J. Math. Kyoto Univ.
1-1 (1961) 109-127.

Some remarks on the Cauchy problem

By

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(Received July 10, 1961)

1. Introduction

Consider the first order partial differential equation

$$M[u] = \frac{\partial}{\partial t}u - \sum_{k=1}^{l} A_k(x, t)\frac{\partial}{\partial x_k}u - B(x, t)u = 0\,, \tag{1.1}$$

where u is a vector-valued function with N components; A_k and B are matrices of order N, infinitely differentiable with respect to t and $x=(x_1, x_2, \cdots, x_l)$. Consider the Cauchy problem for this equation with given initial value at $t=0$. We say that the Cauchy problem for (1.1) is *well posed* for the future in the space $\mathcal{E}$, if i) for any given initial value $u(x, 0)\in \mathcal{E}_x$, there exists a unique solution $u(x, t)\in \mathcal{E}_{x,t}$, $t\geqslant 0$, which takes the given initial value at $t=0$, ii) this linear mapping $u(x, 0)\to u(x, t)$ is continuous from $\mathcal{E}_x$ into $\mathcal{E}_{x,t}$. Here $\mathcal{E}$ is the space of all infinitely differentiable functions with customary topology, (see L. Schwartz [8])[1]: $f_j\to 0$ in $\mathcal{E}$,

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1) In this paper, especially in section 1, we followed the notations of Schwartz's treatise [8]. We used the following linear topological spaces: $\mathcal{E}$, $\mathcal{E}^m(\Omega)$, $\mathcal{B}$, $\mathcal{D}^\infty_{L^2}$, $\mathcal{E}^m_{L^2}(\Omega)$, $\mathcal{E}^m_{L^2(\mathrm{loc})}(\Omega)$, Ω being an open set in R^n, and $\mathcal{E}_-(\mathcal{D}^\infty_{L^2})$. We shall explain briefly. $\mathcal{E}^m(\Omega)$ is the space of all functions $f(x)$, m-times continuously differentiable with the topology: $f_j\to 0$, if $D^\nu f_j(x)$, $|\nu|\leqslant m$, converge to 0 uniformly on any compact in Ω. $\mathcal{E}^m_{L^2}(\Omega)$ is the Hilbert space of all functions $f(x)$ belonging to $L^2(\Omega)$ with their derivatives (in distribution sense) up to order m, with the inner product: $(f, g)=\sum_{|\nu|\leqslant m}(D^\nu f, D^\nu g)_{L^2}$. $\mathcal{E}^m_{L^2(\mathrm{loc})}(\Omega)$ is the space of all functions $f(x)$ such that $D^\nu f(x)\in L^2_{\mathrm{loc}}(\Omega)$, for $|\nu|\leqslant m$, with the topology: $f_j\to 0$, if for any compact K of Ω, $D^\nu f_j(x)\to 0$ in $L^2(K)$, for $|\nu|\leqslant m$. Finally, $f(x, t)\in \mathcal{E}_-(\mathcal{D}^\infty_{L^2})$, if the mapping $t\to f(x, t)\in \mathcal{D}^\infty_{L^2}$ is infinitely differentiable for $t\leqslant 0$.

if for any compact K and any integer ν, $D^\nu f_j$ converge uniformly to 0 on K.

In the study of hyperbolic systems, it is always assumed that (H) $\sum A_k(x, t)\xi_k = A\cdot\xi$ has only *real* eigenvalues for any $(x, t; \xi)$, where ξ is a non-zero real vector.

Our purpose is to prove the following

Theorem 1.1. *Assume that, for some* ξ^0 $(\neq 0)$ *real, the matrix* $\sum A_k(0, 0)\xi_k^0$ *has a non real eigenvalue, then* (1.1) *is not well posed in any small neighborhood of the origin.*

From this it follows the following interesting

Corollary. *Assume the coefficients* $A_k(x, t)$ *and* $B(x, t)$ *to be analytic functions of* x *and* t. *If we make the same assumption as that of Theorem* 1.1, *then there exists a function* $\psi_0(x) \in \mathcal{E}$, *defined in a neighborhood of* $x=0$, *such that there exists no solution* $u(x, t) \in \mathcal{E}$ *of* (1.1) *satisfying* $u(x, 0) = \psi_0(x)$, *in any small neighborhood of the origin.*

Remark 1. This corollary is also true even if we restrict there the domain of existence of $u(x, t)$ to a half-space, that is, for instance for $t \geqslant 0$.

Remark 2. Theorem 1.1 was proved at first by Petrowsky [6] in the case where the coefficients depend only on t. In the case of variable coefficients, P. D. Lax proved this under the condition that this non real eigenvalue is simple [3]. Our proof is a direct extension of that of Petrowsky and quite different from that of Lax[2]. The above corollary was proved by Lax in [3], however we present here our proof.

We shall prove Theorem 1.1 in section 3. Here we prove the corollary, assuming Theorem 1.1 to be true. This proof was borrowed fairly from Hörmander's paper [2], p. 135.

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2) We should mention that our proof is based on the theory of the singular integral operators (in the sense of Calderón and Zygmund). More precisely, we used the localisation of singular integral operators both in x and ξ space. We showed its utility in our previous paper: Le problème de Cauchy pour les systèmes hyperboliques et paraboliques. Mem. Coll. Sci. Univ. Kyoto, 32, 181–212 (1959).

We assume now:

i) The coefficients are analytic functions;

ii) For each $\psi(x)\in\mathcal{E}(\Omega)$, Ω being an arbitrary open set in x-space containing the origin ($x=0$), once fixed for all, there exists a solution $u(x, t)\in\mathcal{E}$ in a neighborhood of the origin, which may depend on ψ.

From Holmgren's Theorem, it follows this: Define a familly of open sets

$$(1.2)\qquad D_\rho = \{(x, t);|t|+|x|^2<\rho\}, \quad 0<\rho<\rho_0 \ \text{(small)},$$

then any solution $u(x, t)\in\mathcal{E}^1$ of $M[u]=0$ with a given initial value is unique in D_ρ. Now we prove

Lemma 1.1. *Under the assumptions* i) *and* ii), *there exists a* $\varepsilon>0$ *such that, for any function* $\psi(x)\in\mathcal{E}(\Omega)$, *there exists a unique solution in* $\mathcal{E}^1(D_\varepsilon)$, *satisfying* $u(x, 0)=\psi(x)$.

