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# Relative Riemannian Geometry

## I. On the affine connection

By

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### Preliminaries

In a previous paper [1] we treated spaces with an analytic distance, in which an analytic distance-function $d(x, y)$ was given and a fundamental tensor $g_{i(j)}(x, y)$ was introduced by means of the distance such that

$$g_{i(j)}=\frac{\partial^2 g(x,\ y)}{\partial x^i \partial y^{(j)}}, \qquad g=-\frac{1}{2}(d(x,\ y))^2 .$$

From this we got the curvature tensor and some of the geometric notions. But it is clear that these can be derived from any function which is not necessarily the function as above given, and hence we can not expect many geometric notions enough to discuss the properties of the space. On the other hand we are under the consideration of the geometric interpretation of a system of integral equations

$$v^i(x)=u^i(x)-\int k^i_{j'}(x,\ y)u^{j'}(y)dy^{1'}\cdots dy^{n'},$$

where the $u$ and $v$ are vectors and the kernel $k^i_{j'}(x, y)$ is the tensor with respect to a pair of points $(x, y)$. Further we should assume from the geometric stand-point that the kernel is of weight one with respect to $(y)$. Thus we meet also with a notion of a tensor with respect to a pair of points.

From these view-points we shall introduce in this paper a notion of *a relative affine connection of a pair of manifolds* $(M, N)$. The connection in $M$ is determined in relation to so-called *observ-*

*ing point* in $N$. This situation is similar to the theory given by *E. Cartan* for Finsler spaces [2], in which the connection depends not only on a point, but also on a supporting element. For Finsler spaces we have an useful condition that the coefficients $C^i_{jk}$ of the connection-forms $\omega^i_j$ are of degree zero with respect to a supporting element [2, the equation (5)], while there is not such a condition in our case. Instead of this condition we shall give a mapping $g$ from the tangent vector space at a point of $M$ to the one at a point of $N$, and impose the condition that the mapping $g$ has its inverse. Hence we have to assume that the dimension of $M$ is equal to the one of $N$ and that the determinant of the tensor $g^i_{j'}(x, y)$, defining the mapping $g$, does not vanish. Further we require that the tensor $g^i_{j'}$ is covariant constant.

Under these considerations we shall define a covariant differentiation and develope the theories, following to the ordinal affine connections. Various curvature tensors are derived according as an observing point displaces or not. In Finsler spaces it is essential that a point displaces to the direction of the supporting element. On the other hand, it will play a role in our case that a point of $M$ displaces to the direction corresponding to the displacement of an observing point by the mapping $g$. From this idea we shall introduce another covariant differentiation and new curvature tensor. These will be thought to be important for the theory of the metric connections, which will be developed in the following papers. Finally we shall define a path with respect to an observing point and a remarkable class of connections.

## 1. The affine connection and the mapping $g$

Let $M$ and $N$ be the differentiable manifolds of dimension $n$, where the differentiable classes are assumed to be $C^\infty$ throughout the paper. We consider points $P(x)$ in $M$ and $Q(y)$ in $N$, the local coordinates of which are given by $(x^i)$ and $(y^{j'})$ respectively. If a set of functions $A^{i_1\cdots i_p k_1'\cdots k_r'}_{j_1\cdots j_q l_1'\cdots l_s'}(x, y)$ is given and obeys the law of transformation

$$\bar{A}^{a_1\cdots a_p c_1'\cdots c_r'}_{b_1\cdots b_q d_1'\cdots d_s'}(\bar{x}, \bar{y}) = A^{i_1\cdots i_p k_1'\cdots k_r'}_{j_1\cdots j_q l_1'\cdots l_s'}(x, y)$$
$$\times \bar{X}^{a_1}_{i_1}\cdots \bar{X}^{a_p}_{i_p} X^{j_1}_{b_1}\cdots X^{j_q}_{b_q} \bar{Y}^{c_1'}_{k_1'}\cdots \bar{Y}^{c_r'}_{k_r'} Y^{l_1'}_{d_1'}\cdots Y^{l_s'}_{d_s'},$$
$$\left(\bar{X}^a_i = \frac{\partial \bar{x}^a}{\partial x^i}, \quad X^i_a = \frac{\partial x^i}{\partial \bar{x}^a}, \text{ etc.}\right),$$

under a change of the local coordinates $(x, y) \to (\bar{x}, \bar{y})$, then we shall call $A^{i_1 \cdots i_p k_1' \cdots k_r'}_{j_1 \cdots j_q l_1' \cdots l_s'}$ components of *a tensor A of (x)-order (p, q) and (y)-order (r, s)*. It is easily seen that the partial derivatives of components of a tensor of $(x)$-order $(p, q)$ and $(y)$-order $(0, 0)$ define a tensor of the same $(x)$-order and $(y)$-order $(0, 1)$.

We shall define the *affine connection at a point P(x) in M with respect to an observing point Q(y) in N* by the following equations in terms of the natural frame $e_i$:

$$dP = dx^i e_i, \tag{1.1}$$

$$de_i = \omega_i^j e_j, \tag{1.2}$$

where the *connection-forms* $\omega_i^j$ are linear forms of $dx$ and $dy$, which are expressed by

$$\omega_i^j = \Gamma_{ik}^j(x, y) dx^k + C_{ik'}^j(x, y) dy^{k'}. \tag{1.3}$$

We should suppose that the coefficients $\Gamma_{ik}^j$ satisfy the law of transformation

$$\bar{\Gamma}_{ac}^b(\bar{x}, \bar{y}) = \Gamma_{ik}^j(x, y) X_a^i \bar{X}_j^b X_c^k + \frac{\partial X_a^i}{\partial \bar{x}^c} \bar{X}_i^b,$$

while the coefficients $C_{jk'}^i$ are components of a tensor of $(x)$-order $(1, 1)$ and $(y)$-order $(0, 1)$. For the fixed observing point $Q(y)$ we have

$$\omega_i^j = \Gamma_{ik}^j(x, y) dx^k .$$

The $\Gamma_{jk}^i$ are called *the translation-components* of the connection. On the other hand, for the fixed $P(x)$ we obtain

$$\omega_i^j = C_{ik'}^j(x, y) dy^{k'},$$

and hence the $C_{jk'}^i$ are called *the rotation-components* of the connection. The connection in $N$ is also defined similar to the above equations and the connection-forms $\omega_{i'}^{j'}$ are written as

$$\omega_{i'}^{j'} = \Gamma_{i'k'}^{j'}(y, x) dy^{k'} + C_{i'k}^{j'}(x, y) dx^k . \tag{1.3'}$$

