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Note on linear processes

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§1. Introduction

As is well known, K. Karhunen [5] introduced the canonical representation of stationary processes which plays an important role in the theory of linear prediction. Let $X(t)$, $-\infty<t<+\infty$, be a mean continuous, purely non-deterministic weakly stationary process with $EX(t)\equiv 0$. Then it can be expressed in the form

$$(1) \qquad X(t)=\int_{-\infty}^{t} F(t-u)dZ(u)$$

with an orthogonal random measure dZ such that $E(dZ(u))^2=du$. This representation is not unique, but there exists essentially one and only one representation which satisfies the condition

$$(2) \qquad \mathfrak{M}_t(X)=\mathfrak{M}_t(Z), \qquad \text{for every } t,$$

where $\mathfrak{M}_t(X)$ and $\mathfrak{M}_t(Z)$ are the closed linear manifolds spanned by $X(\tau)$, $\tau\leq t$, and $Z(\tau)-Z(\sigma)$, $\tau, \sigma\leq t$, respectively. Such a representation (1) is called the canonical representation of $X(t)$ and F is called the canonical kernel. Using the canonical representation, the linear predictor $U(s, t)$ of $X(t)$ based on $X(\tau)$, $\tau\leq s$ $(\leq t)$, i.e. the projection of $X(t)$ into $\mathfrak{M}_t(X)$ can be expressed in the form

$$U(s, t)=\int_{-\infty}^{s} F(t-u)dZ(u).$$

One of us introduced the multiple Markov property for Gaussian processes [3]. This concept can be defined for purely non-deterministic stationary processes which are not always Gaussian, by replacing the independence with the orthogonality. Let $X(t)$ be a stationary process which satisfies the condition that

(M) $U(t_0, t_i)$, $i=1, 2, \cdots, N$, are linearly independent for any t_i's with $t_0 \leq t_1 < \cdots < t_N$, while $U(t_0, t_i)$, $i=1, 2, \cdots, N+1$, are linearly dependent for any t_i's with $t_0 \leq t_1 < \cdots < t_{N+1}$ in the Hilbert space $L^2(\Omega)$.

Then, exactly in the same way as in the Gaussian case (Hida [3]) we can prove that the canonical kernel F can be expressed in the form

$$F(t-u) = \sum_{i=1}^{N} f_i(t) g_i(u), \qquad u \leq t, \tag{3}$$

where f_i's constitute a fundamental system of solutions of an N-th order ordinary differential equation with constant coefficients and g_i's a fundamental system of solutions of adjoint differential equation.

We shall define $X(t)$ to be *linear* if

$$\mathfrak{M}_t(X) \text{ is independent of } \mathfrak{M}_t^{\perp}(X) \text{ for every } t, \tag{4}$$

where $\mathfrak{M}_t^{\perp}(X)$ is the orthogonal complement of $\mathfrak{M}_t(X)$ in $\mathfrak{M}(X) = \bigvee_t \mathfrak{M}_t(X)$; for example an additive process with finite second order moment is a linear process with orthogonal increments and vice versa. A linear process which satisfies the condition (M) is called an *N-ple Markov linear process.*

The main purpose of this paper is to investigate the canonical representation of strictly stationary linear processes with multiple Markov property and the regularity property of the path functions of such processes. Let $X(t)$ be an N-ple Markov linear process and consider its canonical representation (1). Then dZ proves to be an additive random measure i.e. a random measure derived from an additive process. Using Lévy-Itô's decomposition, we have

$$\int_a^t dZ(u) \equiv Z(t) - Z(a) = \sqrt{v}(B_0(t) - B_0(a)) + P(t) - P(a),$$

where $B_0(t)$ is the standard Brownian motion and $P(t)$ is its Poisson part. Therefore $X(t)$ can be decomposed as follows:

$$(5)\qquad X(t) = X_1(t)+X_2(t) = \sqrt{v}\int_{-\infty}^{t} F(t-u)\,dB_0(u)+\int_{-\infty}^{t} F(t-u)\,dP(u)\,.$$

Here the integral with respect to dP can be defined as the sum of stochastic integrals in the sense of K. Itô [4], since in our case F is a degenerated kernel expressed in the form (3). Using the results of Beljaev [1], we can see the regularity property of the path functions of $X_1(t)$; for example if $F(0)=0$ and $F'(0)\neq 0$, almost all path functions of $X_1(t)$ have continuous derivatives, but never have the second order derivatives, and almost all path functions of $X_2(t)$ which is represented in the form

$$(6)\qquad X_2(t) = \sum_{i=1}^{N} f_i(t)\int_{-\infty}^{t} g_i(u)\,dP(u)\,,$$

are continuous and have right derivatives which are no longer continuous.