Proof. Take a sequence $\varepsilon_0=\rho_0>\varepsilon_1>\varepsilon_2>\cdots>\varepsilon_n\cdots\to 0$;Denote by $A_{n,m}$ the set of all functions $\psi(x)$ of $\mathcal{E}(\Omega)$ such that 1) the solution $u(x, t)$ having $\psi(x)$ as initial value has at least D_{ε_n} as its existence domain: more precisely, $M[u]=0$, $u(x, t)\in\mathcal{E}_{L^2}^{[\frac{l}{2}]+2}(D_{\varepsilon_n})$; $u(x, 0)=\psi(x)$ for $x\in D_{\varepsilon_n}\cap(t=0)$,

2) $\|u(x, t)\|_{[\frac{l}{2}]+2}\leq m$, where $\|\ \|_k$ means the customary norm of the functions in $\mathcal{E}_{L^2}^k(D_{\varepsilon_n})$: $\|f\|_k^2=\sum_{|\nu|\leq k}\|D^\nu f\|_{L^2}^2$.

The set $A_{n,m}$ is symmetric convex and closed. The first two properites are evident. We need only to show that $A_{n,m}$ is closed. Suppose that $\psi_n\in A_{n,m}\to\psi_0\in\mathcal{E}(\Omega)$. Corresponding to ψ_n, we have the sequence $u_n(x, t)$. Since u_n is bounded in $\mathcal{E}_{L^2}^{[\frac{l}{2}]+2}(D_{\varepsilon_n})$, we can choose a convergent subsequence u_{n_p} in $\mathcal{E}_{L^2(loc)}^{[\frac{l}{2}]+1}$ (namely, they converge on any compact of D_{ε_n} in $\mathcal{E}_{L^2}^{[\frac{l}{2}]+1}$-topology). Furthermore, if necessary, by taking again its subsequence, we can assume this subsequence to be weakly convergent (as a Hilbert space) in $\mathcal{E}_{L^2}^{[\frac{l}{2}]+1}(D_{\varepsilon_n})$. We denote this limit by $u_0\in\mathcal{E}_{L^2}^{[\frac{l}{2}]+2}(D_{\varepsilon_n})$. Then, $u_{n_p}\to u_0$

weakly implies that u_0 satisfies 2), and by Sobolev, $u_0\in\mathcal{E}^1(D_{\varepsilon n})$, and $M[u_0]=0$. $u_{np}\to u_0$ in $\mathcal{E}_{L^2(loc)}^{[\frac{l}{2}]+1}(D_{\varepsilon n})$ implies that, since u_{np} are uniformly convergent on every compact in $D_{\varepsilon n}\cap(t=0)$, we have $u_0(x, 0)=\psi_0(x)$, which proves that $A_{n,m}$ is closed. Since $\bigcup_{n,m} A_{n,m}=\mathcal{E}(\Omega)$, one of $A_{n,m}$, say A_{n_0,m_0}, is of second category, therefore it contains an open set of $\mathcal{E}(\Omega)$. Since A_{n_0,m_0} is symmetric and convex, it contains a neighborhood of 0 in $\mathcal{E}(\Omega)$. This implies that, for any $\psi(x)\in\mathcal{E}(\Omega)$, there exists a unique solution $u(x, t)\in\mathcal{E}_{L^2}^{[\frac{l}{2}]+2}(D_{\varepsilon n_0})$, a fortiori $\in\mathcal{E}^1(D_{\varepsilon n_0})$ satisfying $u(x, 0)=\psi(x)$. The proof is thus complete.

Lemma 1.2. *The mapping $\psi(x)\in\mathcal{E}(\Omega)\to u(x, t)\in\mathcal{E}^1(D_\varepsilon)$, D_ε being defined in the previous lemma, is continuous.*

Proof. This is the closed graph theorem of Banach.

Proof of the corollary. Assume that. for every $\psi(x)\in\mathcal{E}(\Omega)$, there exists a solution $u(x, t)\in\mathcal{E}$ in a neighborhood of the origin. Then, by Lemma 1.1 and 1,2, there exists D_ε $(\varepsilon>0)$ such that $\psi(x)\in\mathcal{E}(\Omega)\to u(x, t)\in\mathcal{E}^1(D_\varepsilon)$ is continuous. However Theorem 1.1 says that, under the assumtion stated there. this mapping cannot be continuous (as we shall see in the proof of this theorem, we need not to assume $u(x, t)\in\mathcal{E}$, it is enough to assume $u(x, t)\in\mathcal{E}^1$). This contradiction proves the corollary.

We remarked (Remark 1) that the corollary is also true even if we assume that the existence domain of u in $D'_\rho=\{(x, t)\,;\, t\geqq 0, t+|x|^2<\rho\}$, where ρ may depend on the initial value $\psi(x)$. We restrict ourselves to point out how to modify the proof of Lemma 1.1. At first we extend all functions $u(x, t)$ defined in $D'_{\varepsilon n}$ to $\tilde{D}_{\varepsilon n}$ as follows (see, Schwartz, [7]): For $t<0$, define $u_1(x, t)=\sum_{\nu=1}^{p} a_\nu u(x, -\nu t)$, where $p=\left[\frac{l}{2}\right]+2$, a_ν being determined uniquely by the condition that this function has the same traces on the hyperplane $t=0$ as those of $u(x, t)$ up to order $p-1=\left[\frac{l}{2}\right]+1$; namely, $\sum_{\nu=1}^{p}(-\nu)^k a_\nu=1$, $k=0, 1, \cdots, p-1$. Now $\tilde{u}(x, t)=u(x, t)+u_1(x, t)$ is defined in $\tilde{D}_{\varepsilon n}$, and we have $D^\nu\tilde{u}(x, t)=D^\nu u(x, t)+D^\nu u_1(x, t)$

for $|\nu|\leqslant p=\left[\frac{l}{2}\right]+2$, the derivatives being taken in distribution sense, for both traces on $t=0$ of u and u_1 up to order $p-1$ coincide. Moreover $\|\tilde{u}\|_p$ is bounded. Then we choose a subsequence of u_n as follows: $u_{n_p}\rightarrow u_0$ weakly in $\mathcal{E}^{p}_{L^2}(D'_{\varepsilon_n}-(t=0))$. Next consider the subsequence u_{n_p} corresponding to $\{n_p\}$, and choose again a subsequence in such a way that this new subsequence converge in $\mathcal{E}^{p-1}_{L^2(loc)}(\tilde{D}_{\varepsilon_n})$.

