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On the imbeddings of abstract surfaces in projective varieties

By

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Recently Zariski proved that a normal abstract surface can be imbedded in a projective variety if (and only if) there exists an affine variety which carries all singular points of the variety.[1)]

The main purpose of the present paper is to prove the following

THEOREM 1. *There exists a complete normal surface which cannot be imbedded in any projective space.*

In order to prove Theorem 1, we shall prove the following two theorems, from which Theorem 1 follows easily:

THEOREM 2. *Every normal surface can be imbedded in a complete normal surface.*[2)]

THEOREM 3. *There exists a normal abstract surface V with two points P and P' such that, if a function ϕ on V is well defined at both P and P', then ϕ is a constant function.*

§ 1. The proof of Theorem 2.

Let V be a normal abstract surface. Obviously, there exists a normal projective surface V^* such that 1) V^* is birationally equivalent to V and 2) if a point $P^* \varepsilon V^*$ corresponds to a point $P \varepsilon V$, then P^* dominates P (for example, let $V_1, \cdots, V_n$ be projective

1) Zariski treated the complete case at first, as was shown by him in his lecture at Kyoto University (Oct. 1956). By virtue of Theorem 2 (cf. foot-note 2)), he generalized to the non-complete case.

2) Since singularities of a surface are reduced by normalizations and quadratic transformations with singular centers, we may require that the complete surface has no singularity outside of the given surface.

varieties in which affine representatives of V are imbedded respectively and let V^* be the derived normal variety of the join of the V_i's). Therefore the following lemma will prove Theorem 2.

LEMMA. Let V be a normal surface and let V^* be a complete normal surface of the same function field such that if a point $P^* \varepsilon V^*$ corresponds to a point $P \varepsilon V$ then P^* dominates P. If Q is a fundamental point with respect to V^* and if F^* is the total transform of Q in V^*, then $(V^*-F^*) \cup Q$ is a complete normal surface and satisfies the same condition as V^* with respect to V.

Proof. Since F^* is a closed set, V^*-F^* is an abstract surface. On the other hand, since V is a normal surface, there exists a closed set F of V which does not contain Q such that Q is the only one point of $V-F$ which is not biregular with respect to V^*. Then $V-F$ is also an abstract surface and $(V^*-F^*) \cup Q = (V^* -F^*) \cup (V-F)$, hence $(V^*-F^*) \cup Q$ is the union of a finite number of affine varieties. Obviously, every place of the function field of V has one and only one center on $(V^*-F^*) \cup Q$ and therefore $(V^*-F^*) \cup Q$ is a complete surface. Now we see Lemma easily.

§ 2. Preliminaries on a cone.

PROPOSITION. *Let D be the divisor on a normal affine cone V defined by a homogeneous ideal $\mathfrak{a}$. Then D is linearly equivalent to zero on V if (and only if) it is linearly equivalent to zero locally at the vertex P of V.*

Proof. We shall denote also by P the spot (local ring) of P. Then there exists an element $f \varepsilon P$ such that $\mathfrak{a}P = fP$. Let f' be the leading form of f. Then $f' \varepsilon \mathfrak{a}$. Therefore f'/f is a unit in P. Hence $f'P = fP = \mathfrak{a}P$. Since f' is homogeneous, f' generates $\mathfrak{a}$ and D is linearly equivalent to zero on V.

We shall apply this proposition to the cone V^2 defined by $x^3+y^3=z^3$ over the field R of rational numbers. This V^2 can be regarded as the representative cone of the projective curve V^* with the generic point $D^*=(a, b, 1)$, a and b being transcendental numbers such that $a^3+b^3=1$. Let D be the generator of V^2 which goes through D^*. Then we have

LEMMA. For any natural number n, nD is not linearly equivalent to zero locally at the vertex P of V^2.