Finally in § 4, we shall make some important remarks about the concept of the canonical representation. For Gaussian processes, one of us defined it by the condition (2), and it was equivalent to the condition

$$(7)\qquad \boldsymbol{B}_t(X) = \boldsymbol{B}_t(Z)\,,\qquad \text{for ever } t\,,$$

where $\boldsymbol{B}_t(X)$ ($\boldsymbol{B}_t(Z)$) denotes the smallest Borel field with respect to which all the random variables belonging to $\mathfrak{M}_t(X)$ ($\mathfrak{M}_t(Z)$) are measurable. However, for a process represented in the form (1) with additive random measure dP, (2) is stronger than (7) in general. Exanple 3 illustrates this circumstance. Noting this remark, we shall give an example whose nonlinear predictor is really better than the linear one.

We would like to express our hearty thanks to professor K. Itô for his generous help during the preparation of this note. In particular he gave us valuable remarks concerning the definition of linearity.

§ 2. Multiple Markov property of linear processes

We consider a real strictly stationary process $X(t)\equiv X(t, \omega)$,

$-\infty<t<+\infty$, $\omega\in\Omega(P)$, where $\Omega(P)$ is the basic probability space. We assume that $X(t)$ has finite second order moment and is mean continuous. We can assume that

$$EX(t)\equiv 0$$

without loss of generality. Then $X(t)$ can be considered as an element of Hilbert space $L^2(\Omega)$. Let $\mathfrak{M}_t(X)$, $\mathfrak{M}(X)$ and $\mathfrak{M}_t^{\perp}(X)$ be the closed linear manifolds introduced in § 1.

DEFINITION 1. $X(t)$ is called a linear process if $\mathfrak{M}_t(X)$ is independent of $\mathfrak{M}_t^{\perp}(X)$ for any t.[1]

Further we shall assume that $X(t)$ is purely non-deterministic in the sense that

$$\bigcap_t \mathfrak{M}_t(X) = \{0\} .$$

Then $X(t)$ has the canonical representation, that is, it can be expressed in the form (1) and dZ satisfies the condition (2). One of the important examples of linear processes is a Gaussian process.

Example 1. If $X(t)$ is a purely non-deterministic Gaussian process, it is linear. In fact, for a Gaussian system, the orthogonality of two subsystems of $\mathfrak{M}(X)$ is equivalent to their independence. Therefore $\mathfrak{M}_t(X)$ is not only or thogonal to $\mathfrak{M}_t^{\perp}(X)$ but also independent of it.

PROPOSITION 1. *If $X(t)$ is a strictly stationary linear process with the canonical representation, then the random measure dZ is additive and*

$$Z_a(t)\equiv\int_a^t dZ(u) \qquad t\geq a\,, \tag{8}$$

is a temporally homogeneous additive process for any fixed a.

Proof. Since dZ is the random measure associated with the canonical representation, Definition 1 can be replaced with the

1) Two systems $\mathfrak{M}$ and $\mathfrak{N}$ of random variables are called independent if $\boldsymbol{B}(\mathfrak{M})$ and $\boldsymbol{B}(\mathfrak{N})$ are independent (see Doob [2] chap. I), where $\boldsymbol{B}(\mathfrak{M})$ ($\boldsymbol{B}(\mathfrak{N})$) is the smallest Borel field with respect to which all the random variables belonging to $\mathfrak{M}(\mathfrak{N})$ are measurable.

condition that $\mathfrak{M}_t(Z)$ is independent of $\mathfrak{M}_t^{\perp}(Z)$ for every t. Hence dZ is additive. By strict stationarity, we can easily prove that $\{Z_a(t),\ t\geq a\}$ and $\{Z_{a+h}(t+h),\ t\geq a\}$ have the same distribution. Hence the last part of the proposition is proved.

PROPOSITION 2. *Let $X(t)$ be a stationary (not necessarily linear) process expressed in the form* (1). *If $X(t)$ satisfies the condition* (M), *then the canonical kernel F is expressed in the form* (3), *where f_i's are a fundamental system of solutions of the N-th order ordinary differential equation with constant coefficients and g_i's are also a fundamental system of solutions of the adjoint differential equation.*

This proposition can be proved in the same way as in the proofs of Theorem II. 2 and II. 3 in Hida [3], so that the proof is omitted.

