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On the automorphism group of a generic curve of genus >2

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It is our purpose here to prove the assertion made at the end of [2] to the effect that if $n>2$, then there exists a Riemann surface S of genus n having no non-trivial automorphisms (i.e., the only one-to-one conformal mapping of S onto itself is the identity). We believe this result is classically "well-known", but we should like to present a simple and complete proof based upon our recent results [1, 2] and on results of [3] and [4], making use of the fact that the dimension of the variety of moduli of Riemann surfaces of genus n is equal to $3n-3$. Certain of the results which we obtain from facts established in [2] might be proved in an elementary way, not depending on the theory of Chow forms and so on, but such elementary proofs, having little interest of themselves, do not seem worth presenting here.

In what follows, "open variety" will mean a Zariski open subset of a projective variety, and if A is any subset of projective space, A^* will denote its closure in the Zariski topology. Let Γ_n denote the group of $2n\times 2n$ unimodular, integral, symplectic matrices acting on the generalized upper half-plane H_n of degree n, and let K_m denote the subgroup of Γ_n leaving invariant the quotients of m^{th} order θ-zero values, as defined in [1] (here $8p|m$ for some odd prime $p>3$). Then K_m acts without fixed points in H_n and $H_n/K_m=V^{(m)}$ is (realizable as) an open variety. Moreover, there exists an open variety P_m and a regular mapping λ_m of P_m^*

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onto $V^{(m)*}$ with these properties: (1) $P_m=\lambda_m^{-1}(V^{(m)})$; (2) if $v\in V^{(m)}$, then $A_v=\lambda_m^{-1}(v)$ is a normally polarized Abelian variety, every isomorphism class of normally polarized Abelian varieties is represented by A_v for some $v\in V^{(m)}$, and v, $v'\in V^{(m)}$ correspond to the same isomorphism class if and only if they belong to the same orbit of $G_m=\Gamma_n/K_m$. Let E_m denote the set of $v\in V^{(m)}$ such that A_v is the canonically polarized Jacobian variety of some Riemann surface of genus n, and denote by H_m and M_m respectively the subsets of E_m corresponding to the hyperelliptic and non-hyperelliptic Riemann surfaces. We know that E_m is an open variety and that H_m is an analytic subset of E_m. If M_m is not empty, let x be a simple point of M_m, and if $E_m=H_m$, let x be a simple point of H_m. E_m is stable under G_m. Denote by G_x the subgroup of G_m leaving x fixed and let U be a suitable small neighborhood of x on E_m stable under G_x; we may assume that $G_y\subset G_x$ for $y\in U$. Let $E^{(2m)}$ be the $2n$-dimensional identity matrix and denote by e and $-e$ respectively the cosets of $E^{(2n)}$ and $-E^{(2n)}$ modulo K_m. We have e, $-e\in G_y$ for all $y\in U$. Put $A=\lambda_m^{-1}(U)$ and let λ be the restriction of λ_m to A. There is an isomorphism $g\rightarrow\tilde{g}$ of G_x onto a group of automorphisms of A such that $\lambda\tilde{g}=g\lambda$ for all $g\in G_x$, and if $y\in U$, then the group of $\tilde{g}$ for $g\in G_y$ induces the group of all automorphisms of the normally polarized Abelian variety A_y. Moreover, there is an analytic subset C of A such that for $y\in U$, $C_y=\lambda^{-1}(y)\cap C$ is a non-singular algebraic curve of degree d (independent of y) and of genus n, A_y is the Jacobian variety of C_y, and C_y is canonically imbedded in A_y (for example, we may for each y choose, in a suitable manner, C_y to be one of the finite number of canonically imbedded curves C_y^i in A_y such that for a fixed θ-divisor X on A, the Abelian sum of the points of $X\cdot C_y^i$ is zero).