Remark 3. It is desired that the corollary is proved in the following form: there exists a function $\psi_0(x)\in\mathcal{E}$, such that there exists no solution $u(x, t)\in\mathcal{E}^1$. We don't know whether this is true of not. However, as the proof shows, we can say that there exists no solution $u(x, t)\in\mathcal{E}^{\left[\frac{l}{2}\right]+2}$, l being the dimension of the space.

Theorem 1.1. is of course true for higher order single equations, not only for kowalewskians but also for p evolution equations. We say here that the linear single equation

$$(1.3)\qquad M[u]=\frac{\partial^m}{\partial t^m}u+a_1\left(x, t;\frac{\partial}{\partial x}\right)\frac{\partial^{m-1}}{\partial t^{m-1}}u+a_2\left(x, t;\frac{\partial}{\partial x}\right)\frac{\partial^{m-2}}{\partial t^{m-2}}u+\cdots+a_m\left(x, t;\frac{\partial}{\partial x}\right)u=0$$

is p-evolution (p: positive integer), if

(p) $a_i(x, t;\xi)$ is a polynomial in ξ of degree $\leqslant ip$. We donote this highest homogeneous part (principal part as p-evolution) of degree ip by $h_i(x, t:\xi)$.

There are infinitely many possible choices of p for a given equation. Namely, if (1.3) is p_0-evolution, then it is of course p-volution for any $p>p_0$. However, in this case, the corresponding principal part becomes $h_i\equiv 0$ $(i=1, 2, \cdots, m)$ which is a trivial choice of p. Taking into account of this remark, we have

Theorem 1.2. *Consider the characteristic equation (as p evolution):*

$$P(\lambda)=\lambda^m+h_1(x, t;i\xi)\lambda^{m-1}+h_2(x, t;i\xi)\lambda^{m-2}+\cdots+h_m(x, t;i\xi)=0\,.$$

We denote the roots by $\lambda_i(x, t;\xi)$ (ξ real $\neq 0$).

If for some x_0, ξ_0, one of roots, say λ_1 has a positive real part:

(1. 4) $\qquad$ Real part $\lambda_1(x_0, 0; \xi_0) > 0$,

then the forward Cauchy problem with initial time $t=0$ *is not well posed in* $\mathcal{D}_{L^2}^{\infty}$ *in any small neighborhood of* $t=0$.

The theorem is true even if we replace there $\mathcal{D}_{L^2}^{\infty}$ by $\mathcal{B}$ or $\mathcal{E}$[1]. However, we should remark the following fact: In the case of the equations with constant coefficients, if (1. 3) is not kowalewskian, then the Cauchy problem is not well posed in $\mathcal{E}$, see Gårding [1].

We can consider the corollary corresponding to that of Theorem 1. 1. We can say the following fact: For the parabolic equation of the form

$$M[u] = \frac{\partial}{\partial t}u - \sum_{i,j=1}^{l} a_{ij}(x, t)\frac{\partial^2}{\partial x_i \partial x_j}u - \sum a_i(x, t)\frac{\partial}{\partial x_i}u - c(x, t)u = 0; \tag{1. 5}$$

where $\sum a_{ij}(x, t)\xi_i\xi_j \geqslant c|\xi|^2$, c is a positive constant; the coefficients are in $\mathcal{B}$, consider the backward Cauchy problem in $\mathcal{D}_{L^2}^{\infty}$. Then there exists a $\psi(x) \in \mathcal{D}_{L^2}^{\infty}$ such that there exists no solution $u(x, t) \in \mathcal{E}_{-}(\mathcal{D}_{L^2}^{\infty})$[1] with the initial value $\psi(x)$ (at $t=0$) in any small neighborhood of $t=0$. In fact, we know that the backward Cauchy problem for this equation is unique in this space, see [4].

We shall give a brief proof of Theorem 1. 2 in the last section.

Now we give a rough sketch of the proof of Theorem 1. 1. At first, we remark that, if necessary by replacing ξ^0 by $-\xi^0$, we can assume

(1. 5) $\qquad$ Imaginary part $\lambda_1(0, 0; \xi^0) < 0$, $\quad |\xi^0| = 1$.

Then, consider the convolution operator $M_0: M_0 = \frac{\partial}{\partial t} - iA_0\xi^0\Lambda$, where $A_0\xi^0 = \sum A_k(0, 0)\xi_k^0$. We shall prove in section 2 that, so far as we restrict the sequence of the initial values in a convenient way, then this operator gives a good approximation to M. More explicitely, we define $u_n(x, 0) = \exp(in\xi^0 x)\psi(x)$, where $\psi(x)$ is an analytic function whose Fourier transform has its support located only in a small neighborhood of ξ^0. Now we assume the continuity in $\mathscr{E}$, then

(3.7) $$\max_{x\in\Omega} |u_n(x, t)| \leqslant 0(n^h) \qquad \text{for} \quad 0 \leqslant t \leqslant T',$$

Ω being a compact chosen closely to the origin. Finally using the inequality obtained in section 2, we show that (3.7) is a contradiction.

2. Operator M_0 approximating to M.

Take a function $\beta(x) \in C^\infty$ and $\alpha(\xi) \in C^\infty$ of small supports, which take the value 1 in a neighborhood of $x=0$ and ξ^0 respectively. We shall define these sizes later. Define

(2.1) $$\alpha_n(\xi) = \alpha(\xi/n),$$

and, denoting the Fourier inverse image of this function by $\alpha_n(x)$, define

(2.2) $$\alpha_n f = \alpha_n(x) * f(x).$$

Our aim is to show that the convolution operator

(2.3) $$M_0 = \frac{\partial}{\partial t} - iA_0 \xi^0 \Lambda$$

is, in a certain sense, a first approximation of M.

At first, apply $\beta(x)$ to (1.1),

(2.4) $$M[\beta(x)u] = \sum A_k \beta_k(x) u,$$

where $\beta_k(x) = \frac{\partial}{\partial x_k}\beta(x)$.

Now, since M acts on $\beta(x)u$, it does not change if we modify the coefficients A_k and B outside the support of $\beta(x)$. Therefore we can write (2.4) in the form

(2.5) $$\left(\frac{\partial}{\partial t} - \sum \tilde{A}_k \frac{\partial}{\partial x_k} - \tilde{B}\right)[\beta u] = \sum \tilde{A}_k \beta_k(x) u,$$

where the oscillations of $\tilde{A}_k$ and $\tilde{B}$ become small, when the size of Supp (β) become small.

