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A generalization of the imbedding problem of an abstract variety in a complete variety

To Professor Y. Akizuki for celebration of his 60th birthday

By

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In a previous paper [4], we proved that every abstract variety is an open subset of a complete abstract variety. In the present paper, we try to generalize this result to the case of a Neotherian scheme of finite type[1]. Namely, we consider first a ground Neotherian scheme S which is covered by a finite number of open Neotherian affine schemes S_i. Then a scheme we like to say to be of finite type over S is a scheme M over S such that M is covered by a finite number of open affine schemes M_j so that for a suitable choice of S_i, the morphism $M \rightarrow S$ induces morphisms $M_j \rightarrow S_i$ and the affine ring $\mathfrak{o}_j$ of M_j is finitely generated over the natural image of the affine ring of S_i in $\mathfrak{o}_j$.

Our main theorems imply that:

If M is a Noetherian scheme of finite type over a Noetherian ground scheme S, then M is an open subset of a proper scheme (Noetherian and of finite type) over S.

In our treatment, we use valuation-theoretic method, hence the usual definition of a scheme is not nicely suited to our proof. Therefore we give a valuation-theoretic definition of a Neotherian scheme of finite type. Then our method in our paper [4] can be adapted and we have our main results.

As for terminology on rings, we shall use mainly the one in our book [3].

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1) As for the definition of a scheme, see Grothendieck [1].

1. Definition of a scheme

A ring is assumed always to be a commutative ring with identity. We do not assume that the identity of a subring coincides with that of the original ring.

Since we are interested only in Noetherian schemes of finite type in this paper, schemes defined below is a Noetherian scheme and a scheme M over a scheme S is nothing but a scheme M of finite type over S.

Let $\mathfrak{o}$ be a Noetherian ring. Then the set A of local rings $\mathfrak{o}_{\mathfrak{p}}$, where $\mathfrak{p}$ runs through all prime ideals of $\mathfrak{o}$, is called the *affine scheme* of $\mathfrak{o}$; $\mathfrak{o}$ is called the *affine ring* of the affine scheme A ($\mathfrak{o}$ is uniquely determined by A). An affine scheme is the affine scheme of some Noetherian ring.

When a ring R is given, we consider the set of pairs $(\mathfrak{p}, v)$ of prime ideals $\mathfrak{p}$ of R and valuation rings v of the field of quotients of $R/\mathfrak{p}$. This set is denoted by $ZM(R)$. A local ring R' contained in R is said to be *dominated* by $(\mathfrak{p}, v)$ $(\in ZM(R))$ if $R'/(\mathfrak{p} \cap R')$ is different from 0 and is dominated by v. Two local rings R' and R'' contained in R are defined to correspond to each other if there is a member $(\mathfrak{p}, v)$ of $ZM(R)$ which dominates both R' and R''.

We shall define also the notion of weak domination as follows: We say that a ring R' *dominates* another ring R'' *weakly* if (i) they are contained in a ring R and (ii) denoting by e' and e'' the identities of R' and R'' respectively, it holds that $e'e''=e'$ and $R''e'$ is dominated by R'.

We consider from now on the case where R is an Artin ring $\mathfrak{A}$. $\mathfrak{A}$ is the direct sum of a finite number of local rings whose maximal ideals are nilpotent.

A set M of local rings is called a *scheme* if (1) all local rings are contained in an Artin ring $\mathfrak{A}$,[2] (2) M is the union of a finite

2) For a given set of rings, there is an Artin ring $\mathfrak{A}$ such that there is an isomorphism ϕ_λ from every member R_λ of the set into $\mathfrak{A}$. If we identify R_λ with $\phi_\lambda(R_\lambda)$, then we have a definite relationship among elements of rings R_λ. Thus our assumption that all local rings are contained in an Artin ring is made in order to define a definite relationship among elements of these local rings. In the usual definition (sheaf-theoretical definition) of a scheme, relationship among elements of local rings (stalks) is given by sections.

number of affine schemes and (3) if a local ring $R' \in M$ corresponds to an $R'' \in M$ (i.e., if R' and R'' are dominated by a member of $ZM(\mathfrak{A})$), then $R'=R''$.

The ring A' generated by all local rings of M is again an Artin ring. This Artin ring A' is called the *function ring* of M.[3] Then one can see that an affine scheme is a scheme (cf. [2, I]).