Proof. Let E^* be the point $(1, -1, 0)$ on V^*. Now, we assume the contrary. Then by Proposition, nD is linearly equivalent to zero on V^2. Let f be the fomogeneous form which defines nD. Let m be the degree of f. Then $f/(x+y)^m$ is a function on V^* whose zero and pole are nD^* and $3mE^*$ respectively and we have $nD^* \sim 2mE^*$, hence $n(D^*-E^*)\sim 0$ $(3m=n)$, which is a contradiction because E^* is rationl over R and D^* is a generic point of V^* over R.

§ 3. The proof of Theorem 3.

Let again a and b be transcendental numbers such that $a^3+b^3=1$ and let k be a field containing a and b. Then V^2 in § 2 can be defined by

$$x^3+y^3+3(ax^2+by^2)z+3(a^2x+b^2y)z^2=0 \qquad (1)$$

and the divisor D is defined by $x=y=0$. Set $\mathfrak{o}=k[x, y, z]$, $\mathfrak{b}=x\mathfrak{o}+y\mathfrak{o}$, $\mathfrak{a}=x\mathfrak{o}+(y^2+3byz+3b^2z^2)\mathfrak{o}$. Let F be the divisor defined by $\mathfrak{a}$. Furthermore, we set

$$\left.\begin{array}{l} t=(a^2x+b^2y)/b^2x^2,\ u=(tx-(a/b)^2)t, \\ v=[(tx-(a/b)^2)^3+1]/3b^2z. \end{array}\right\} \qquad (2)$$

Then: $y=(tx-(a/b)^2)x,\ x=(a^2t+b^2u)/b^2t^2,$

$z=[(tx-(a/b)^2)^3+1]/3b^2v.$

LEMMA 1. The mapping $(x, y, z)\xrightarrow{\sigma}(t, u, v)$ defines an involution of $k(x, y, z)$.

Proof. By the relations above, we see immediately that σ: $(x, y)\rightarrow(t, u)$ defines an involution in $k(x, y)$. By the involution, tx is mapped to tx itself, and therefore we see easily that the involution can be extended to an involution of $k(x, y, z)$ which mapps (x, y, z) to (t, u, v), observing that (t, u, v) satisfies the same relation as (x, y, z).

LEMMA 1. $k[x, y, z, t, u, v]$ $(=k[x, z, t, v])$ defines the affine model $V-F$.[3)]

Proof. Since $a^2x+b^2y \in \mathfrak{b}^{(2)}$ and since $x\mathfrak{o}=\mathfrak{b}\cap\mathfrak{a}$, we see that $t\in\mathfrak{a}^{-2}$ (i.e., $\mathfrak{a}^2t\subseteq\mathfrak{o}$). Let A be the affine variety defined by $k[x, y, z, t]$.

3) If we make use of the notion of $\mathfrak{a}$-transform, defined by Nagata "A treatise on the 14-th problem of Hillert" Memoirs Kyoto University vol. 30, No. 1 (1956), this assertion means that $k[x, z, t, v]$ is the $\mathfrak{a}$-transform of $\mathfrak{o}$.

Since $\mathfrak{a}\mathfrak{o}[t]$ is generated by x and z^2, $A-$(the divisor defined by $x=z=0$) coincides with $V-F$. Since $1+(tx-(a/b)^2)^3=1+(y/x)^3 =(x^3+y^3)/x^3=-3[by^2z+ax^2z+(a^2x+b^2y)z^2]/x^3$, we see that $x^3v \varepsilon \mathfrak{o}$. Obviously $zv \varepsilon \mathfrak{o}[t]$ and therefore, for the affine variety A^*, defined by $k[x, y, z, t, v]$, the same property, as stated above for A, holds good. But, obviously the definition of v shows that $x\mathfrak{o}[t, v]+z\mathfrak{o}[t, v]$ contains 1. Therefore $A^*=V-F$ and the assertion is proved.

Now, by the involution σ, we can consider V^σ, D^σ, F^σ. Then we see that $V-F=V^\sigma-F^\sigma$ by Lemma 2. Furthermore, we see that the union M of V and V^σ is an abstract surface, which is obviously normal. For, let $\mathfrak{v}$ be a place of $k(x, y, z)$ which has centers in V and V^*. Since $k[t, u, v, x, y, z]$ defines $V-F$, the center of $\mathfrak{v}$ is in $V-F$. Therefore the centers of $\mathfrak{v}$ in V and V^* conincides with each other. Therefore M is an abstract variety. Let P and P^σ be the spots of the vertices of the cones V and V^σ. In order to prove Theorem 3, it is sufficient to show that $P\cap P^\sigma =k$. P and P^σ will denote also the vertices.