Next, we shall define the *g-mapping*, which carries the tangent vector space at $P$ in $M$ to the one at $Q$ in $N$. Let $g_j^{i'}(x, y)$ be components of a tensor of $(x)$-order $(0, 1)$ and $(y)$-order $(1, 0)$. For any vector $V^i$ at $P$, we obtain a vector $V^{i'}$ at $Q$ such that

$$V^{j'} = V^i g_i^{j'}(x, y) . \tag{1.4}$$

Hence the tensor $g_i^{j'}$ defines a linear mapping $g: T_P \to T_Q$, where the $T_P$ and $T_Q$ are the tangent vector spaces at $P$ and $Q$ respectively. If there exists such a relation (1.4) for two vectors $V^i$ and $V^{j'}$ at $P$ and $Q$ respectively, then these vectors are said to be *g-related*. In particular infinitesimal displacements $dx$ and $dy$ are $g$-related if and only if the equations

$$dy^{j'} = dx^i g_i^{j'}(x, y) \tag{1.5}$$

are satisfied. We should like to deal with the manifold $N$ equal to the manifold $M$, so that we assume that the mapping $g$ has its inverse and hence the *det.* $|g_i^{j'}|$ *does not vanish.* The inverse $g_{j'}^i$ is clearly the tensor of $(x)$-order $(1, 0)$ and $(y)$-order $(0, 1)$, and we get the inverse mapping $g^{-1}: T_Q \to T_P$.

The notion of $g$-related vectors may be extended to tensors of any type. For an example, a tensor $A_{j'}^i(x, y)$ is said to be $g$-related to a tensor $A_l^{k'}(x, y)$ if

$$A_l^{k'} = A_{j'}^i g_i^{k'} g_l^{j'} .$$

The mapping $g$ may be expressed in terms of the natural frames. In fact, from (1.4) we get

$$e_{j'} v^{j'} = e_{j'} g_i^{j'} v^i ,$$

and hence the mapping $g$ is thought of as

$$g: e_i \to g_i^{j'} e_{j'} .$$

On the other hand, we take a point $P' = P(x+dx)$ and the connection in $M$ defines the mapping $\varphi_M: T_{P'} \to T_P$ as follows:

$$\varphi_M: e_i(x+dx) \to e_i(x) + de_i .$$

Now we require that the condition $\varphi_M g^{-1} = g\varphi_N$ holds for the connection and the $g$-mapping. We have immediately

$$\varphi_M g^{-1}(e_{i'}(y+dy)) = g_{i'}^j e_j + g_{i'}^k \omega_k^j e_j + dg_{i'}^j e_j ,$$
$$g\varphi_N(e_{i'}(y+dy)) = g_{i'}^j e_j + \omega_{i'}^{k'} g_{k'}^j e_j ,$$

and hence the above condition is expressible by the equations:

$$\frac{\partial g^{i}_{j'}}{\partial x^{k}} = -g^{h}_{j'}\Gamma^{i}_{hk} + g^{i}_{h'}C^{h'}_{j'k}\,, \tag{1.6}$$

$$\frac{\partial g^{i}_{j'}}{\partial y^{k'}} = -g^{h}_{j'}C^{i}_{hk'} + g^{i}_{h'}\Gamma^{h'}_{j'k'}\,. \tag{1.7}$$

Since the $g^{i'}_{j}$ are the inverse of the $g^{j}_{i'}$, it follows from the above equations that

$$\frac{\partial g^{i'}_{j}}{\partial x^{k}} = -g^{h}_{j'}C^{i'}_{h'k} + g^{i'}_{h}\Gamma^{h}_{jk}\,, \tag{1.6$'$}$$

$$\frac{\partial g^{i'}_{j}}{\partial y^{k'}} = -g^{h'}_{j}\Gamma^{i'}_{h'k'} + g^{i'}_{h}C^{h}_{jk'}\,. \tag{1.7$'$}$$

Now we gave all of the suppositions in order to develop our theories. We shall call by the theories of the *relative affine connection of the pair* $(M, N)$ the objects derived from the connection-forms $\omega^{i}_{j}$, $\omega^{i'}_{j'}$ and the $g$-tensors $g^{i}_{j'}$, $g^{i'}_{j}$ which satisfy the above four equations.

We shall introduce in this place *a covariant differentiations* $( , )$ with respect to $x^{i}$ or $y^{j'}$. For a tensor $A^{ik'}_{jl'}$, the rule of the operation with respect to $x^{m}$ is given by the equation

$$A^{ik'}_{jl',m} = \frac{\partial A^{ik'}_{jl'}}{\partial x^{m}} + A^{ak'}_{jl'}\Gamma^{i}_{am} - A^{ik'}_{al'}\Gamma^{a}_{jm} + A^{ia'}_{jl'}C^{k'}_{a'm} - A^{ik'}_{ja'}C^{a'}_{l'm}\,.$$

The one with respect to $y^{j'}$ is defined by the similar rule to the above, where the coefficients $\Gamma^{i'}_{j'k'}$ and $C^{i}_{jk'}$ are used instead of the $\Gamma^{i}_{jk}$ and $C^{i'}_{j'k}$. The covariant derivatives of a tensor of $(x)$-order $(p, q)$ and $(y)$-order $(r, s)$ with respect to $x^{i}$ define a tensor of $(x)$-order $(p, q+1)$ and the same $(y)$-order. This is easily verified from the law of transformation of the coefficients of the connection-forms. Then the four equations (1.6), (1.7), (1.6′) and (1.7′) imply that the $g$-tensors are covariant constant with respect to both of the variables $x^{i}$ and $y^{j'}$.

## 2. The torsion and curvature tensors.

We shall define in this section a torsion tensor and three kinds of curvature tensors. We put

$$\Omega^{i} = -\omega^{i}_{j}\wedge dx^{j}\,, \tag{2.1}$$

where the sign $(\wedge)$ denotes the operation of exterior product.