We now prove that C is a complex submanifold of A. Let $y\in U$, $b\in C_y$, and let H be a hyperplane cutting C_y properly in b. It may be easily proved that the Chow point of C_y is a continuous analytic function of y, because U is a complex manifold. Then for y' near y, H cuts $C_{y'}$ properly in a point b' near b in such a way that b' is a continuous function of y'. Let R be an affine space containing b such that b is the origin of R; we may assume H is given by $L=0$, L being a linear form in the coordinates of R. For any complex number ζ with $|\zeta|$ sufficiently small, denote by H_ζ the hyperplane $L=\zeta$. Let b'_ζ be the point near b in which H_ζ cuts $C_{y'}$ for y' near y. Consider the mapping

$$f:\ (y',\ \zeta)\rightarrow b'_\zeta\,.$$

It follows from the factorization of the Chow form of $C_{y'}$ into linear factors that f is single valued and continuous in a neighborhood of $(y, 0)$. It follows from the implicit function theorem that in the same neighborhood, b'_ζ is an analytic function of ζ for fixed y', and from the analyticity of the mapping λ that b'_ζ is an analytic function of y' for fixed ζ (this is because of a well-known theorem on the removable singularities of analytic functions). Conversely, it is easily seen that f^{-1} is analytic. Hence a neighborhood of b on C is isomorphic with a neighborhood of $(y, 0)$ in the product of U with the complex plane, and therefore C is a manifold at b, which complete our proof. Henceforth, we denote $\lambda|C$ by λ when no confusion can arise.

The kernel of the representation of G_x as an automorphism group of U contains e and $-e$; if $y\in U$, $a\in A_y$, then $-ea=-a$. If D is a subset of A, let $D^*=\{-d|d\in D\}$. Suppose first of all that C_x is not hyperelliptic. In this case we may assume C_y is not hyperelliptic for $y\in U$. Then if $g\in G_y$, $y\in U$, we have $\tilde{g}C_y=(C_y)_{-a}$ or $\tilde{g}C_y=(C^*_y)_{-a}$ for a unique $a\in A_y$. Let $U(g)=\{y\in U|gy=y\}$, let $T=T(g)=\{a|a\in A_y,\ y\in U(g),\ \tilde{g}C_y=(C_y)_{-a}\}$, and put $T'=T'(g)=\{a|a\in A_y,\ y\in U(g),\ \tilde{g}C_y=(C^*_y)_{-a}\}$. Clearly T and T' are analytic subsets of A, $\lambda(T)\cup\lambda(T')=U(g)$, and $\lambda(T)\cap\lambda(T')$ is empty, so that if we assume $U(g)$ to be connected (as a point set), we see that either T or T' must be empty. Put $C(g)=C\cap\lambda^{-1}(U(g))$. If T is not empty, define $\varphi^{(y)}_g:\ C_y\rightarrow C_y$ by $\varphi^{(y)}_g(c)=\tilde{g}c+a_y$, $a_y=\lambda^{-1}(y)\cap T$, if $y\in U(g)$, and define $\varphi_g:\ C(g)\rightarrow C(g)$ by letting $\varphi_g|C_y=\varphi^{(y)}_g$. It is seen that T is a submanifold of A (because of the one-to-one analytic map λ of T onto the complex manifold U) and that $C(g)$ is a submanifold of C, since $U(g)$ is a submanifold of U. Therefore φ_g is an analytic, fibre-preserving automorphism of $C(g)$. We define φ_g to be φ_{-eg} if T' is non-empty. Finally, if C_x is hyperelliptic, then C_y is hyper-elliptic for all $y \in U$, $U(-e)=U$, and we may attach to $-e$ in a manner similar to the above an automorphism φ_{-e} of C. Hence, for all y sufficiently near to x, the automorphism group of C_y may be viewed as a subgroup of the automorphism group of C_x.