Next, we define $\exp(-\delta'\Lambda t)$, $(\delta'>0)$, by

(2.6) $$(\exp(-\delta'\Lambda t)u)^{\wedge} = \exp(-\delta'|\xi|t)\hat{u}(\xi).$$

This is the elementary solution of the evolution equation, for $t\geqslant 0$,

$$\left(\frac{\partial}{\partial t}+\delta'\Lambda\right)u = 0.$$

Denote

(2.7) $$\gamma_n(t) = \exp(-\delta'\Lambda t)\alpha_n.$$

Apply this convolution operator to (2.5), then

(2.8) $$\left(\frac{\partial}{\partial t} - \sum \tilde{A}_k\frac{\partial}{\partial x_k} - \delta'\Lambda - \tilde{B}\right)[\gamma_n\beta(x)u]$$
$$= \gamma_n \sum \tilde{A}_k\beta_k u - \sum (\gamma_n\tilde{A}_k - \tilde{A}_k\gamma_n)\frac{\partial}{\partial x_k}[\beta(x)u]$$
$$-(\gamma_n\tilde{B}-\tilde{B}\gamma_n)(\beta(x)u).$$

Now we consider the left-hand side. We denote this operator by $\tilde{M}-\delta'\Lambda I$. Consider the operator $\sum \tilde{A}_k\frac{\partial}{\partial x_k}=\mathcal{H}\Lambda$. Since $\mathcal{H}\Lambda$ acts on the functions of the form $\alpha_n v$, we can replace there $\mathcal{H}$ by $\tilde{\mathcal{H}}$. $\tilde{\mathcal{H}}$ is defined as follows: We call the support of $\alpha(\xi)$ on the unit sphere the projection from the origin. Then we change the symbol $i\sum A_k(x, t)\xi_k/|\xi|$ outside the support of $\alpha(\xi)$ on the sphere. Thus we obtain a singular integral operator $\tilde{\mathcal{H}}$. Here we can assume that the oscillation of $\sigma(\mathcal{H})$ becomes smaller and smaller if we choose the sizes of the supports of α and β smaller and smaller.

Now we want to examine this situation more precisely. Consider the eigenvalues $\lambda_1^0, \cdots, \lambda_N^0$ of $iA_0\xi^0=i\sum A_k(0, 0)\xi_k^0$. If eventually some of them are pure imaginary, we translate them by $-\varepsilon$ ($\varepsilon>0$, small), then we can assume that

(2.9) $$\begin{aligned}&\text{Real part } \lambda_1^0-\varepsilon, \cdots, \lambda_{N_1}^0-\varepsilon\geqslant 6\delta, \quad \delta>0, \quad N_1\geqslant 1,\\ &\text{Real part } \lambda_{N_1+1}^0-\varepsilon, \cdots, \lambda_N^0-\varepsilon\leqslant -6\delta.\end{aligned}$$

Then, by a known theorem (see for instance, [6] p. 24, Lemma 5), we can find a non-singular matrix N_0 such that

$$N_0 \cdot iA_0\xi^0 \cdot N_0^{-1} = \begin{pmatrix} \lambda_1^0 & & & \\ & \lambda_1^0 & 0 & \\ & & \ddots & \\ & a_{ij}^0 & & \ddots \\ & & & & \lambda_N^0 \end{pmatrix} = \mathfrak{D}_0 ,$$

where $|a_{ij}^0| < \frac{\delta}{4N}$, δ being defined in (2. 9).

Denote

$$\sigma(\mathcal{H}) = iA_0\xi^0 + \sigma(\mathcal{H}_\varepsilon) . \quad \text{Then} \tag{2. 11}$$

$$N_0\mathcal{H} = (\mathfrak{D}_0 + N_0\mathcal{H}_\varepsilon N_0^{-1})N_0 = (\mathfrak{D}_0 + \mathfrak{D}_\varepsilon)N_0 . \tag{2. 12}$$

Now we define the sizes of the supports of β and α and T: If we take these sizes and T small, we shall have, denoting the (i, j)-element of $\mathfrak{D}_\varepsilon$ by $d_{ij}^{(\varepsilon)}$,

$$|\sigma(d_{ij}^{(\varepsilon)})(x, t ; \xi)| \leqslant \frac{\delta}{4N}, \qquad \text{for} \quad 0 \leqslant t \leqslant T . \tag{2. 13}$$

Now we want to derive an inequality. At first we state a lemma on singular integral operators.

Lemma 2. 1. Let H be a singular integral operator (in the sense of Calderón and Zygmund).

i) If $\sigma = \inf_{x, \xi}$. Real part $\sigma(H)(x ; \xi) > 0$, then

$$\text{Real part } (H\Lambda u, u) \geqslant \frac{\sigma}{2}(\Lambda u, u) - C_1 \|u\|^2 ,$$

where C_1 is a constant.

ii) Let $M = \sup_{x, \xi} |\sigma(H)(x ; \xi)|$, then

$$|(H\Lambda u, u)| \leqslant 2M(\Lambda u, u) + C_2 \|u\|^2 ,$$

C_2 being a constant.

We don't give the proof. The reader will easily verify by consulting our previous paper [5].

Now we assume in (2. 8), $\delta' = \varepsilon + \frac{\delta}{2^p}$, $p = 0$, positive integer, then, it can be written

$$(2.14)\qquad \left\{\frac{\partial}{\partial t}-(\mathfrak{D}_0+\mathfrak{D}_\varepsilon-\delta' I)\Lambda-B_0\right\}[v^{(n)}] = N_0 f,$$

where f is the right-hand side of (2.8); $B_0(x, t)=N_0B(x, t)N_0^{-1}$, and

$$(2.15)\qquad v^{(n)} = \gamma_n\beta(x)N_0u\,.$$

Hereafter we omit the suffix (n) of $v^{(n)}$.
Consider now

$$(2.16)\quad S(t) = \sum_{i=1}^{N_1}||v_i(t)||^2-\sum_{j=N_1+1}^{N}||v_j(t)||^2,$$ where $v_i(t)=v_i(x, t)$ is the i-th component of v.