DEFINITION 2. If a linear process satisfies the condition (M), it is called an N-ple Markov linear process.

Let $X(t)$ be an N-ple Markov linear process expressed in the form (1). Then, by Proposition 2, we can express it in the form

$$X(t) = \sum_{i=1}^{N} f_i(t) \int_{-\infty}^{t} g_i(u)\, dZ(u)\,. \tag{9}$$

By Proposition 1, dZ is additive and hence each integral appearing in (9) is an additive process. Therefore $X(t)$ can be considered as the sum of N different simple Markov processes. In this case, there exist N functions $a_i(t\,;\ t_1, t_2, \cdots, t_N)$ of t for any $t_1 < t_2 < \cdots < t_N$ $(<t)$ such that

$$\sum_{i=1}^{N} a_i(t\,;\ t_1, t_2, \cdots, t_N)\, U(t_1, t_i) = U(t_1, t)\,. \tag{10}$$

Using the linear independence of g_i's, we have

$$\sum_{i=1}^{N} a_i(t\,;\ t_1, t_2, \cdots, t_N) f_j(t_i) = f_j(t)\,, \qquad j = 1, 2, \cdots, N\,. \tag{11}$$

It should be noted that these a_i's are uniquely determined for any fixed t_i's.

Next we shall give a sufficient condition for $X(t)$ to be a linear process. Let $X(t)$ be expressed in the form (1) and satisfy (M). And assume that

(12) $$\{X(t)-\sum_{i=1}^{N} a_i(t\,;\,t_1,\,\cdots,\,t_N)\,U(t_1,\,t_i),\ t>t_N\}$$
is independent of $\mathfrak{M}_{t_1}(X)$,

where a_i's are the functions determined appropriately. Then $X(t)$ is linear. We shall prove this fact. By Proposition 2, $F(t-u)$ is expressed in the form (3) and a_i's must be determined so that (11) holds. Writing $U_i(t)=\int_{-\infty}^{t} g_i(u)\,dZ(u)$, (12) can be stated as follows:

$$\{X(t)-\sum_{i=1}^{N} a_i(t\,;\,t_1,\,t_2,\,\cdots,\,t_N)\sum_{j=1}^{N} f_j(t_i)\,U_j(t_1)$$
$$= X(t)-\int_{-\infty}^{t_1} F(t-u)\,dZ(u),\ t>t_N\}\ \text{is independent}$$
of $\mathfrak{M}_{t_1}(X)$, that is,
$$\{X(t)-U(t_1,\,t),\ t\geq t_1\}\ \text{is independent of}\ \mathfrak{M}_{t_1}(X).$$

Therefore $\mathfrak{M}_{t_1}^{\perp}(X)$, which is spanned by $\{X(t)-U(t_1,\,t),\ t\geq t_1\}$, is independent of $\mathfrak{M}_{t_1}(X)$.

§ 3. Representations of multiple Markov linear processes

In this section, we shall investigate the representations of stationary multiple Markov linear processes using stochastic integral and study the regularity prorerty of its path functions.

Let $X(t)$ be a strictly stationary N-ple Markov linear process expressed in the form (1). Since $Z_a(t)$ defined by (8) is a temporally homogeneous additive process with $E(Z_a(t))^2=t-a$, it can be considered as Lévy process by taking an appropriate version. Appealing to Lévy-Itô theorem (K. Itô [4]) $Z_a(t)$ can be expressed as

(13) $$Z_a(t,\,\omega)=\sqrt{v}\,(B_0(t,\,\omega)-B_0(a,\,\omega))+P(t,\,\omega)-P(a,\,\omega)\qquad t\geq a\,,$$ [2)]

where $B_0(t)$ is the standard Brownian motion and the Poisson part $P(t)-P(a)$ is expressed by the following stochastic integral

(14) $$P(t,\,\omega)-P(a,\,\omega)=\int_a^t\int_{|s|>0} f(s)\,q(du\,ds,\,\omega)\,.$$

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2) To denote the path function of $Z_a(t)$, we use the notation $Z_a(t,\omega)$ writing ω explicitly. If otherwise, it is considered as the element of $L^2(\Omega)$.