These *2*-forms are called the *torsion-forms* and expressed as follows:

(2. 2) $$\Omega^i = T^i_{jk}dx^j \wedge dx^k + C^i_{jk'}dx^j \wedge dy^{k'} .$$

where the $T^i_{jk}$ is given by

(2. 3) $$T^i_{jk} = \frac{1}{2}\Gamma^i_{[jk]}{}^* .$$

It is clear that the $T^i_{jk}$ are the components of a tensor of $(x)$-order $(1, 2)$ and $(y)$-order $(0, 0)$. Next we have from (1. 3)

(2. 4) $$d\omega^j_i = \omega^k_i \wedge \omega^j_k + \Omega^j_i ,$$

where the sign $(d)$, operating to a form, denotes the exterior differentiation, and the $\Omega^j_i$ are the *2*-forms expressed by

(2. 5) $$\Omega^j_i = -\frac{1}{2}R^j_{i\cdot kl}dx^k \wedge dx^l - R^j_{i\cdot kl'}dx^k \wedge dy^{l'} - \frac{1}{2}R^j_{i\cdot k'l'}dy^{k'} \wedge dy^{l'} .$$

These coefficients are given in terms of the coefficients of the connection-forms $\omega^j_i$ as follows:

(2. 6) $$R^j_{i\cdot kl} = \frac{\partial \Gamma^j_{i[k}}{\partial x^{l]}} + \Gamma^h_{i[k}\Gamma^j_{|h|l]} ,$$

(2. 7) $$R^j_{i\cdot kl'} = \frac{\partial \Gamma^j_{ik}}{\partial y^{l'}} - \frac{\partial C^j_{il'}}{\partial x^k} + \Gamma^h_{ik}C^j_{hl'} - C^h_{il'}\Gamma^j_{hk} ,$$

(2. 8) $$R^j_{i\cdot k'l'} = \frac{\partial C^j_{i[k'}}{\partial y^{l']}} + C^h_{i[k'}C^j_{|h|l']} .$$

The fact that these quantities $R^j_{i\cdot kl}$, $R^j_{i\cdot kl'}$ and $R^j_{i\cdot k'l'}$ define tensors will be verified later on.

We consider two infinitesimal parallel circuits in $M$ and $N$ consisting of the four vertices $P(x)$, $P_1(x+dx)$, $P_2(x+dx+\delta(x+dx))$, $P_1'(x+\delta x)$ and of the $Q(y)$, $Q_1(y+dy)$, $Q_2(y+dy+\delta(y+dy))$, $Q_1'(y+\delta y)$ respectively. We develope the tangent affine space at $P_2$ along the infinitesimal side $P_2P_1$ and then $P_1P$ on the tangent affine space at $P$ by means of the usual process, using (1. 1) and (1. 2),

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* We use for brevity the sign $[ij]$ throughout the paper, which means
$$\Gamma^i_{[jk]} = \Gamma^i_{jk} - \Gamma^i_{kj} .$$

and then we get the point $P'$ and the vectors $e_i'$ as the images of $P_2$ and the frame vectors $e_i$ at $P_2$ respectively. By the same process using the point $P_1'$ instead of $P_1$, we have then the point $P''$ and the vectors $e_i''$ as the images. The differences $\Delta P = P' - P''$ and $\Delta e_i = e_i' - e_i''$ are given by

$$\Delta P = (\delta d - d\delta)P, \quad \Delta e_i = (\delta d - d\delta)e_i .$$

By the direct calculation we obtain their expressions:

$$\Delta P = (2T^i_{jk}dx^j\delta x^k + C^i_{jk'}dx^j\delta y^{k'} - C^i_{jk'}\delta x^j dy^{k'})e_i , \tag{2.9}$$

$$\begin{aligned}\Delta e_i = (&R^j_{i\cdot kl}dx^k\delta x^l + R^j_{i\cdot kl'}dx^k\delta y^{l'} \\ &- R^j_{i\cdot kl'}\delta x^k dy^{l'} + R^j_{i\cdot k'l'}dy^{k'}\delta y^{l'})e_j .\end{aligned} \tag{2.10}$$

We see from (2.9) that the $\Delta P$ is equal to zero for the fixed observing point $Q$ if and only if the tensor $T^i_{jk}$ vanishes, and hence we denote this tensor by the *torsion tensor.*

Next we consider the latter equation (2.10). In the first place, for the fixed $Q(y)$ we have

$$\Delta e_i = R^j_{i\cdot kl}dx^k\delta x^l e_j .$$

The tensor $R^j_{i\cdot kl}$ is expressed in terms of the translation-parts of the connection alone and has the similar geometric meaning to the curvature tensor in Riemannian spaces. We call that the *translation-curvature.* On the other hand, if the point $P(x)$ in $M$ is fixed, then we get

$$\Delta e_i = R^j_{i\cdot k'l'}dy^{k'}\delta y^{l'}e_j .$$

which expresses the difference between the two rotations of the frame vectors $e_i$ at $P(x)$ according as the observing point $Q_2$ displaces along the sides $Q_2 \to Q_1 \to Q$ and $Q_2 \to Q_1' \to Q$. The tensor $R^j_{i\cdot k'l'}$ constitutes of the rotation-parts of the connection, and hence we call this the *rotation-curvature.* Finally we take $\delta x = 0$ and $dy = 0$, and then we get

$$\Delta e_i = R^j_{i\cdot kl'}dx^k\delta y^{l'}e_j .$$

In this case the frame vectors $e_i$ at $P_1$ enjoy first the rotation according to the displacement $Q_1' \to Q$ of the observing point, and then the vectors as thus obtained are carried into the vectors $e_i'$ of the tangent space at the origin $P$ for the fixed $Q$. Next, the frame vectors at $P_1$ are carried first into the vectors of the tangent space at $P$ with respect to the fixed observing point $Q_1'$, and then the images enjoy a rotation according to the displacement $Q_1'\to Q$ and become the vectors $e_i''$. The above equations give the difference $\Delta e_i = e_i' - e_i''$ of the vectors as thus obtained. We shall refer to the tensor $R^j_{i\cdot kl'}$ as the *mixed curvature.*

In terms of the covariant differentiations $(,)$ the mixed curvature and the rotation-curvature are written in the forms respectively

$$R^j_{i\cdot kl'} = \frac{\partial \Gamma^j_{ik}}{\partial y^{l'}} - C^j_{il',k} - C^j_{ih'} C^{h'}_{l'k}, \tag{2.7'}$$

$$R^j_{i\cdot k'l'} = C^j_{i[k',l']} - C^h_{i[k'} C^j_{|h|l']} + 2C^j_{ih'} T^{h'}_{k'l'}. \tag{2.8'}$$

The second equation shows that the $R^j_{i\cdot k'l'}$ is a tensor of $(x)$-order $(1, 1)$ and $(y)$-order $(0, 2)$. And, since it is easily seen that the $\partial\Gamma^j_{ik}/\partial y^{l'}$ are components of a tensor, it follows from the first equation that the $R^j_{i\cdot kl'}$ is a tensor of $(x)$-order $(1, 2)$ and $(y)$-order $(0, 1)$.