Let M and S be schemes such that all local rings of M and S are contained in an Artin ring $\mathfrak{A}$. Then the correspondence of local rings defined above gives a multi-valued function defined on a certain subset of M with values in S. This function is the *transformation* from M into S and is denoted by $T_{M\to S}$. $T_{M\to S}$ is defined at $R' \in M$ if and only if R' corresponds to some local ring of S; $T_{M\to S}(R')$ is then the set of local rings of S which correspond to R'. In the other case, we define $T_{M\to S}(R')$ to be the empty set.

Now we say that $T_{M\to S}$ is *regular* at $R' \in M$ if there is $R'' \in S$ which is dominated by R' weakly; in this case, $T_{M\to S}(R')=R''$ as is easily seen (cf. [2, I]). We say that $T_{M\to S}$ is *biregular* at R' if $R' \in S \cap M$. $T_{M\to S}$ is said to be *regular* or to be a *morphism* if it is regular at every local ring of M; in this case, we say that *M dominates S weakly*. If furthermore every $R' \in M$ dominates $T_{M\to S}(R')$, then we say that *M dominates S* (*strongly*).

These notions being defined, we can define the notions of a *specialization*, the *locus* of a local ring in a scheme, *Zariski topology*, etc. as in our paper [2, I].

We say that M is a *scheme over a* (*ground*) *scheme* S if (i) M and S are schemes whose local rings are contained in an Artin ring $\mathfrak{A}$, (ii) M dominates S weakly, (iii) for suitable finite affine open coverings M_{ij} $(i=1, \cdots, n;\ j=1, \cdots, m(i))$ and S_i $(i=1, \cdots, n' \geq n)$ of M and S respectively, the affine ring $\mathfrak{o}_{ij}$ of M_{ij} is finitely generated over $e_{ij}\mathfrak{o}_i$, where e_{ij} is the identity of $\mathfrak{o}_{ij}$ and $\mathfrak{o}_i$ is the affine ring of S_i and we assume that $e_{ij} \in e_{ij}\mathfrak{o}_i$.

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3) The definition of a scheme depends in appearance on $\mathfrak{A}$ but really depends only on the function ring of the scheme, as is easily seen. We should note here that, since, for instance, in the primary decomposition of ideals in a Noetherian ring, primary components belonging to imbedded prime divisors are not uniquely determined, hence form a given Noetherian scheme in the sense of [1], we may have many different function rings of the scheme depending on how we imbed stalks in which Artin ring.

Many theorems in [2] can be adapted. We shall state one of these results:

Proposition 1.1. *Let M be a scheme over a ground scheme S and let M' be a non-empty subset of M. Then the following* 3 *conditions are equivalent to each other*:

(1) *M' is an open subset of M.*

(2) *M' is a scheme over S.*

(3) *The following two conditions hold good*:

(i) *If $R \notin M'$ $(R \in M)$, then every specialization of R is not in M' (i.e., if $R' \in M'$, then the affine scheme defined by R' is contained in M').* (ii) *If $R' \in M'$, then there is a closed subset F' of the locus F of R' in M such that $R' \notin F'$ and $F - F' \subseteq M'$.*

2. Birationality

Let M be a scheme. A local ring R in M is called a *component of zero* if all non-units in R are zero-divisors. The set of components of zero of a scheme is obviously a finite set.

Let M and M' be schemes. We say that $T_{M \to M'}$ is *rational* if it is regular at every components of zero of M. We say that $T_{M \to M'}$ is *quasi-rational* if it is regular at every component of zero of M which is contained in $T_{M' \to M}(M')$. We say that M is *birational* to M' if the set of components of zero of M coincides with that of M'. We note here that in the case of schemes with function fields, rationality and birationality of transformations are nicely defined by virtue of function fields. But, function rings of schemes do not tell us much (cf. foot-note 3)), and we should observe components of zero.