$$\begin{aligned}S'(t) &= \sum_{i=1}^{N_1}2Re(v_i, (v_i)_t)-\sum_{j=N_1+1}^{N}2Re(v_j, (v_j)_t)\\
&= \sum_i 2Re(v_i, (\lambda_i^0-\delta'+d_{ij}^{(\varepsilon)})\Lambda v_i)-\sum_j 2Re(v_j, (\lambda_j^0-\delta'+d_{ij}^{(\varepsilon)})\Lambda v_j)\\
&+\sum_i\sum_{k\neq i} 2Re(v_i, (a_{ik}^0+d_{ik}^{(\varepsilon)})\Lambda v_k)-\sum_j\sum_{k\neq j} 2Re(v_j, (a_{jk}^0+d_{jk}^{(\varepsilon)})\Lambda v_k)\\
&+\sum_i 2Re(v_i, f_i')-\sum_j 2Re(v_j, f_j')-\sum_i 2Re(v_i, \sum_k b_{ik}v_k)-\sum_j 2Re(v_j, \sum_k b_{jk}v_k),\end{aligned}$$

where f_i' is the i-th component of N_0f, and $B_0=(b_{ij})$.
By the previous lemma, the first row of the right-hand side is greater than

$$2\delta(\sum_i(\Lambda v_i, v_i)+\sum_j(\Lambda v_j, v_j))-C||v||^2\ ^{3)}.$$

In the same way, the second row is greater than

$$-N\frac{2\delta}{2N}(\sum_i(\Lambda v_i, v_i)+\sum_j(\Lambda v_j, v_j))-C||v||^2.$$

The last row is greater than

$-C||v||^2-||v||\,||N_0f||$. Hence

$$\begin{aligned}S'(t) &\geqslant \delta\{\sum_{i=1}^{N}(\Lambda v_i, v_i)\}-C||v||^2-2||v||\,||N_0f||\\
&\geqslant \delta\{\text{distance }(0, \text{Supp}(\hat{v}))-C\}\,||v||^2-2||v||\,||N_0f||.\end{aligned}$$

3) Throughout this paper we may use the symbol C in order to re- present positive constants. Sometimes it expresses a positive constant which can be chosen independently of n.

Assuming distance $(0, \mathrm{Supp}\,(\hat{v}))>4C$, we have for $v^{(n)}$,

(2.17) $$\frac{d}{dt}S(t\,; v^{(n)}) \geqslant \frac{\delta}{2}n\|v^{(n)}\|^2 - C\|v^{(n)}\|\,\|f\|\,.$$

In fact, distance $(0, \mathrm{Supp}\,(\hat{v}^{(n)})) \geqslant (3/4)n$,

3. Proof of Theorem 1.1.

We prove this theorem by contradiction. We assume therefore (1.1) to be well posed in $\mathcal{E}$.

At first we define a series of solutions $u_n(x, t)$ of (1.1). Namely we define their initial values. Let $\hat{\psi}(\xi)$ be a function whose support is located in a small neighborhood of ξ^0; On the support of $\hat{\psi}(\xi)$, $\alpha(\xi)=1$; we assume $\int|\hat{\psi}(\xi)|^2d\xi=1$. We define then

(3.1) $$\hat{\psi}_n(\xi) = \hat{\psi}(\xi - n\xi^0)\,. \quad \text{Namely}$$

(3.2) $$\psi_n(x) = \exp\,(in\xi^0x)\psi(x)\,.$$

Now we define $u_n(x, t)$ by

(3.2) $$M[u_n] = 0\,, \quad u_n(x, 0) = N_0^{-1}\begin{pmatrix}\psi_n(x)\\ 0\\ \vdots\\ 0\end{pmatrix}, \quad n = 1, 2, \cdots$$

To consider $\alpha_n u_n(x, t)$ means that we take only a high frequency part of $u_n(x, t)$ (t being parameter). Now we want to show that

(3.3) $\|\alpha_n\beta(x)N_0u_n(x, 0)\| = c + 0\left(\frac{1}{n}\right)$, c is a positive constant.

$$\alpha_n\beta(x)u_n(x, 0) = \alpha_n\beta(x)\psi_n(x) = \beta(x)\alpha_n\psi_n(x) + (\alpha_n\beta(x) - \beta(x)\alpha_n)\psi_n(x)\,.$$

Here $\alpha_n\psi_n(x)=\psi_n(x)$, for $\alpha_n(\xi)=1$ on the support of $\hat{\psi}_n(\xi)$. Hence the first term of the right-hand side is $\beta(x)\psi_n(x)= \exp\,(in\xi^0x)\times \beta(x)\psi(x)$. Since $\psi(x)$ is analytic we have

(3.4) $$\|\beta(x)\alpha_n\psi_n(x)\| = (\int|\beta(x)\psi(x)|^2dx)^{\frac{1}{2}} = c>0\,.$$

Now we look at the last term. We know that

$$(\alpha_n\beta(x)-\beta(x)\alpha_n)u = \int \alpha_n(x-y)\{\beta(y)-\beta(x)\}\, u(y)dy\,.$$

By Taylor expansion : $\beta(y)-\beta(x)=\sum_{1\leqslant|\nu|\leqslant p}\frac{(y-x)^\nu}{\nu!}\beta^{(\nu)}(x)+\sum_{|\nu|=p+1}\frac{(y-x)^\nu}{\nu!}\times$ $\beta_\nu(x, y)$, we see that the last integral is equal to

$$\sum_{1\leqslant|\nu|\leqslant p}\frac{(-1)^{|\nu|}}{\nu!}\beta^{(\nu)}(x)(x^\nu\alpha_n)*u+\sum_{|\nu|=p+1}\frac{(-1)^{|\nu|}}{\nu!}\times \int\beta_\nu(x, y)(x-y)^\nu\alpha_n(x-y)u(y)dy\,. \tag{3.6}$$

When we put $u=\psi_n(x)$, the first terms corresponding to $\sum_{1\leqslant|\nu|\leqslant p}$ are all zero, since $(x^\nu\alpha_n)*\psi_n(x)\underset{\mathcal{F}}{\rightarrow}$ const. $\alpha_n^{(\nu)}(\xi)\hat{\psi}_n(\xi)=0$.

Next, the last terms are majorized by $c\|u\|\int|(x^\nu\alpha_n)(x)|dx$. Now $|(x^\nu\alpha_n)(x)|$ is majorized by const. $\int|\alpha_n^{(\nu)}(\xi)|d\xi=$const. $\left(\frac{1}{n}\right)^{|\nu|}\int_{\text{Supp}(\alpha_n)}\left|\alpha^{(\nu)}\left(\frac{\xi}{n}\right)\right|d\xi\leqslant$const. $\left(\frac{1}{n}\right)^{|\nu|-l}$, l being the dimension of the space. On the other hand, for $|x|\geqslant 2$

$$(x^\nu\alpha_n)(x) = \frac{1}{|x|^{2k}}(x^\nu|x|^{2k}\alpha_n)(x)\leqslant\frac{\text{const.}}{|x|^{2k}}\times \int|\Delta^k\alpha_n^{(\nu)}(\xi)|d\xi\leqslant\frac{\text{const.}}{|x|^{2k}}\left(\frac{1}{n}\right)^{|\nu|+2k-l}.$$

This shows that, if we take $|x|\geqslant 1$, then

$$\int|(x^\nu\alpha_n(x))(x)|dx = 0\left(\frac{1}{n^{|\nu|-l}}\right). \tag{3.6}$$

Thus we proved (3.3).