If, in place of the manifold $M$, we consider the manifold $N$ and $M$ is regarded as the observing manifold, we obtain similarly the torsion tensor $T^{i'}_{j'k'}$ and the curvature tensors $R^{j'}_{i'\cdot k'l'}$, $R^{j'}_{i'\cdot k'l}$ and $R^{j'}_{i'\cdot kl}$ of $N$, which are given by the similar equations to (2.3), (2.6), (2.7) and (2.8).

The condition of integrability of the covariant differentiation $(,)$ is immediately obtained as follows:

$$\begin{aligned} A^{ik'}_{jl',[h,m]} = {} & A^{ak'}_{jl'} R^i_{a\cdot hm} - A^{ik'}_{al'} R^a_{j\cdot hm} + A^{ia'}_{jl'} R^{k'}_{a'\cdot hm} \\ & - A^{ik'}_{ja'} R^{a'}_{l'\cdot hm} - 2A^{ik'}_{jl',a} T^a_{hm}, \end{aligned} \tag{2.11}$$

$$\begin{aligned} A^{ik'}_{jl',[h',m']} = {} & A^{ak'}_{jl'} R^i_{a\cdot h'm'} - A^{ik'}_{al'} R^a_{j\cdot h'm'} + A^{ia'}_{jl'} R^{k'}_{a'\cdot h'm'} \\ & - A^{ik'}_{ja'} R^{a'}_{l'\cdot h'm'} - 2A^{ik'}_{jl',a'} T^{a'}_{h'm'}, \end{aligned} \tag{2.12}$$

$$\begin{aligned} A^{ik'}_{jl',[h,m']} = {} & A^{ak'}_{jl'} R^i_{a\cdot hm'} - A^{ik'}_{al'} R^a_{j\cdot hm'} - A^{ia'}_{jl'} R^{k'}_{a'\cdot m'h} \\ & + A^{ik'}_{ja'} R^{a'}_{l'\cdot m'h} - A^{ik'}_{jl',a} C^a_{hm'} + A^{ik'}_{jl',a'} C^{a'}_{m'h}. \end{aligned} \tag{2.13}$$

(We should remark that the algebraic sign of the third term in the right hand member of the last equation is minus.) If we apply the rule (2.11) to a vector $V^i$, then we get

$$V^i_{,[j,k]} = V^a R^i_{a\cdot jk} - 2V^i_{,a} T^a_{jk},$$

from which we see that the $R^i_{j\cdot kl}$ is a tensor of $(x)$-order $(1, 3)$ and vanishing $(y)$-order.

## 3. Various identities satisfied by the torsion and curvature tensors

We shall find the identities satisfied by the torsion tensor, the curvature tensors and their covariant derivatives. First of all we have directly from definitions

$$R^j_{i\cdot(kl)} = 0\,, \qquad R^j_{i\cdot(k'l')} = 0^* \tag{3.1}$$

Next, applying to the $g$-tensor $g^{i'}_j$ the equations (2.11), (2.12) and (2.13), and making use of the fact that the tensor is covariant constant, we get

$$g^{a'}_j R^{i'}_{a'\cdot kl} - g^{i'}_a R^a_{j\cdot kl} = 0\,, \quad g^{a'}_j R^{i'}_{a'\cdot k'l'} - g^{i'}_a R^a_{j\cdot k'l'} = 0\,,$$
$$-g^{a'}_j R^{i'}_{a'\cdot k'l} - g^{i'}_a R^a_{j\cdot lk'} = 0\,,$$

and further we have also the similar equations for the inverse $g^i_{j'}$. Hence we can define the tensors such that

$$\begin{aligned} R^{i'}_{j\cdot kl} &= g^{a'}_j R^{i'}_{a'\cdot kl} = g^{i'}_a R^a_{j\cdot kl}\,,\\ R^{i'}_{j\cdot k'l'} &= g^{a'}_j R^{i'}_{a'\cdot k'l'} = g^{i'}_a R^a_{j\cdot k'l'}\,,\\ R^i_{j'\cdot kl} &= g^a_{j'} R^i_{a\cdot kl} = g^i_{a'} R^{a'}_{j'\cdot kl}\,,\\ R^i_{j'\cdot k'l'} &= g^a_{j'} R^i_{a\cdot k'l'} = g^i_{a'} R^{a'}_{j'\cdot k'l'}\,, \end{aligned} \tag{3.2}$$

and moreover

$$\begin{aligned} R^{i'}_{j\cdot k'l} &= g^{a'}_j R^{i'}_{a'\cdot k'l} = -g^{i'}_a R^a_{j\cdot lk'}\,,\\ R^i_{j'\cdot kl'} &= g^a_{j'} R^i_{a\cdot kl'} = -g^i_{a'} R^{a'}_{j'\cdot l'k}\,, \end{aligned} \tag{3.3}$$

from which we see

$$\begin{aligned} R^i_{j\cdot kl} &= R^{a'}_{b'\cdot kl} g^i_{a'} g^{b'}_j\,,\\ R^i_{j\cdot k'l'} &= R^{a'}_{b'\cdot k'l'} g^i_{a'} g^{b'}_j\,, \end{aligned} \tag{3.4}$$

$$-R^i_{j\cdot lk'} = R^{a'}_{b'\cdot k'l} g^i_{a'} g^{b'}_j\,. \tag{3.5}$$

The equation (3.4) shows that, in so far as the contravariant

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* The sign $(ij)$ is used to mean
$$R^j_{i\cdot(kl)} = R^j_{i\cdot kl} + R^j_{i\cdot lk}\,.$$

index and the first covariant index, the translation-curvature of $M(N)$ is $g$-related to the rotation-curvature of $N(M)$.

In order to find the identities satisfied by the covariant derivatives of the torsion tensor, we operate to (2.1) the exterior differentiation and then we get as a consequence of (2.4)

$$d\Omega^i + \Omega^i_j \wedge dx^j = -\omega^k_j \wedge \omega^i_k \wedge dx^j . \tag{3.6}$$

Comparing the coefficients of the both sides we obtain the following two equations:

$$R^{i}_{(j\cdot kl)} = 2T^{i}_{(jk,l)} - 4T^{i}_{h(j}T^{h}_{kl)}{}^{*} \tag{3.7}$$

$$R^{j}_{[i\cdot k]l'} = 2T^{j}_{ik,l'} - C^{j}_{[i|l'|,k]} - 2T^{h}_{ik}C^{j}_{hl'} - 2T^{j}_{h[i}C^{h}_{k]l'} - C^{j}_{[i|h'|}C^{h'}_{|l'|k]} . \tag{3.8}$$