To obtain our desired result, it will therefore be sufficient to show that if $g \in G_x$, then the set of $y \in U(g)$ such that φ_g induces a non-trivial automorphism of C_y is a proper analytic subset of U. So suppose that for some $g \in G_x$ this is not so. Then φ_g is a fibre-preserving automorphism of $C(g)$, and it is a trivial matter to show that the fixed points of φ_g form a finite number b of submanifolds of $C(g)$, each being a cross-section of the fibering $(C(g), \lambda, U)$. Denote these manifolds by $B_1, \cdots, B_b$. Let H be the finite group of automorphisms of $C(g)$ generated by $\tilde{g}$ and put $C'=C(g)/H$. Clearly we have a fibre space (C', λ', U) induced from $(C(g), \lambda, U)$, and each fibre is a Riemann surface of the same genus n'. It is clear, in fact, that $\tilde{g}$ has the same order $m=\text{ord}(H)$ at each of its fixed points, and by the usual formula expressing the genus in terms of the numbers of edges, faces, and vertices of a triangulation of a Riemann surface, it is easily seen that

$$n-1 = m(n'-1)+\frac{1}{2}(m-1)b\,. \tag{1}$$

By constructing a C^∞ connection in the fibering (C', λ', U) one may easily see that if M is the C^∞-manifold underlying each of the fibers C'_y for $y \in U$, then, for appropriate choice of U, (C', λ', U) is a C^∞ product fibering, $C'=U\times M$. We choose a canonical basis $\alpha_1, \cdots, \alpha_{2n'}$ of integral cycles on M. By considering the holomorphic 1-forms on C invariant under H, it is easily seen that we may find holomorphic differential 1-forms $\omega_1, \cdots, \omega_{n'}$ on C' such that the restrictions $\omega_i(y)$ of these to each fibre C'_y for $y \in U$ is a basis of holomorphic 1-forms on C'_y, and it follows from [3, pp. 164–165] that the integrals

$$\int_{\alpha_j} \omega_i(y) = p_{ij}(y)$$

are holomorphic functions on U. Since $\alpha_1, \cdots, \alpha_{2n'}$ is a canonical basis of 1-cycles, $\det(p_{ij})_{i,j=1,\cdots,n'} \neq 0$, and it is therefore clear that we may assume by appropriate normalization of $\omega_1, \cdots, \omega_{n'}$ that

$$(p_{ij}(y)) = (E'\ Z'(y))$$

for $y \in U$, where E' is the $n' \times n'$ identity matrix and $Z'(y) \in H_{n'}$. Thus we have a holomorphic mapping $\psi: U \rightarrow H_{n'}$ defined by $\psi(y) = Z'(y)$. Since $\psi(U)$ is contained in the Zariski-closure of the subset of $H_{n'}$ corresponding to the normalized period matrices of Jacobian varieties of Riemann surfaces of genus n', we have for $a \in \psi(U)$,

$$\dim \psi^{-1}(a) \geq \dim U - d(n') = 3n - 3 - d(n'), \tag{2}$$

where $d(0)=0$, $d(1)=1$, and $d(n')=3n'-3$ if $n'>1$. We shall now prove:

(I) $\dim \psi^{-1}(a) \leq b$, and if $n'=0$, $m=2$, then $\psi^{-1}(a) \leq b-3$.

(II) if $n>2$, $b+d(n')<3n-3$, but if $n'=0$ and $m=2$, we have only $b<3n$.

Clearly (I) and (II) imply a contradiction of (2) if $n>2$.

To prove (I), it is sufficient to prove the inequality at every regular point of $\psi^{-1}(a)$. Let y_0 be a regular point of $\psi^{-1}(a)$ and let U'' be a small neighborhood of y_0 on $\psi^{-1}(a)$. Let (C'', λ'', U'') denote the restriction of the fibering (C', λ', U) to U''. From our previous considerations, we see that (C'', λ'', U'') is a regular fibering. Since, by Torelli's theorem, all the fibers are complex analytically isomorphic, we may assume (C'', λ'', U'') is complex analytically a product fibering $C'' = C_0' \times U''$; if $n'=0$, this follows from a result of [4][1]. We now prove this for $n'>0$. Let J_0 be