Now we put $u=u_n(x, t)$ in (2.8) and $\delta'=\varepsilon+\delta$. By hypothesis, there exist a positive integer h and a neighborhood (in x, t space) Ω of $(x, t)=0$ and small T' such that

$$\max_{(x,t)\in\Omega}|u_n(x, t)|\leqslant 0(n^h) \qquad \text{for} \quad 0\leqslant t\leqslant T'\,. \tag{3.7}$$

By taking the support of $\beta(x)$ small, we can assume that the support of $\beta(x)$ is contained in Ω, therefore we have

$$\|\beta(x)u_n(x, t)\|\leqslant 0(n^h)\,, \qquad \text{for} \quad 0\leqslant t\leqslant T'\,. \tag{3.8}$$

Now we consider the right-hand side of (2. 8), we want to show that it is expressed, modulo bounded functions (in L^2-sense) with respect to n, as linear combination of

$$(3.9)\quad \sqrt{n}\ \exp\left\{-\left(\varepsilon+\frac{\delta}{2}\right)\Lambda t\right\}\sqrt{n}^{\,|\gamma|}\alpha_n^{(\gamma)}\times\begin{cases}\beta(x)u_n\\ \left(\dfrac{1}{\sqrt{n}}\right)\beta_k(x)u_n\,,\end{cases}$$

whose coefficients being bounded operators in L^2 (bounded with respect to n too). Namely

$$(3.10)\quad \sqrt{n}\sum C_{\gamma,\kappa}^{(n)}(t)\exp\left\{-\left(\varepsilon+\frac{\delta}{2}\right)\Lambda t\right\}\times$$
$$\sqrt{n}^{\,|\gamma|}\alpha_n^{(\gamma)}\left(\frac{1}{\sqrt{n}}\right)^{(\kappa)}\beta^{(\kappa)}(x)u_n(x,t)+0(1)\,,$$

where $\|C_{\gamma,\kappa}^{(n)}(t)\|_{\mathcal{L}(L^2;L^2)}\leqslant$const. independent of n; $1\leqslant|\gamma|+|\kappa|\leqslant m=2(h+l)$, $|\kappa|\leqslant 1$.

To see this, take the most delicate part of (2. 8), $(\gamma_n\tilde{A}_k-\tilde{A}_k\gamma_n)\times\dfrac{\partial}{\partial x_k}[\beta(x)u_n]$. Denoting $\tilde{A}_k$ merely by A, and using, as previously, the Taylor expansion, this can be expressed as

$$(3.11)\quad \sum_{|\nu|=1}^{m}\frac{(-1)^{|\nu|}D^\nu A}{\nu!}(x^\nu\gamma_n)\frac{\partial}{\partial x_k}[\beta(x)u_n]$$
$$+\sum_{|\nu|=m+1}\int A_\nu(x,y)(x^\nu\gamma_n)(x-y)\frac{\partial}{\partial y_k}[\beta(y)u_n(y,t)]dy\,,$$

where

$$(3.12)\qquad m=2(h+l)\,,$$

where we chose m so as to make the last terms of (3. 11) bounded (with respect to n). Take at first a term of the first part: $(x^\nu\gamma_n)\dfrac{\partial}{\partial x_k}[\beta(x)u_n]$. The convolution operator $(x^\nu\gamma_n)\dfrac{\partial}{\partial x_k}$ has its Fourier image const. $D_\xi^\nu\gamma_n(\xi)\xi_k=$const. $D_\xi^\nu[\exp\{-(\varepsilon+\delta)|\xi|t\}\alpha_n(\xi)]\xi_k$. By Leibniz,

$$(3.13)\quad D_\xi^\nu[\exp\{-(\varepsilon+\delta)|\xi|t\}\alpha_n(\xi)]$$
$$=\sum_{\nu_1+\nu_2=\nu}C_{\nu_1}^{\nu}D_\xi^{\nu_1}[\exp\{-(\varepsilon+\delta)|\xi|t\}]D^{\nu_2}\alpha_n(\xi)\,.$$

In general, we have

$$D_\xi^\nu \exp\{-\delta'|\xi|t\} = \psi_\nu(\xi, t)\exp\{-\delta'|\xi|t\}, \quad (\delta' = \varepsilon+\delta),$$

where $|\psi_\nu(\xi, t)| \leqslant C_\nu\{\sum_{k=1}^{|\nu|}(t|\xi|)^k\}/|\xi|^{|\nu|}$, for $|\xi| \geqslant 1$.

Since $(\sum_{k=1}^{|\nu|}(t|\xi|)^k)\exp\left(-\frac{\delta}{2}|\xi|t\right)$ is bounded, we have finally

$$\text{(3. 14)} \quad D_\xi^\nu \exp\{-(\varepsilon+\delta)|\xi|t\} = \varphi_\nu(\xi, t)\exp\left\{-\left(\varepsilon+\frac{\delta}{2}\right)|\xi|t\right\},$$

where $|\varphi_\nu(\xi, t)| \leqslant \frac{c_\nu}{|\xi|^{|\nu|}}$, for $|\xi| \geqslant 1$.