Further, by means of the same process from (2.4), we have

$$d\Omega^j_i = \omega^k_i \wedge \Omega^j_k - \Omega^k_i \wedge \omega^j_k , \tag{3.9}$$

and the following four equations:

$$R^{j}_{i\cdot(kl,m)} = 2R^{j}_{i\cdot h(k}T^{h}_{lm)} , \tag{3.10}$$

$$R^{j}_{i\cdot kl,m'} - R^{j}_{i\cdot[k|m'|,l]} = 2R^{j}_{i\cdot hm'}T^{h}_{kl} + R^{j}_{i\cdot h[k}C^{h}_{l]m'} + R^{j}_{i[k|h'|}C^{h'}_{|m'|l]} , \tag{3.11}$$

$$R^{j}_{i\cdot l'm',k} + R^{j}_{i\cdot k[l',m']} = -2R^{j}_{i\cdot kh'}T^{h'}_{l'm'} - R^{j}_{i\cdot h[l'}C^{h}_{|k|m']} - R^{j}_{i\cdot[l'|h'|}C^{h'}_{m']k} \tag{3.12}$$

$$R^{j}_{i\cdot(k'l',m')} = 2R^{j}_{i\cdot h'(k'}T^{h'}_{l'm')} . \tag{3.13}$$

These equations are the generalizations of the Bianchi's identities in Riemannian geometry.

## 4. The notions of the $g$-torsion and $g$-curvature

We shall restrict our consideration in this section within the case where the displacement $dy$ of an observing point $Q(y)$ in $N$ is $g$-related to the $dx$ of a point $P(x)$ in $M$. Then the connection-forms $\omega^j_i$ and $\omega^{j'}_{i'}$ are written in the forms

$$\theta^j_i = \Lambda^j_{ik} dx^k , \qquad \theta^{j'}_{i'} = \Lambda^{j'}_{i'k'} dy^{k'} , \tag{4.1}$$

where we put

$$\Lambda^j_{ik}(x, y) = \Gamma^j_{ik} + C^j_{ih'} g^{h'}_k , \quad \Lambda^{j'}_{i'k'}(y, x) = \Gamma^{j'}_{i'k'} + C^{j'}_{i'h} g^h_{k'} . \tag{4.2}$$

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* We use, for brevity, the sign $(ijk)$ which means

$$T^i_{h(j}T^h_{kl)} = T^i_{hj}T^h_{kl} + T^i_{hk}T^h_{lj} + T^i_{hl}T^h_{jk} .$$

By means of these coefficients $\Lambda^j_{ik}$ and $\Lambda^{j'}_{i'k'}$ we can introduce new operations (/) of covariant differentiations, the rule of which are written, for a tensor $A^{ik'}_{jl'}$, in the forms

$$(4.3)\qquad \begin{aligned} A^{ik'}_{jl'/h} &= \frac{\partial A^{ik'}_{jl'}}{\partial x^h} + A^{ak'}_{jl'}\Lambda^i_{ah} - A^{ik'}_{al'}\Lambda^a_{jh}, \\ A^{ik'}_{jl'/h'} &= \frac{\partial A^{ik'}_{jl'}}{\partial y^{h'}} + A^{ia'}_{jl'}\Lambda^{k'}_{a'h'} - A^{ik'}_{ja'}\Lambda^{a'}_{l'h'}. \end{aligned}$$

For a vector $V^i$ we put

$$V^i_{/j'} = \frac{\partial V^i}{\partial y^{j'}}.$$

However, we have to remark that the $g$-tensors are *not* covariant constant with respect to these differentiations (/) and we get from (1.6) and (1.7)

$$(4.4)\qquad \begin{aligned} g^i_{j'/k} &= g^i_{h'}C^{*h'}_{j'k}, \qquad & g^i_{j'/k'} &= -g^h_{j'}C^{*i}_{hk'}, \\ g^{i'}_{j\,/k} &= -g^{h'}_j C^{*i'}_{h'k}, \qquad & g^{i'}_{j\,/k'} &= g^{i'}_h C^{*h}_{jk'}, \end{aligned}$$

where putting

$$(4.5)\qquad C^{*i}_{jk'} = C^i_{jk'} + C^{a'}_{b'c}g^i_{a'}g^{b'}_j g^c_{k'}. \quad C^{*i'}_{j'k} = C^{*a}_{bc'}g^{i'}_a g^b_{j'}g^{c'}_k.$$

We shall use at present this operation (/) to write simply the following equations. The condition of integrability of the (/)-differentiation is given by

$$(4.6)\qquad A^{ik'}_{jl'/[h/m]} = A^{ak'}_{jl'}Q^i_{a\cdot hm} - A^{ik'}_{al'}Q^a_{j\cdot hm} - 2A^{ik'}_{jl'/a}S^a_{hm}.$$

$$(4.7)\qquad A^{ik'}_{jl'/[h'/m']} = A^{ia'}_{jl'}Q^{k'}_{a'\cdot h'm'} - A^{ik'}_{ja'}Q^{a'}_{l'\cdot h'm'} - 2A^{ik'}_{jl'/a'}S^{a'}_{h'm'},$$

$$(4.8)\qquad A^{ik'}_{jl'/[h/m']} = A^{ak'}_{jl'}\Lambda^i_{a\cdot hm'} - A^{ik'}_{al'}\Lambda^a_{j\cdot hm'} - A^{ia'}_{jl'}\Lambda^{k'}_{a'\cdot m'h} + A^{ik'}_{ja'}\Lambda^{a'}_{l'\cdot m'h},$$

where we put

$$(4.9)\qquad S^i_{jk} = \frac{1}{2}\Lambda^i_{[jk]}, \qquad \Lambda^i_{j\cdot kl'} = \frac{\partial \Lambda^i_{jk}}{\partial y^{l'}},$$

$$(4.10)\qquad \begin{aligned} Q^i_{j\cdot kl} &= \frac{\partial \Lambda^i_{j[k}}{\partial x^{l]}} + \Lambda^h_{j[k}\Lambda^i_{|h|l]} \\ &= R^i_{j\cdot kl} - C^i_{jh',[k}g^{h'}_{l]} + C^h_{ja'}C^i_{hb'}g^{a'}_{[k}g^{b'}_{l]} + 2C^i_{ja'}g^{a'}_b T^b_{kl}, \end{aligned}$$

and further the $S^{i'}_{j'k'}$, $Q^{i'}_{j'\cdot k'l'}$ and $\Lambda^{i'}_{j'\cdot k'l}$ are defined by the similar equations in terms of the $\Lambda^{i'}_{j'k'}$ instead of the $\Lambda^{i}_{jk}$.