1) In line with the belief one should always give elementary proofs for elementary facts, we observe that $C''=C_0'\times U''$ may be proved by taking three nonintersecting cross-sections σ_0, σ_1, and σ_∞ of the regular fibering (C'', λ'', U'') and considering the map of C'' onto C_0' which carries each fiber conformally onto C_0' and these cross-sections onto the points 0, 1, and ∞, respectively.

the Jacobian variety of C_0', let C_0' be canonically imbedded in J_0, and let $C^* = C_0' \times U'' \subset J_0 \times U'' = J^*$. Denote by λ^* the projection of J^* onto U''. Let $\omega_1, \cdots, \omega_{n'}$ have the same meaning as before. Since (C'', λ'', U'') is a regular fibering, there exists, locally at least, an analytic cross-section of this fibering. Therefore, we may use $\int \omega_1, \cdots, \int \omega_{n'}$ to define a mapping

$$\omega : C'' \to J^*$$

such that $\lambda^* \omega = \lambda''$ and such that for $y \in U''$, $\omega(C_y'')$ is canonically imbedded in $J_0 \times y$. We define

$$T_1 = \{(a, y) \mid y \in U'', \ a \in J_0, \ \omega(C_y'')_a = C_0' \times y''\} ,$$

and

$$T_2 = \{(a, y) \mid y \in U'', \ a \in J_0, \ \omega(C_y'')_a = C_0'^* \times y''\} ,$$

where $C_0'^* = \{-c \mid c \in C_0'\}$. Clearly T_1 and T_2 are analytic sets and $\lambda_1^* : T_1 \to U''$ is one-to-one into as is also $\lambda_2^* : T_2 \to U''$. Since (we may assume) U'' is connected, it follows that T_1 or T_2 is empty. We assume (without loss of generality) that T_2 is empty. Since U'' is a manifold, it follows that T_1 is a submanifold of J^*. We define

$$i : C'' \to C_0' \times U''$$

by $i(c) = \omega(c) + \lambda_1^{*-1}(y)$ for $c \in C_y^*$. Clearly i is an isomorphism of complex manifolds, and $\lambda^* i = \lambda''$, which completes the proof of our assertion that $C'' = C_0' \times U''$.