Here the measure $q(E, \omega)$, $E \in \mathcal{B}(R^2)$ and the function f in (14) are the same as in Itô [4] § 9. Since $Z_a(t)$ has finite variance

$$\int f(s)^2 \frac{ds}{s^2} < \infty$$

must hold. Therefore the representation (1) can be expressed in the form

$$(15) \quad X(t) = U(a, t) + \sqrt{v}\int_a^t F(t-u)\,dB_0(u) + \int_a^t \int F(t-u) f(s)\, q(du\, ds)$$

$$= U(a, t) + \textstyle\sum_{i=1}^N f_i(t)\Big[\sqrt{v}\int_a^t g_i(u)\,dB_0(u)$$

$$+ \int_a^t \int g_i(u) f(s)\, q(du\, ds)\Big],$$

letting $a \to -\infty$,

$$= \textstyle\sum_{i=1}^N f_i(t)\Big[\sqrt{v}\int_{-\infty}^t g_i(u)\,dB_0(u)\Big]$$

$$+ \textstyle\sum_{i=1}^N f_i(t)\Big[\int_{-\infty}^t \int g_i(u) f(s)\, q(du\, ds)\Big].$$

Here it should be noted that the terms represented by the stochastic integrals with respect to q are defined in the L^2-sense. If we take appropriate versions, they coincide with the ones defined in Itô's sense, as is easily seen. The sum of such integrals will be called the Poisson part of $X(t)$ and the other sum will be called the Gaussian part. Applying the same argument to (13), we get the representation of path functions of $X(t)$ as follows

$$(15') \qquad X(t, \omega) = \textstyle\sum_{i=1}^N f_i(t)\Big[\sqrt{v}\int_{-\infty}^t g_i(u)\,dB_0(u, \omega)$$

$$+ \int_{-\infty}^t \int g_i(u) f(s)\, q(du\, ds, \omega)\Big].$$

Now we can investigate the regularity property of the path functions of $X(t)$. The path function in the square bracket in (15′) is continuous except possibly for discontinuity of the first kind (see Itô [4] § 9) for almost all ω, and it follows from this fact that $X(t, \omega)$ has the same regularity property. Taking an appropriate version, we can assume that

$$U_i(t, \omega) = \int_{-\infty}^t \int g_i(u) f(s)\, q(du\, ds, \omega)$$

is right continuous in t. Then we have

THEOREM. *A strictly stationary N-ple Markov linear process* $X(t)$ *of the form* (1) *can be decomposed into two processes* $X_1(t)$ *and* $X_2(t)$, *which are called the Gaussian part and the Poisson part respectively. Then the path function* $X(t, \omega)$ *is the sum of two functions* $X_1(t, \omega)$ *and* $X_2(t, \omega)$ *with the following regularity property*:

i) *If* $F(0) \neq 0$, *then* $X_1(t, \omega)$ *is continuous but not derivable, while almost every sample path of* $X_2(t, \omega)$ *is continuous except for discontinuities of the first kind, and the position and the height of the jumps arise from those of* Z *and vice versa.*

ii) *If* $F(0)=F'(0)=\cdots=F^{(k)}(0)=0$ *and* $F^{(k+1)}(0) \neq 0$, $0 \leq k \leq N-2$, *then* $X(t, \omega)$ *has derivatives up to* $k+1$*-th order and* $X^{(k+1)}(t, \omega)$ *has the same regularity as that of* $X(t, \omega)$ *in* i) (*note that the* $k+1$*-th derivative should be a right derivative*).

Proof. The decomposition of $X(t)$ and $X(t, \omega)$ has already been proved. For the Gaussian part, the continuity of path functions has investigated by Hunt and Beljaev [1][3]. Therefore we shall prove the theorem only for the Poisson part.

Let us fix an ω for which (15′) holds. Then the jump points of $Z(t, \omega)$ are denumerable, say $\{t_\nu\}$. For $t=t_\nu$, $Z(t, \omega)$ has discontinuities of the first kind with jump $s_\nu = Z(t_\nu+0, \omega) - Z(t_\nu-0, \omega)$. Then $\int_{-\infty}^{t}\int g_i(u) f(s) q(du\, ds, \omega)$ has also discontinuity of the first kind at t_ν with jump $f_i(t_\nu)s_\nu$. Therefore $X_2(t, \omega)$ has discontinuities of the first kind at t_ν, too, with jump

$$\sum_{i=1}^{N} f_i(t_\nu) g_i(t_\nu) s_\nu = F(0) s_\nu \neq 0 . \tag{16}$$

This proves that every jump point of Z is also the jump point of X_2 and the converse is true. Thus i) is proved.