Now we set

(4.11) $$\Theta^i = -\theta^i_j \wedge dx^j, \qquad \Theta^{i'} = -\theta^{i'}_{j'} \wedge dy^{j'}.$$

which are written in the forms

(4.12) $$\Theta^i = S^i_{jk} dx^j \wedge dx^k, \qquad \Theta^{i'} = S^{i'}_{j'k'} dy^{j'} \wedge dy^{k'}.$$

These coefficients $S^i_{jk}$ and $S^{i'}_{j'k'}$ are clearly the components of tensors, which are said the *g-torsion* of $M$ and $N$ respectively.

If we take generally a function $f(x, y)$, then we obtain

$$df = \frac{\partial f}{\partial x^i} dx^i + \frac{\partial f}{\partial y^{j'}} dy^{j'}.$$

Hence, if we use the sign $\Delta$ for $g$-related $dx^i$ and $dy^{j'}$ instead of the $d$, then we get

$$\Delta f = \left(\frac{\partial f}{\partial x^i} + \frac{\partial f}{\partial y^{j'}} g^{j'}_i\right) dx^i = \left(\frac{\partial f}{\partial y^{j'}} + \frac{\partial f}{\partial x^i} g^i_{j'}\right) dy^{j'}.$$

Therefore if we put

$$\frac{\Delta f}{\Delta x^i} = \frac{\partial f}{\partial x^i} + \frac{\partial f}{\partial y^{j'}} g^{j'}_i, \quad \frac{\Delta f}{\Delta y^{j'}} = \frac{\partial f}{\partial y^{j'}} + \frac{\partial f}{\partial x^i} g^i_{j'},$$

then we have

$$\Delta f = \frac{\Delta f}{\Delta x^i} dx^i = \frac{\Delta f}{\Delta y^{j'}} dy^{j'},$$

and

(4.13) $$\frac{\Delta f}{\Delta x^i} = \frac{\Delta f}{\Delta y^{j'}} g^{j'}_i.$$

Let $\omega$ be a $p$-form of $g$-related $dx^i$ and $dy^{j'}$, and then it follows by direct calculation that

(4.14) $$\Delta\omega = (-1)^p \frac{\Delta\omega}{\Delta x^i} \wedge dx^i = (-1)^p \frac{\Delta\omega}{\Delta y^{j'}} \wedge dy^{j'},$$

where the $\Delta/\Delta x^i$ and $\Delta/\Delta y^{j'}$ operating to a form denote the $\Delta$-differentiation of its coefficients alone.

Applying to (4.1) the $\Delta$-differentiation and making use of (4.14), we obtain

$$\Delta\theta^i_j = \theta^k_j \wedge \theta^i_k + \Theta^i_j\,, \qquad (4.15)$$
$$\Delta\theta^{i'}_{j'} = \theta^{k'}_{j'} \wedge \theta^{i'}_{k'} + \Theta^{i'}_{j'}\,,$$

where we put

$$\Theta^i_j = -\frac{1}{2} P^i_{j\cdot kl}\, dx^k \wedge dx^l\,, \qquad (4.16)$$
$$\Theta^{i'}_{j'} = -\frac{1}{2} P^{i'}_{j'\cdot k'l'}\, dy^{k'} \wedge dy^{l'},$$

and the coefficients are given by

$$P^i_{j\cdot kl} = \frac{\Delta\Lambda^i_{j[k}}{\Delta x^{l]}} + \Lambda^h_{j[k}\Lambda^i_{|h|l]} = Q^i_{j\cdot kl} + \Lambda^i_{j\cdot[k|h'|}g^{h'}_{l]}\,, \qquad (4.17)$$

and the $P^{i'}_{j'\cdot k'l'}$ are defined by the similar equations in terms of the $\Lambda^{i'}_{j'k'}$ instead of the $\Lambda^i_{jk}$. The coefficients $P^i_{j\cdot kl}$ define a tensor which is called the *g-curvature tensor* of *M*.

As a consequence of (1.6), (1.7), (1.6′) and (1.7′) we get

$$\frac{\Delta g^i_{j'}}{\Delta x^k} = -g^h_{j'}\Lambda^i_{hk} + g^i_{h'}\Lambda^{h'}_{j'k}\,,$$
$$\frac{\Delta g^i_{j'}}{\Delta y^{k'}} = -g^h_{j'}\Lambda^i_{hk'} + g^i_{h'}\Lambda^{h'}_{j'k'}\,,$$

and the similar equations for $g^{i'}_j$, where we set

$$\Lambda^{i'}_{j'k} = \Lambda^{i'}_{j'h'}g^{h'}_k\,, \qquad \Lambda^i_{jk'} = \Lambda^i_{jh}g^h_{k'}\,.$$

Therefore if we introduce the *covariant g-differentiation* ( ; ) such that, for a tensor $A^{ik'}_{jl'}$

$$A^{ik'}_{jl';h} = \frac{\Delta A^{ik'}_{jl'}}{\Delta x^h} + A^{ak'}_{jl'}\Lambda^i_{ah} - A^{ik'}_{al'}\Lambda^a_{jh} + A^{ia'}_{jl'}\Lambda^{k'}_{a'h} - A^{ik'}_{ja'}\Lambda^{a'}_{l'h}\,, \qquad (4.18)$$

and the similar equation for $A^{ik'}_{jl';h'}$, then these derivatives define clearly tensors and that we have

$$g^i_{j';k} = 0\,, \qquad g^i_{j';k'} = 0\,,$$
$$g^{i'}_{j;k} = 0\,, \qquad g^{i'}_{j;k'} = 0\,,$$

Moreover the covariant $g$-differentiation has a useful property:

$$A^{ik'}_{jl';h'} = A^{ik'}_{jl';m}g^m_{h'}\,, \qquad (4.19)$$

which is the direct consequence of (4.13). And we get also from (4.3)

$$A^{ik'}_{jl';h} = A^{ik'}_{jl'/h} + A^{ik'}_{jl'/m'} g^{m'}_{h}. \tag{4.20}$$

If we put

$$\begin{aligned} D^{i}_{jk} &= S^{i}_{jk} - S^{a'}_{b'c'} g^{i'}_{a'} g^{b'}_{j} g^{c'}_{k}, \\ D^{i'}_{j'k'} &= S^{i'}_{j'k'} - S^{a}_{bc} g^{i'}_{a} g^{b}_{j'} g^{c}_{k'}, \end{aligned} \tag{4.21}$$

then we have

$$D^{i'}_{j'k'} = -D^{a}_{bc} g^{i'}_{a} g^{b}_{j'} g^{c}_{k'},$$

and

$$\begin{aligned} \frac{\Delta g^{i}_{[j'}}{\Delta y^{k']}} &= -2D^{i}_{j'k'} \quad (= -2D^{i}_{ab} g^{a}_{j'} g^{b}_{k'}), \\ \frac{\Delta g^{i'}_{[j}}{\Delta x^{k]}} &= -2D^{i'}_{jk} \quad (= -2D^{i'}_{a'b'} g^{a'}_{j} g^{b'}_{k}). \end{aligned} \tag{4.22}$$

