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On the dimension of local rings

By

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Let A be a local ring in Krull's sense [3][1] and $\mathfrak{p}$ any prime ideal of A. Then in general the sum of the rank and the dimension of $\mathfrak{p}$ is at most equal to the dimension of A. But there are several types of local rings in which, for any prime ideal, the sum of the rank and the dimension is exactly equal to the dimension of the ring itself; that is, for regular local rings this was proved by Krull [3], for geometrical local rings by C. Chevalley [1] and for complete local domains by I. S. Cohen [2]. In this short paper we shall give the most general result which includes each case above-mentioned.

Let A be a local ring and $\hat{A}$ its completion; and let n be the dimension of A. Then as is well known $\dim \hat{A} = \dim A = n$. For any prime ideal $\mathfrak{p}$ of A we have

$$\dim A/\mathfrak{p} + \dim A_{\mathfrak{p}} \leq n^{2)},$$

as is mentioned above. But now we shall prove that if $\dim \hat{A}/\hat{\mathfrak{p}}_i^{(0)}$ is equal to n for every minimal prime divisor $\hat{\mathfrak{p}}_i^{(0)}$ of (0) in $\hat{A}$, then the equality holds.

First we assume that A is an integral domain. Let us put $\dim A_{\mathfrak{p}} = r$, then there exists a set of elements $\{a_i \in \mathfrak{p}\,;\, i=1, 2, \cdots, r\}$ such that $\mathfrak{p}$ is a minimal prime divisor of $(a_1, a_2, \cdots, a_r)A$. For,

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1) Numbers in brackets refer to the bibliography at the end of this paper.

2) Throughout this paper $A_{\mathfrak{p}}$ is meant by the ring of quotients in Chevally's sense. That is, let S be the complementary set of the prime ideal $\mathfrak{p}$ in A, then the set $N = \{a \in A\,;\, ab=0 \text{ for some } b \in S\}$ is an ideal of A and there exists a natural homomorphism ϕ of A into A/N. As is easily seen, $\phi(S)$ has no zero divisors in $\phi(A)$ and hence we can define the ring of quotients in Grell's sense with respect to $\phi(A)$. This is denoted by $A_{\mathfrak{p}}$.

Cf. Chevalley: On the notion of the ring of quotients of a prime ideal, Bull. Amer. Math. Soc., Vol, 50 (1944).

let a_1 be an arbitrary element of $\mathfrak{p}$, then we can take an element $a_2 \in \mathfrak{p}$ such that a_2 does not belong to all the minimal prime divisors of $(a_1)A$ contained in $\mathfrak{p}$. Therefore each minimal prime divisor of $(a_1, a_2)A$ in $\mathfrak{p}$ includes some minimal prime divisor of $(a_1)A$ strictly, and moreover is of rank 2 by Krull's Theorem.[3] Hence we can take an element $a_3 \in \mathfrak{p}$ such that a_3 does not belong to all the minimal prime divisors of $(a_1, a_2)A$ contained in $\mathfrak{p}$. Thus, by repeating the same process, we can get $a_1, a_2, \cdots, a_r$.

Let $\mathfrak{q}$ be the primary component of $(a_1, a_2, \cdots, a_r)A$ with prime associated ideal $\mathfrak{p}$ and e a positive integer such that $\mathfrak{p}^e \subset \mathfrak{q}$. It can easily be shown that there exists an element $x \notin \mathfrak{p}$ in $[(a_1, a_2, \cdots, a_r)A : \mathfrak{p}^e]$ and that there exists an element $y \notin \hat{\mathfrak{p}}$ and a positive integer s such that $y^s \cdot \hat{\mathfrak{p}}^s \subset \mathfrak{p}\hat{A}$, where $\hat{\mathfrak{p}}$ is any minimal prime divisor of $\mathfrak{p}\hat{A}$. Putting $c = x \cdot y^{es}$, we have $c \notin \hat{\mathfrak{p}}$. Hence it follows that

$$c \cdot \hat{\mathfrak{p}}^{es} = x \cdot y^{es} \cdot \hat{\mathfrak{p}}^{es} \subset x(\mathfrak{p}\hat{A})^e \subset (a_1, a_2, \cdots, a_r)\hat{A},$$

and therefore $\phi(\hat{\mathfrak{p}}^{es})$ is contained in $(\phi(a_1), \phi(a_2), \cdots, \phi(a_r))\hat{A}_{\hat{\mathfrak{p}}}$. This shows that $\dim \hat{A}_{\hat{\mathfrak{p}}} \leq r$. But now, since the equality

$$\dim \hat{A}/\hat{\mathfrak{p}} + \dim \hat{A}_{\hat{\mathfrak{p}}} = n$$

holds ([2], Theorem 19), we have

$$\dim A/\mathfrak{p} \geq \dim \hat{A}/\hat{\mathfrak{p}} \geq n - r.$$

Hence we have

$$\dim A/\mathfrak{p} = \dim \hat{A}/\hat{\mathfrak{p}} = n - r.$$

Next we consider the general case when A has zero divisors. There exists a minimal prime divisor $\mathfrak{p}^{(0)}$ of (0) in A such that $\dim A_{\mathfrak{p}} = \dim (A/\mathfrak{p}^{(0)})_{\mathfrak{p}/\mathfrak{p}^{(0)}}$ holds. Every minimal prime divisor $\mathfrak{p}^{(0)\prime}$ of $\mathfrak{p}^{(0)}\hat{A}$ in $\hat{A}$ is also one of the minimal prime divisors of (0) in $\hat{A}$. Because $\mathfrak{p}^{(0)\prime}$ contains at least one of the minimal prime divisors of (0), say, $\hat{\mathfrak{p}}^{(0)}$, but since $\mathfrak{p}^{(0)\prime} \cap A = \mathfrak{p}^{(0)}$, it follows that $\hat{\mathfrak{p}}^{(0)} \cap A = \mathfrak{p}^{(0)}$. This shows that $\hat{\mathfrak{p}}^{(0)} \supset \mathfrak{p}^{(0)}\hat{A}$, hence $\hat{\mathfrak{p}}^{(0)} = \mathfrak{p}^{(0)\prime}$. Now we can take $A/\mathfrak{p}^{(0)}$ in place of A, for $A/\mathfrak{p}^{(0)}$ also satisfies the required condition. Thus the general case is reduced to the former one.

Thus we have proved the following theorem:

THEOREM *Let A be a local ring and $\hat{A}$ its completion. If, for every minimal prime divisor $\hat{\mathfrak{p}}^{(0)}$ of* (0) *in $\hat{A}$, we have $\dim \hat{A}/\hat{\mathfrak{p}}^{(0)} = \dim A$, then it holds that*

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3) This means the Theorem 7* of [3]

$$\dim A/\mathfrak{p} + \dim A_{\mathfrak{p}} = \dim A\,,$$

were $\mathfrak{p}$ *is an arbitrary prime ideal of* A.

BIBLIOGRAPHY

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3. W. Krull: Dimensionstheorie in Stellenringen, Crelles J., 179 (1938).