Next, let us assume that $F(0)=0$ and $F'(0) \neq 0$. Then (16) shows that $X_2(t, \omega)$, consequently $X(t, \omega)$, is continuous. Furthermore, noting that the difference

3) Since the Fourier transform of F is a rational function, we can examine the order of decrease of the spectral distribution function at ∞. Hence Beljaev's results are applicable.

$$X_2(t_\nu+h, \omega)-X_2(t_\nu-, \omega) = \sum_{i=1}^N f_i(t_\nu+h)\int_{t_\nu-}^{t_\nu+h}\int g_i(u)f(s)q(du\,ds, \omega)$$
$$+\sum_{i=1}^N (f_i(t_\nu+h)-f_i(t_\nu))\int_{-\infty}^{t_\nu}\int g_i(u)f(s)q(du\,ds, \omega)$$
$$= \sum_{i=1}^N f_i(t_\nu+h)g_i(t_\nu)(s_\nu+o(1))$$
$$+\sum_{i=1}^N (hf'_i(t_\nu)+o(h))\int_{-\infty}^{t_\nu}\int g_i(u)f(s)q(du\,ds, \omega)$$

for sufficiently small $h>0$, we can prove that $X_2(t, \omega)$ is not derivable. But replacing $t_\nu-$ with t_ν in the above equation, we can see that the first term is negligible and $X(t, w)$ has right derivative

$$X'_2(t, \omega) = \sum_{i=1}^N f'_i(t)\int_{-\infty}^{t}\int g_i(u)f(s)q(du\,ds, \omega).$$

This is not derivable as we proved before, since $\sum_{i=1}^N f'_i(t)g_i(t) = F'(0)\neq 0$. As to $X_1(t, \omega)$, it has continuous derivative, since the spectral density function $F(\lambda)$ is of order λ^{-4} as $\lambda\to\infty$ (Beljaev [1]). Thus we have proved ii) in case $k=0$. Similarly we can prove ii) for every k, $k=1, 2, \cdots, N-2$.

Example 2. Let $P^i(t)$,[4] $i=1, 2$, $t\geq 0$, be mutually independent Poisson processes with $EP^i(t)=t$. Then we can form an additive random measure dP from

$$P(t) = \begin{cases} P^1(t)-t, & \text{if } t\geq 0, \\ -P^2(-t)+t, & \text{if } t<0. \end{cases}$$

Define $X(t)$ by

$$X(t, \omega) = \int_{-\infty}^{t} (e^{-(t-u)}-e^{-2(t-u)})\,dP(u, \omega).$$

The kernel $(e^{-(t-u)}-e^{-2(t-u)})$ is a canonical kernel. Almost all sample functions of $X(t)$ are continuous and $P(t, \omega)$ can be formed by the right derivatives of $X(t, \omega)$. This is the simplest example of the processes considered in the theorem above.

§4. Concluding remark

As we have seen in the course of obtaining (16) in the previous section, all the jumps of $Z(t, \omega)$ can be obtained from the

4) $P^2(t, \omega)$ should have been modified to be left continuous.

Poisson part by linear operation for its path function. This is true even if $F(0)=0$ (Theorem 3. ii)). This fact suggests how to define the concept of the canonical representation of stochastic processes introduced by P. Lévy [6] §3. 8. Denoting by $\boldsymbol{B}_t(X)$ and $\boldsymbol{B}_t(Z)$ the smallest Borel fields with respect to which all the $X(\tau)$ and $Z(\tau)$, $\tau\leq t$, are measurable respectively, we shall call the representation *canonical in the sense of Borel field*,[5] if

$$(17)\qquad \boldsymbol{B}_t^*(X)=\boldsymbol{B}_t^*(Z) \quad \text{for every } t \quad (\boldsymbol{B}_t^*(\cdot)=\bigcap_n \boldsymbol{B}_{t+1/n}(\cdot)).$$

The representation (15) of $X(t)$, in particular the representation (14) of $P(t)-P(a)$, is canonical in this sense. We shall prove this fact. The measure $q(\cdot)$ is defined by jumps (jump points and heights), which are determined by the jumps of $X(t, \omega)$ or its derivative of suitable order. Hence we can form $X_2(t, \omega)$. At the same time we can see that $\boldsymbol{B}_t^*(X_2)=\boldsymbol{B}_t^*(q)$ holds. Subtracting $X_2(t, \omega)$ from $X(t, \omega)$, we get $X_1(t, \omega)$ which is a sample functions of Gaussian part of $X(t)$. Concerning $X_1(t)$, one of the authors studied its representation [3] and we know that there exists Wiener's random measure $dB_0(t)$ satisfying

$$\mathfrak{M}_t(X_1)=\mathfrak{M}_t(B_0), \qquad \text{consequently } \boldsymbol{B}_t(X_1)=\boldsymbol{B}_t(B_0)$$

Thus we can prove

$$\boldsymbol{B}_t^*(X)\supset\boldsymbol{B}_t^*(Z)$$

The opposite inclusion relation is obvious. Thus we have proved (17).