LEMMA 3. $\mathfrak{o}\cap\mathfrak{o}^\sigma=k$.

Proof. Regard V to be an affine representative of the projective cone defined by the same relation. Then the infinity plane section is irreducible. Therefore $\mathfrak{o}\cap k[1/x, y/x, z/x]=k$. As is easily seen, t, u, v are in $k[1/x, y/x, z/x]$ (as for v, use the relation derived in the proof of Lemma 2). Therefore $\mathfrak{o}\cap\mathfrak{o}^\sigma=k$.

LEMMA 4. If a divisor C on M does not go through any of P and P^σ, then $C=0$.

Proof. We may assume at first that C is irreducible. Let K be a field of definition of C. Let k be a purely transcendental extension of $R(a, b)$ contained in K such that K is algebraic over k. Let C' be the sum of all conjugates of C over k. Since P and P^σ are rational points over k, C' does not go through any of P and P^σ, and C' is a prime rational cycle over k. Observe that $\sqrt{-3}\notin k$. Now, it is sufficient to prove that $C'=0$. Let $\mathfrak{q}$ be the prime ideal in $k[t, v, x, z]$ which defines C'. If $\mathfrak{q}=k[t, v, x, z]$, then the assertion is obvious and we assume that $\mathfrak{q}$ is of rank 1. Since $P\notin C'$, $\mathfrak{q}$ contains an element $f(x, y, z)$ such that $f(0, 0, 0)=1$. Let $\mathfrak{f}$ be the derived normal ring of $k[tx, tz]$. Since tx and tz are functions on the cubic curve V^* in § 2, we see that $\mathfrak{f}$ is of dimension 1 and t is transcendental over $\mathfrak{f}$. (It will be not hard to see that $\mathfrak{f}=k[tx, tz]$.) For a sufficiently large n, t^nf is a monic polynomial in t with coefficients in $\mathfrak{f}$. Therefore $\mathfrak{q}$ contains a prime element in $\mathfrak{k}[t]$. $\mathfrak{k}[t, 1/t, 1/tx]=k[x, z, t, v, 1/t, 1/x]$ and t, x are prime elements in $k[x, z, t, v]$ (for, Lemma 2 shows that $xk[x, z, t, v]$ defines the divisor D; $t=x^{\sigma}$). Therefore we see that $\mathfrak{q}$ is generated by one element, say q. Assume that $q \notin \mathfrak{o}$ and let i be the least integer such that $x^i q \in \mathfrak{o}$. Then the divisor on V defined by $x^i q$ is $iD+C'$ (because F is prime rational over k and x defines $D+F$). This shows tnat iD is linearly equivalent to zero locally at P, which is a contradiction. Similarly we have $q \in \mathfrak{o}^{\sigma}$. Thus $q \in \mathfrak{o} \cap \mathfrak{o}^{\sigma}=k$. Thus Lemma 4 is proved.

Now we come to the proof of $P \cap P'=k$. Let ϕ be a function on M which is well defined at both P and P'. Then the pole of ϕ does not goes through any of P and P', hence ϕ has no pole by Lemma 4, i.e., ϕ is well defined at every point on M. By Lemma 3, we see now that ϕ is constant. Thus Theorm 3 is proved completely.

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Added in Proof. The writer proved recently that (1) let L be a function field of dimension not less than 2, then there exists a normal complete abstract variety of L which cannot be imbedded in any projective space, provided that the ground field is sufficiently large; if dim L is greater than 2, then such a variety exists without any condition on the ground field and (2) if n is a natural number greater than 2, then there exists a non-singular complete variety of dimension n which cannot be imbedded in any projective space.

The details will be published in a forthcoming paper.