1} on $\mathfrak{B}^{r-1}$.

§ 4. The Stiefel classes of a submanifold and some remarks

Similar consideration as in the preceding section may be applyed to the tangent bundle over R^m. We shall only sketch its outline. Any element of the $(m-s+1)$-frame bundle $\mathfrak{B}^{s+p-1}$ is now denoted by $Pe_{s+1}e_{s+2}\cdots e_m e_{m+1}$, and the natural projection $\mu: \mathfrak{B}^{s+p-1} \to \mathfrak{B}^{s+p}$ is defined by

$$\mu Pe_{s+1}e_{s+2}\cdots e_m e_{m+1} = Pe_{s+1}e_{s+2}\cdots e_m .$$

The form Π_λ^{s+p-1} on $\mathfrak{B}^{s+p-1}$ is now given by

$$\Pi_\lambda^{s+p-1} = \sum_{(A)} \epsilon_{A_1A_2\cdots A_s\, s+1\cdots m+1\, A_{s+1}\cdots A_{s+p-1}} \times \Omega^{(s+p)}_{A_1A_2}\cdots\Omega^{(s+p)}_{A_{2\lambda-1}A_{2\lambda}}\, \omega_{A_{2\lambda+1}\, m+1}\cdots\omega_{A_{s+p-1}\, m+1},$$

with

$$\Omega^{(s+p)}_{AB} = \Omega_{AB} + \sum_{h=s+1}^{m} \omega_{Ah}\omega_{hB} .$$

Then it holds that

$$\theta^{(s)}_{ij} = \Omega^{(s+p)}_{ij} - \sum_{\alpha=m+1}^{n} \theta_{i\alpha}\theta_{j\alpha}$$

in $\mathfrak{T}^0$. Taking a cross-section $g: L^s \to \mathfrak{T}^s$, we have now in $N(g\sigma^s)$

$$\Pi_\lambda^{s+p-1} = (-1)^{p-1}(s+p-2\lambda-1)! \sum_{(i)} \sum_{l_1+l_2+\cdots+l_p=s-2\lambda} \frac{1}{l_1!l_2!\cdots l_p!} \epsilon_{i_1i_2\cdots i_s\, s+1\cdots n} \times \Omega^{(s+p)}_{i_1i_2}\cdots\Omega^{(s+p)}_{i_{2\lambda-1}i_{2\lambda}}\, \theta_{i_{2\lambda+1}\cdots i_s;\, m+1(l_1)\, m+2(l_2)\cdots n(l_p)}\, v^{l_1}_{m+1} v^{l_2}_{m+2}\cdots v^{l_p}_{n} A^{p-1}$$

where $v_\alpha = v_{m+1,\alpha}$, and the form $A^{p-1} = (-1)^{p-1}\omega_{m+2,m+3\cdots n;\, m+1(p-1)}$ gives the surface element of the $(p-1)$-sphere $\bar{S}^{p-1}$ described by the normal vectors Pe_{m+1} at a point $P \in R^m$. By a same calculation as in the preceding section we get finally

$$(30) \qquad -\int_{S^{p-1}} \Pi^{s+p-1} = \Theta^s,$$

and it follows that

$$(31) \qquad (-1)^s C_s \cdot Z^s \overset{*}{=} \int_{gz^s} \Theta^s = -\int_{N(gz^s)} \Pi^{s+p-1}.$$

We have thus obtained the formula which expresses the Stiefel classes of a submanifold in terms of the forms Π^{r-1} on $\mathfrak{B}^{r-1}$. When $s=m$, (31) coincides with Chern's formula.[4] The case $s=1$ is trivial, since the class is zero.

It can be proved that the formula (30) and (31) also hold in the case $m=n-1$ which was excepted in our considerations: that is, we can regard the formula (5) as special case of (30) when $m=n-1$. In fact, (5) may be rewritten as

$$(32) \qquad -\{\Pi^r + (-1)^r \Pi^r\} = \Omega^r \qquad (r < n).$$

On the other hand, by changing the orientation of the vector $Pe_{(r+1)'}$, the forms $\omega_{A,(r+1)'}$ are transformed to $-\omega_{A,(r+1)'}$ and so Π^r to $(-1)^r \Pi^r$. Hence, the form in the braces of the left hand side of (32) is nothing but the integrated form of Π^{s+p-1} over the 0-sphere S^0 consisting of two normal vectors of R^m at a point. Furthermore the relation (4) can be regarded as the formula (30) for $m=n$.

Consequently it has been made clear that the forms Ω^r, Θ^s and Φ^s, which represent the characteristic classes of various bundles induced from the tangent sphere bundle over R^n, are systimatically derived from the forms Π^r which are essential to represent the deformation cochains of frame fields in R^n.

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4) S. S. Chern: *On the curvatura integra in a Riemannian manifold*, Ann. of Math., **46** (1945), p. 679.