It should be noted that the assumption that F is the canonical kernel is unnecessary if $X(t)$ contains no Gaussian part as we can see in the proof of the theorem in the previous section.

Now we can state an interesting remark about prediction theory of stationary processes. Let $Y(t)$ be defined by

$$(18)\qquad Y(t, \omega)=\int_{-\infty}^{t}\widetilde{F}(t-u)\,dP(u, \omega),$$

where dP is the random measure in Example 2 and $\widetilde{F}$ is a non-

5) C.f. P. Lévy [6] Chap. III. N. Wiener [7] also studied nonlinear prediction in this view point.

canonical kernel with $\tilde{F}(0) \neq 0$, satisfying all the conditions that appeared in Proposition 2.

Then we have

$$\boldsymbol{B}_t^*(Y) = \boldsymbol{B}_t^*(P), \qquad \text{for every } t,$$

by the discussions above; while

$$\mathfrak{M}_t(Y) \subsetneqq \mathfrak{M}_t(P), \qquad \text{for every } t, \tag{19}$$

holds owing to the non-canonical property of the representation (18). Hence the conditional expectation $E(Y(t)/\boldsymbol{B}_s)$, $s \leq t$, $(\boldsymbol{B}_s = \boldsymbol{B}_s(Y))$ of $Y(t, \omega)$ under assumption that $\{Y(\tau, \omega); \tau \leq s\}$ are known is given by

$$E(Y(t)/\boldsymbol{B}_s) = E(Y(t)/\boldsymbol{B}_s(P)) = \int_{-\infty}^{s} \tilde{F}(t-u)\,dP(u, \omega), \tag{20}$$

which is a sort of predictors, indeed a nonlinear predictor, of $Y(t, \omega)$ based on $Y(\tau, \omega)$'s before s. On the other hand, forming the canonical representation

$$Y(t) = \int_{-\infty}^{t} F(t-u)\,dZ(u)$$

of $Y(t)$ in the usual L^2-sense (dZ is merely the orthogonal random measure), we get the linear predictor of $Y(t)$ based on $Y(\tau)$'s before s:

$$U(s, t) = \int_{-\infty}^{s} F(t-u)\,dZ(u).$$

Thus we have got two different kinds of predictors. $E(Y(t)/\boldsymbol{B}_s)$ is better than $U(s, t)$ in the following sense;

$$E[Y(t) - E(Y(t)/\boldsymbol{B}_s)]^2 < E[Y(t) - U(s, t)]^2 \tag{21}$$

holds. This inequality can be proved, if we note that

$$\begin{aligned} E(Y(t)/\boldsymbol{B}_s) &= \text{Projection of } Y(t) \text{ on } \mathfrak{M}_t(P) \\ &= U(s, t) + \text{Projection of } Y(t) \text{ on } \mathfrak{M}_t(P) \ominus \mathfrak{M}_t(Y) \end{aligned}$$

and that the second term of the last equation has a positive variance since $\tilde{F}$ is a non-canonical kernel.

Such a circumstance seems to characterize the linearity of the process, noting that $Y(t)$ discussed above is not a linear process.

Example 3. Consider a process defined by

$$Y_1(t, \omega) = \int_{-\infty}^{t} [3e^{-(t-u)} - 4e^{-3(t-u)}] \, dP(u, \omega) . \tag{22}$$

The kernel in the square bracket is not a canonical one. This is not a linear process, but in the same way as in the theorem, we can form $P(t, \omega)$ from $Y_1(t, \omega)$ and hence (22) is the canonical representation in the sense of Borel field.
Another process defined by

$$Y_2(t, \omega) = \int_{-\infty}^{t} [2e^{-(t-u)} - e^{-3(t-u)}] \, dP(u, \omega) \tag{23}$$

is a double Markov linear process. By simple computations we can see that $Y_1(t)$ and $Y_2(t)$ have the common covariance function, but they are not the same process. As to $Y_1(t)$ the nonlinear predictor is really better than linear one, while the linear predictor of $Y_2(t)$ coincide with its nonlinear one.

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