Making use of these equations we have the conditions of integrability of $\Delta$-differentiation as follows:

$$\begin{aligned} \frac{\Delta}{\Delta x^{[k}}\left(\frac{\Delta f}{\Delta x^{j]}}\right) &= -2\frac{\partial f}{\partial y^{h'}} D^{h'}_{jk}, \\ \frac{\Delta}{\Delta y^{[k'}}\left(\frac{\Delta f}{\Delta y^{j']}}\right) &= -2\frac{\partial f}{\partial x^{h}} D^{h}_{j'k'}. \end{aligned} \tag{4.23}$$

We get immediately, in virtue of (4.23), for a vector $V^i$

$$\begin{aligned} V^{i}_{;[j;k]} &= V^{a} P^{i}_{a\cdot jk} - 2V^{i}_{;a} S^{a}_{jk} - 2\frac{\partial V^{i}}{\partial y^{a'}} D^{a'}_{jk}, \\ V^{i}_{;[j';k']} &= V^{a} P^{*i}_{a\cdot j'k'} - 2V^{i}_{;a'} S^{a'}_{j'k'} - 2\frac{\partial V^{i}}{\partial x^{a}} D^{a}_{j'k'}, \end{aligned} \tag{4.24}$$

where the $P^{i}_{a\cdot jk}$ are components of the $g$-curvature, while the $P^{*i}_{a\cdot j'k'}$ are given by

$$P^{*i}_{a\cdot j'k'} = \frac{\Delta \Lambda^{i}_{a[j'}}{\Delta x^{k']}} + \Lambda^{h}_{a[j'} \Lambda^{i}_{|h|k']}.$$

The first of (4.24) shows that the $P^{i}_{a\cdot jk}$ is certainly a tensor, because the $\partial V^i/\partial y^{a'} = V^{i}_{/a'}$ is a tensor. We see, however, from the second of (4.24) that the $P^{*i}_{a\cdot j'k'}$ is not a tensor, because the $\partial V^i/\partial x^a$ is not a tensor. Making use of the covariant (/)-differentiation we may rewrite the second of (4.24) in the form

$$= V^a P^{*i}_{a\cdot j'k'} - 2V^i_{;a'} S^{a'}_{j'k'} - (V^i_{/a} - V^b \Lambda^i_{ba}) D^a_{j'k'}$$
$$= V^a (P^{*i}_{a\cdot j'k'} + 2\Lambda^i_{ab} D^b_{j'k'}) - 2V^i_{;a'} S^{a'}_{j'k'} - 2V^i_{/a} D^a_{j'k'},$$

which shows that the quantities

$$P^i_{j\cdot k'l'} = P^{*i}_{j\cdot k'l'} + 2\Lambda^i_{ja} D^a_{k'l'}$$

define a tensor. However we get

$$P^i_{j\cdot k'l'} = P^i_{j\cdot ab} g^a_{k'} g^b_{l'}, \tag{4.25}$$

which is the immediate result from the equations (4.19) and (4.24). Consequently we establish the condition of integrability of the covariant $g$-differentiation as follows:

$$\begin{aligned} A^{ik'}_{jl';[h;m]} = {} & A^{ak'}_{jl'} P^i_{a\cdot hm} - A^{ik'}_{al'} P^a_{j\cdot hm} + A^{ia'}_{jl'} P^{k'}_{a'\cdot hm} - A^{ik'}_{ja'} P^{a'}_{l'\cdot hm} \\ & - 2A^{ik'}_{jl';a} S^a_{hm} - 2A^{ik'}_{jl'/a'} D^{a'}_{hm}. \end{aligned} \tag{4.26}$$

If we want to get the expressions for the $A^{il'}_{jk';[h';m']}$ and $A^{ik'}_{jl';[h;m']}$ they are easily established by means of (4.19) and (4.26). Applying (4.26) to the $g$-tensors and making use of (4.4), we obtain

$$\begin{aligned} g^h_{j'} P^i_{h\cdot kl} - g^i_{h'} P^{h'}_{j'\cdot kl} &= -2g^h_{j'} C^{*i}_{hm'} D^{m'}_{kl}, \\ g^{i'}_h P^h_{j\cdot kl} - g^{h'}_j P^{i'}_{h'\cdot kl} &= -2g^{i'}_h C^{*h}_{jm'} D^{m'}_{kl}. \end{aligned} \tag{4.27}$$

Remembering the definition of the $D^{i'}_{jk}$, $C^{*i}_{jk'}$ and (4.25), the above equations give that, *if the g-torsion of M is g-related to the one of N, or if the rotation parts of the connection of M is different from the tensor g-related to the one of N in a point of algebraic sign alone, then the g-curvature of M is g-related to the one of N,*

Finally we shall find the identities satisfied by the covariant $g$-derivatives of the $g$-torsion and $g$-curvature. Operating $\Delta$ to (4.11) and making use of (4.15), we get

$$\Delta\Theta^i = -\theta^k_j \wedge \theta^i_k \wedge dx^j - \Theta^i_j \wedge dx^j. \tag{4.28}$$

Comparing the coefficients of the both sides we have

$$P^i_{(j\cdot kl)} = 2S^i_{(jk;l)} - 4S^i_{a(j} S^a_{kl)}. \tag{4.29}$$

This is of the same form as (3.7). Next, if we operate $\Delta$ to (4.14) and substitute from (4.23), then we have the general formula

$$\Delta^2\omega = \frac{\partial\omega}{\partial y^{j'}} \wedge D^{j'}. \tag{4.30}$$

where we put $D^{j'}=D^{j'}_{k\ l}dx^k\wedge dx^l$. Applying this formula to the form (4.15), we get

$$\Delta\Theta^i_j=\theta^k_j\wedge\Theta^i_k-\Theta^k_j\wedge\theta^i_k+\frac{\partial\theta^i_j}{\partial y^{k'}}\wedge D^{k'}. \tag{4.31}$$

Comparison of the coefficients of the both sides gives us

$$P^{\ i}_{j\cdot(kl\,;\,m)}=2P^{\ i}_{j\cdot h(k}S^{\ h}_{l\,m)}-2\Lambda^{\ i}_{j\cdot(k|h'|}D^{h'}_{l\,m)}. \tag{4.32}$$

This is of a little different form from (3.10). However, if the $g$-torsions are $g$-related, then they has the same form.