Hence, there is no question as regards to the terms $|\nu_1| \geqslant 1$ in the right-hand side of (3. 13). As regards to the terms $\nu_1 = 0$, (hence $|\nu_2| \geqslant 1$), we have $\alpha_n^{(\nu_2)}(\xi)\xi_k = n\left(\frac{\xi_k}{n}\right)\alpha_n^{(\nu_2)}(\xi)$. Since ξ_k/n is uniformly bounded (with respect to n) on the support of $\alpha_n(\xi)$, we see that (3. 10) is true for any of the first terms in (3. 11). Therefore we need only to show that the last terms of (3. 11) is 0(1). These terms are majorized by the form (by Young's Theorem)

$$\text{(3. 16)} \quad c\left(\int\left|\frac{\partial}{\partial x_k}(x^\nu\gamma_n)(x)\right|dx\right)||\beta(x)u_n||.$$

Now we want to estimate the above integral. Take Fourier transform of the integrand: $c'\xi_k\gamma_n^{(\nu)}(\xi)$. This is majorized by $c'|\xi|\,|\gamma_n^{(\nu)}(\xi)|$. Taking into account of $|\alpha_n^{(\nu)}(\xi)| \leqslant c\left(\frac{1}{n}\right)^{|\nu|}$, and of (3. 14), we have

$$\text{(3. 17)} \quad |\xi|\,|\gamma_n^{(\nu)}(\xi)| \leqslant c\left(\frac{1}{n}\right)^{|\nu|-1}.$$

This implies

$$\text{(3. 18)} \quad \left|\frac{\partial}{\partial x_k}(x^\nu\gamma_n)(x)\right| \leqslant \text{const.}\int_{\mathrm{Supp}(\alpha_n)}|\xi|\,|\gamma_n^{(\nu)}(\xi)|d\xi \leqslant \text{const.}\left(\frac{1}{n}\right)^{|\nu|-1-l}$$
$$= c\left(\frac{1}{n}\right)^{m-l} = c\left(\frac{1}{n}\right)^h\left(\frac{1}{\sqrt{n}}\right)^m.$$

On ther other hand, for $|x| \geqslant 2$, putting

$\frac{\partial}{\partial x_k}(x^\nu\gamma_n)(x)=\frac{1}{|x|^{2k'}}|x|^{2k'}\frac{\partial}{\partial x_k}(x^\nu\gamma_n)(x)$, we see that this is majorized by

$$c\int|\Delta^{k'}[\xi_k\gamma_n^{(\nu)}(\xi)]|d\xi\leqslant c\left(\frac{1}{n}\right)^{2k'+|\nu|-1}\int_{\mathrm{Supp}(\alpha_n)}d\xi\leqslant \text{const.}\left(\frac{1}{n}\right)^{2k'+|\nu|-1-2l},$$

choosing $k'=l$, we have

$$\left|\frac{\partial}{\partial x_k}(x^\nu\gamma_n)(x)\right|\leqslant c\frac{1}{|x|^{2l}}\left(\frac{1}{n}\right)^m, \qquad \text{for } |x|\geqslant 2, \tag{3.19}$$

where c does not depend on n.

Finally we have

$$(3.16)\leqslant\left(\frac{1}{\sqrt{n}}\right)^m 0\left(\frac{1}{n^h}\right)\|\beta(x)u_n(x,t)\|=0(1)\left(\frac{1}{\sqrt{n}}\right)^m. \tag{3.20}$$

We can see that the same reasoning on the rest terms of (2.8) will give (3.10).

In view of (3.10), we consider next all the functions appearing there:

$$\exp\left\{-\left(\varepsilon+\frac{\delta}{2}\right)\Lambda t\right\}\sqrt{n}^{-|\gamma|}\alpha_n^{(\gamma)}\sqrt{n}^{-|\kappa|}\beta^{(\kappa)}(x)u_n(x,t),$$

where γ and κ satisfy the condition mentioned there. Namely, we make the same process for these functions as for $\gamma_n\beta(x)u$: we replace in the above reasoning $\beta(x)\to\begin{cases}\beta(x)\\ \beta_k(x)\end{cases}$, $\exp\{-(\varepsilon+\delta)\Lambda t\}\alpha_n\to\exp\left\{-\left(\varepsilon+\frac{\delta}{2}\right)\Lambda t\right\}\alpha_n^{(\gamma)}$. In total, we are led to consider all functions of the form:

$$\theta^{(n)}(p,\gamma,\kappa)u_n=\sqrt{n}^{-|\gamma|-|\kappa|}\exp\left\{-\left(\varepsilon+\frac{\delta}{2^p}\right)\Lambda t\right\}\alpha_n^{(\gamma)}\beta^{(\kappa)}(x)u_n, \tag{3.21}$$

where $p\leqslant|\gamma|+|\kappa|\leqslant m$, $p=0,1,\cdots,m$.

We shall then have the equations analogous to (2.8) for all these functions. Namely, we have the equations of the form (2.14), where

i) $v^{(n)}=\theta^{(n)}(p,\gamma,\kappa)N_0u_n$,

ii) the right-hand side N_0f is expressed, modulo bounded functions, as a linear combination of the functions (3.21), whose coefficients are $\sqrt{n}\times$(uniformly bounded operators in L^2).

Now we return to section 2. For each $\theta^{(n)}(p, \gamma, \kappa)N_0u_n(x, t)$, consider $S(t)$, denoted by $S(t; \theta^{(n)}(p, \gamma, \kappa)N_0u_n)$. Finally define

$$(3.32)\qquad S_n(t) = \sum S(t; \theta^{(n)}(p, \gamma, \kappa)N_0u_n(x, t));$$
$$\sigma_n(t) = \sum \|\theta^{(n)}(p, \gamma, \kappa)N_0u_n(x, t)\|^2,$$

where the summation is extended over all functions in (3.21). Taking into account of the fact that the support of $(\theta^{(n)}(p, \gamma, \kappa) N_0u_n)^\wedge(\xi, t)$ is contained in that of $\alpha_n(\xi)$, if we apply the inequality (2.17), we shall have

$$(3.33)\qquad S_n'(t)\geqslant\frac{1}{2}\delta n\sigma_n(t)-C\sqrt{n}\,\sigma_n(t)-0(1),$$

where C is a constant. Since for n large, $\frac{1}{2}\delta n-C\sqrt{n}\geqslant\frac{\delta}{4}n$,

$$(3.34)\qquad \left(\exp\left(-\frac{\delta}{4}nt\right)S_n(t)\right)_t'\geqslant-0(1)\exp\left(-\frac{\delta}{4}nt\right).$$

Now we know that, since at $t=0$ all functions $\theta^{(n)}(p, r, \kappa)N_0u_n$ have their components 0 except the first components, we have $S_n(0)\geqslant S(0; \gamma_n\beta(x)N_0u_n)=\|\alpha_n\beta(x)\psi_n(x)\|^2$, and this is equal to $c^2+0\left(\frac{1}{n}\right)$, by (3.3). Hence

$$(3.35)\qquad S_n(0)\geqslant c^2/2 \qquad \text{for } n \text{ large.}$$

By integration of (3.34), we have

$$(3.36)\quad S_n(t)\geqslant\frac{c^2}{2}\exp\left(\frac{\delta}{4}nt\right)-0\left(\frac{1}{n}\right)\qquad \text{for } n \text{ large}\quad 0\leqslant t\leqslant T'.$$

On the other hand, taking into account of (3.21) and (3.7), we see that $\|\theta^{(n)}(p, \gamma, \kappa)N_0u_n\|^2\leqslant$const. $n^{|\gamma|-|\kappa|}\|\beta^{(\kappa)}(x)u_n(x,t)\|^2$ $=0(n^{m+2h})=0(n^{4h+2l})$.
Since $S_n(t)\leqslant\sigma_n(t)$, it follows

$$(3.37)\qquad S_n(t)\leqslant 0(n^{4h+2l})\qquad \text{for}\quad 0\leqslant t\leqslant T.$$

(3.36) and (3.37) cannot be compatible unless $t=0$. The proof of Theorem 1.1 is thus complete.