Let $B_1, \cdots, B_b$ have the same meaning as before. Define $\pi : U'' \to C_0'^b$ by $\pi(y) = (B_1 \cap \lambda''^{-1}(y), \cdots, B_b \cap \lambda''^{-1}(y))$. π is an analytic mapping. If $\beta \in C_0'^b$, we have, if $\pi^{-1}(\beta)$ is not empty, $\dim \pi^{-1}(\beta) \geq \dim U'' - b$. We now prove $\dim \pi^{-1}(\beta) = 0$. Let y_0^* be a simple point of $\pi^{-1}(\beta)$ and let U^* be a small neighborhood of y_0^* on $\pi^{-1}(\beta)$. The restriction of $(C(g), \lambda, U)$ to U^* is a C^∞ product fibering $C_0 \times U^*$ and $B_i^* = B_i \cap (C_0 \times U^*)$ are submanifolds of $C_0 \times U^*$. If $y^* \in U^*$, let $p_i(y^*) = B_i^* \cap (C_0 \times y^*)$, and if $\rho : C_0 \to C_0'$ is the quotient mapping (modulo the group H), let $p_i(y^*)' = \rho(p_i(y^*))$. If $\gamma_1, \cdots, \gamma_k$ is a set of generators of the fundamental group $\pi_1(C_{y_0^*} - \{p_1(y_0^*), \cdots, p_b(y_0^*)\})$, we see that these are also a set of generators of $\pi_1(C_{y^*} - \{p_1(y^*), \cdots, p_b(y^*)\})$ for all $y^* \in U^*$ if U^* is small enough. By hypothesis, we have $p_i(y_0^*)'=p_i(y^*)'$ for $y^*\in U^*$. Let $p_i(y_0^*)'=p_i'$. Then for each $y^*\in U^*$, $C_{y^*}-\{p_1(y^*), \cdots, p_b(y^*)\}$ is an unramified covering of m sheets of $C_0'-\{p_1', \cdots, p_b'\}$, the natural mapping ρ is complex analytic, and the associated subgroup of $\pi_1(C_0'-\{p_0', \cdots, p_b'\})$ is in each case the same. Therefore, $C_{y^*}-\{p_1(y^*), \cdots, p_b(y^*)\}$ may be mapped in a one-to-one, complex analytic manner onto $C_{y_0^*}-\{p_1(y_0^*), \cdots, p_b(y_0^*)\}$, in such a way that this mapping may be extended to a conformal isomorphism of C_{y^*} onto $C_{y_0^*}$. But we know that any subset of E_m such that the corresponding Riemann surfaces are all conformally isomorphic to each other is discrete. Hence $\dim \pi^{-1}(\beta)=\dim U^*=0$, which completes the proof of (I), except for the case $n'=0$, $m=2$. If $n'=0$, $m=2$, C_y is a two-sheeted covering of the Riemann sphere C_y', and this covering has $b=2n+2$ branch points. Projectively equivalent sets of branch points give conformally isomorphic Riemann surfaces. We allow the complex, 1-dimensional, projective group PG to act on each factor of $C_0'^b$. Since $\dim \pi^{-1}(\beta)=0$, we have $\dim U''=\dim \pi(U'')$. Then $\pi(U'')$ can meet any orbit of PG in at most a discrete set of points. Since $\dim PG=3$, it follows from the dimension theorem that $\dim U''\leq b-3$, and (I) is proved completely.

To prove (II), we consider the cases $n'=0$, $n'=1$, $n'>1$.

Suppose $n'=0$. Then $n=\frac{1}{2}(m-1)(b-2)$. If $m=2$, $b=2n+2$, so $b<3n$ if $n>2$. Assume $m\geq 3$. Since $d(n')=0$, we must prove $3n-3-b>0$. If $m=3$, $b=n+2$; since $n>2$, $3n-3-b=2b-9>0$. If $m=4$, $n=\frac{3}{2}(b-2)$, so b must be even and $b\geq 4$; thus $3n-3-b=\frac{7}{2}b-12>0$. If $m\geq 5$, $3n-3-b\geq 6(b-2)-b=5b-12$; since $n>0$, we have $b>2$, so $5b-12>0$.

Suppose $n'=1$. Then $3n-3=\frac{3}{2}(m-1)b$ and $d(n')=1$. Hence $3n-3-b-1=\frac{3}{2}(m-1)b-b-1$. Since $n>2$, we have $(m-1)b\geq 4$. If $m=2$, $b\geq 4$, so $\frac{1}{2}b-1>0$. If $m\geq 3$, $\frac{3}{2}(m-1)b-b-1\geq 2b-1>0$, since $b>0$.

Suppose $n'>1$. Then $d(n')=3n'-3$, and we have $3n-3-(3n'-3)-b=m(3n'-3)+\frac{3}{2}(m-1)b-(3n'-3)-b=(m-1)(3n'-3)+$

$\left(\frac{3}{2}(m-1)-1\right)b \geq 3n' - 3 + \frac{1}{2}b > 0$, because $m > 1$.

This completes our proof of (II), and therewith the proof of our main assertion.

Theorem. If $n > 2$, there exists a compact Riemann surface of genus n having no conformal automorphism other than the identity.

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Added in Proof: In proving (II), the case $n'=0$, $m=5$ must be treated separately. In this case we actually have $3n-3-b=5b-15$, $n=2(b-2)$, and since $n>2$, we must have $b \geq 4$, and the right conclusion follows. If $m \geq 6$, we have $3n-3-b \geq \frac{13}{2}b-18>0$, because $b \geq 3$.