## 5. The notion of paths

We shall define a parallelism of vectors. Let $V^i(x, y)$ be a vector field. It is natural from (1.2) that the $V^i$ is said to enjoy the *parallel displacement* if the equations

$$dV^i+V^j(\Gamma^i_{jk}(x, y)dx^k+C^i_{jk'}(x, y)dy^{k'})=0 \tag{5.1}$$

are satisfied. If the displacement $dy$ of the observing point $Q(y)$ is $g$-related to the $dx$ of the origin $P(x)$, then the above equation is reducible to

$$dV^i+V^j\Lambda^i_{jk}(x, y)dx^k=0. \tag{5.2}$$

We shall use this in order to define a path. That is, we consider a curve $C$: $x^i=x^i(t)$ in $M$ and, if the equations

$$\frac{d^2x^i}{dt^2}+\Lambda^i_{jk}(x, y)\frac{dx^j}{dt}\frac{dx^k}{dt}=0 \tag{5.3}$$

are satisfied under the suitable choice of a parameter $t$, then the $C$ is denoted the *path* in $M$ *with respect to the observing point* $Q(y)$. This definition is probably natural and useful, because the displacement $dy$ of the observing point $Q(y)$ is not contained in (5.3).

On the other hand, we have a remarkable class of curves, such that every tangent vector $dx^i/dt$ of a curve $C$: $x^i=x^i(t)$ in $M$ is $g$-related to a constant vector $V^{j'}$ at an observing point $Q(y)$ in $N$. We differentiate the equations

$$\frac{dx^i}{dt}=g^i_{j'}(x, y)V^{j'}$$

with respect to $t$ and make use of (1. 6), and then it follows that

$$\frac{d^2x^i}{dt^2} + (\Gamma^i_{jk}(x, y) - g^{a'}_j C^{b'}_{a'k}(y, x) g^i_{b'}) \frac{dx^j}{dt}\frac{dx^k}{dt} = 0\,. \tag{5.4}$$

The curve $C$ satisfying the above equations is called the *central path.*

Now we see that the equation (5. 4) does not generally coincide with the equations (5. 3) of a path. However, I think to be natural from the stand-point of our geometry that the condition of coincidence of (5. 3) and (5. 4) is imposed. The condition is written by

$$\Lambda^i_{(jk)} = \Gamma^i_{(jk)} - g^{a'}_{(j} C^{b'}_{|a'|k)} g^i_{b'}\,,$$

that is

$$g^{a'}_{(j} C^{*i}_{k)a'} = 0 \tag{5.5}$$

Thus we arrive at a connection such that the $C^{*i}_{jk'}$ defined by (4. 5) vanish, that is

$$C^i_{jk'} = -C^{a'}_{b'c} g^i_{a'} g^{b'}_j g^c_{k'}\,. \tag{5.6}$$

In the following we show that such a connection may be defined. If the translation-parts $\Gamma^i_{jk}$ of the $\omega^i_j$ are arbitrarily given, then the rotation-parts $C^{i'}_{j'k}$ of the $\omega^{i'}_{j'}$ are uniquely determined such that the equations (1. 6) hold. While we have yet an arbitrariness in order to take the $C^i_{jk'}$ and $\Gamma^{i'}_{j'k'}$ satisfying (1. 7). Hence if the $C^i_{jk'}$ are determined by (5. 6) in terms of the $C^{i'}_{j'k}$, then the $\Gamma^{i'}_{j'k'}$ are uniquely taken to satisfy (1. 7). The pair ($\omega^j_i$, $\omega^{j'}_{i'}$) as thus determined is called to be *$(-g)$-connection.*

In this case we see some interesting circumstances. First of all, since (5. 5) holds, a central path is always a path. Next, we see from (4. 4) that the $g$-tensors are covariant constant with respect to the covariant (/)-differentiations. Hence, applying to the $g$-tensor the formulae (4. 6), (4. 7) and (4. 8), we have

$$Q^i_{j\cdot kl} = Q^{i'}_{j'\cdot k'l'} = 0\,, \tag{5.7}$$

$$g^h_{j'} \Lambda^i_{h\cdot kl'} + g^i_{h'} \Lambda^{h'}_{j'\cdot l'k} = 0\,. \tag{5.8}$$

As a consequence of (4. 10) we have the expression of the translation-curvature as follows:

(5.9) $$R^{i}_{j\cdot kl}=C^{i}_{jh',[k}g^{h'}_{l]}-C^{h}_{ja'}C^{i}_{hb'}g^{a'}_{[k}g^{b'}_{l]}-2C^{i}_{ja'}g^{a'}_{b}T^{b}_{kl},$$

and from (4.17) we obtain for the $g$-curvature

(5.10) $$P^{i}_{j\cdot kl}=\Lambda^{i}_{j\cdot[k|h'|}g^{h'}_{l]}$$

and from (4.27)

(5.11) $$P^{i'}_{j'\cdot kl}=P^{a}_{b\cdot kl}g^{i'}_{a}g^{b}_{j'}.$$

It follows from (4.25) and (5.11) that the $g$-curvature tensor of $M$ is $g$-related to the one of $N$.

Finally, we give a simple example of a $(-g)$-connection. We take a vector field $\rho_i$ and put

$$C^{i}_{jk'}=\rho_j g^{i}_{k'},$$

and then the $\Gamma^{i'}_{j'k'}$ are uniquely determined such that the equations (1.7) are satisfied. We define the $C^{i'}_{j'k}$ by

$$C^{i'}_{j'k}=-\rho_{j'}g^{i'}_{k},$$

where the vector $\rho_{j'}$ is $g$-related to the $\rho_i$. It follows easily that the equation (5.6) holds good. And further, if we determine the $\Gamma^{i}_{jk}$ by (1.6), then we get the special type of a $(-g)$-connection. In this case by means of (5.9), we obtain the simple expression of the translation-curvature

$$R^{i}_{j\cdot kl}=\rho_{j,[k}\delta^{i}_{l]}-\rho_j\rho_{[k}\delta^{i}_{l]}-2\rho_j T^{i}_{kl},$$

and for the rotation-curvature we have from its definition (2.8′)

$$R^{i}_{j\cdot k'l'}=-\rho_{j,[k'}g^{i}_{l']}-\rho_j\rho_{[k'}\rho^{i}_{l']}+2\rho_j T^{h'}_{k'l'}g^{i}_{h'}.$$

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