4. Proof of Theorem 1.2.

In principle, this proof is the same as that of Theorem 1.1. Therefore, we want to explain how to reduce this proof to the previous one. We treat here the kowalewskians ($p=1$).

At first, denoting $u_1=u$, $u_2=\frac{\partial}{\partial t}u, \cdots, u_m=\frac{\partial^{m-1}}{\partial t^{m-1}}u$, we consider an equivalent system to (1.3):

$$(4.1) \qquad L[u] = \frac{\partial}{\partial t}u-(A+J)[u] = 0\,, \quad \text{where}$$

$$A = \begin{pmatrix} & & \\ & 0 & \\ & & \\ -a_m & -a_{m-1}\cdots & -a_1 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & \ddots \\ & & & \ddots\ 1 \\ & & & 0 \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ \vdots \\ u_m \end{pmatrix}.$$

Apply $\beta(x)$ to $L[u]$, then

$$(4.2) \qquad L[\beta(x)u]+\sum_{1\leqslant|\nu|\leqslant m} C_\nu\Big(x, t\,; \frac{\partial}{\partial x}\Big)[\beta^{(\nu)}(x)u] = 0\,,$$

$$\text{where, denoting } C_\nu = \begin{pmatrix} & & \\ & 0 & \\ & & \\ c_{1,\nu} & c_{2,\nu}\cdots & c_{m,\nu} \end{pmatrix},$$

(4.3) $c_{i,\nu}(x, t\,; \xi)$ is of order (in ξ)$\leqslant m+1-i-|\nu|$.

Now we apply $\gamma_n(t)$ to (4.2) just in the same way as in the section 2, we then have

$$(4.4) \quad L[\gamma_n\beta(x)u]+(\gamma_nL-L\gamma_n)[\beta(x)u]+\gamma_n\sum C_\nu[\beta^{(\nu)}(x)u] =0\,.$$

Taking into account of (4.1), we have here

$$(4.5) \qquad \gamma_nL-L\gamma_n = \gamma_nA-A\gamma_n\,.$$

Now we insert a process which was not done in the case of the first order system. Denote

$$(3.6) \qquad E_m(\Lambda) = \begin{pmatrix} (\Lambda+1)^{m-1} & & & \\ & (\Lambda+1)^{m-2} & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

Instead of u itself, we consider $E_m(\Lambda)u$. Then it follows from (4.2),

$$(4.7)\quad (E_m L E_m^{-1})[E_m \gamma_n \beta(x) u] + (\gamma_n A - A\gamma_n) E_m^{-1}[E_m \beta(x) u] + \gamma_n \sum_{1\leqslant|\nu|\leqslant m} C_\nu\Big(x, t; \frac{\partial}{\partial x}\Big) E_m^{-1}[E_m \beta(x) u] = 0 .$$

Now consider the first term. We can express it by singular integral operators:

$$(4.8)\qquad \Big(\frac{\partial}{\partial t} - \mathcal{H}\Lambda - \mathcal{B}\Big)[\gamma_n E_m \beta(x) u],$$

where

i) $\mathcal{B}$ is a bounded operator in L^2;

$$(4.9)\quad \text{ii)}\ \sigma(\mathcal{H}) = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & \ddots & \\ & & & \ddots & \ddots \\ & & & 0 & 1 \\ -h_m & -h_{m-1} & & & -h_1 \end{pmatrix}; \quad h_i = h_i(x, t; i\xi/|\xi|) .$$

Next, we see that $(\gamma_n A - A\gamma_n)E_m^{-1}$ *has just the same property as* $\sum (\gamma_n A_k - A_k \gamma_n)\dfrac{\partial}{\partial x_k}$ *in* (2.8). Finally, denoting $C_\nu E_m^{-1} = C'_\nu$, we see that the term $c'_{i,\nu}$ corresponding to $c_{i,\nu}$ has the order of differentiation $\leqslant (m+1-i-|\nu|)-(m-i)=1-|\nu|$. This shows that $\gamma_n \sum C'_\nu[\beta^{(\nu)}(x) u]$ *has the same property as* $\gamma_n \sum A_k \beta_k u$ *in* (2.8).

Finally, we define $u_n(x, 0)$ by

$$(4.10)\qquad N_0 E_m(\Lambda) u_n(x, 0) = \begin{pmatrix} \psi_n(x) \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

where N_0 is defined by $\sigma(\mathcal{H})$ in the same way as in (2.10). (4.10) can be written

$$(4.10)'\qquad u_n(x, 0) = E_m(\Lambda)^{-1} N_0^{-1} \begin{pmatrix} \psi_n(x) \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Obviously, $u_n(x, 0) \in \mathcal{D}^\infty_{L^2} \cap \mathcal{B}$. Finally we remark that

$$(4.11)\quad \|\alpha_n N_0 E_m \beta(x) u_n(x, 0)\|_{L^2} \geqslant \|\beta(x) \alpha_n N_0 E_m u_n(x, 0)\| - \|[\beta(x)(\alpha_n N_0 E_m) - (\alpha_n N_0 E_m)\beta(x)] u_n(x,0)\| = \|\beta(x)\psi_n(x)\| - 0\left(\frac{1}{n}\right), \text{ by the same reason as in (3.6)}.$$

Thus we can verify easily that, replacing $\alpha_n^{(\gamma)}\beta^{(\kappa)}(x) N_0 u_n$ in the previous section by $(N_0 E_m(\Lambda))\alpha_n^{(\gamma)}\beta^{(\kappa)}(x) u_n$ the analogous reasoning can be carried out.

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