We then define the notion of the *join* $J(A, A')$ of subsets A, A' of schemes M, M' (respectively) over a ground scheme S as in our paper [2, I]. Then $J(R, R')$ $(R \in M,\ R' \in M')$ is not empty if and only if R corresponds to R'. $T_{J(M,M') \to M}$ is regular. If $T_{M \to M'}$ is rational, then $J(M, M')$ is birational to M. If A and A' are subsets of M, then $J(A, A') = A \cap A'$. Note also that in the definition of the join, it is independent of particlular choice of a ground scheme S. Consequently,

Proposition 2. 1. *If A and A' are affine schemes contained in M, then $A \cap A'$ is an affine scheme.*

Furthermore, we can prove easily the following (cf. [2, I]):

Proposition 2. 2. *If M and M' are schemes over a ground scheme S and if they are birational to each other, then $M \cap M'$ is a scheme over S which is birational to M. If we drop the birationality from the assumption, then $M \cap M'$ is either empty or a scheme over S. Namely, the set of local rings in M at which $T_{M \to M'}$ is not biregular is a closed subset of M.*

Let again M be a scheme whose rings are contained in an Artin ring $\mathfrak{A}$. Let $\mathfrak{p}_1, \cdots, \mathfrak{p}_n$ be all of prime ideals of $\mathfrak{A}$ and consider $(\mathfrak{p}_i, \mathfrak{A}/\mathfrak{p}_i)$. We denote by 1 the identity of $\mathfrak{A}$. For each $\mathfrak{p}_i$, there is a uniquely determined idempotent e_i by the condition that $1-e_i \in \mathfrak{p}_i$, $1-e_i \notin \mathfrak{p}_j$ for every $j \neq i$. Now let L be the function ring of M and let e be the identity of L. Then we have

Proposition 2. 3. *There is a local ring R in M dominated by $(\mathfrak{p}_i, \mathfrak{A}/\mathfrak{p}_i)$ if and only if $ee_i \neq 0$ (i.e. $Le_i \neq 0$).*

Proof. The only if part is obvious. Assume that $ee_i \neq 0$. Then there is a local ring R' in M such that $e_iR' \neq 0$, whence $R' \not\subseteq \mathfrak{p}_i$. Then $R = R'_{(\mathfrak{p}_i \cap R')}$ is in M and dominated by $(\mathfrak{p}_i, \mathfrak{A}/\mathfrak{p}_i)$. This proves the assertion.

A local ring R of M is called a *pseudo-component of zero* of M if R is dominated by some $(\mathfrak{p}_i, \mathfrak{A}/\mathfrak{p}_i)$.

We should note here that

Lemma 2. 4. *A component of zero of M is a pseudo-component of zero of M.*

In order to give an explicit proof of Lemma 2. 4, we give a remark:

Frist we introduce a notation. When e' is an idempotent of $\mathfrak{A}$, then we denote by Me' the set of Re' such that $R \in M$ and $Re' \neq 0$. Now we have

Lemma 2. 5. *Let e' be an idempotnet of $\mathfrak{A}$ such that $Le' \neq 0$. Then,* (1) *Me' is a scheme whose function ring is Le',* (2) *$Re' = R'e' \neq 0$ $(R, R' \in M)$ implies $R = R'$,* (3) *if R is a component of zero of*

M, then either Re' is a component of zero of Me' or $R(1-e')$ is a component of zero of $M(1-e')$ and (4) if $R_1, \cdots, R_n$ are all of the pseudo-components of zero of M, then all of non-zero R_ie' form the set of all pseudo-components of zero of Me'.

The proof of Lemma 2.5 is straightforward.

Corollary 2.6. *If R is a component of zero of M, then there is a primitive idempotent e' of $\mathfrak{A}$ (i.e., e' is an idempotent such that $\mathfrak{A}e'$ is a local ring) such that Re' is a component of zero of Me' ($Le' \neq 0$).*

Note that in Corollary 2.6, since the function ring of Me' is a local ring, Re' is unique component of zero of Me' and is also unique pseudo-component of zero of Me'. Therefore, R is a pseudo-component of zero of M by (2) and (4) in Lemma 2.5.

Now we define, with respect to the Artin ring $\mathfrak{A}$ of consideration, the notions of *strong rationality*, *strong quasi-rationality*, and *strong birationality* by the same way as above, but replacing "components of zero" by "pseudo-components of zero". Then it is obvious that Proposition 2.2 can be adapted to strongly birational schemes. Namely:

Proposition 2.7. *Let M and M' be schemes over a scheme S. If M is strongly birational to M', then $M' \cap M$ is a scheme over S which is strongly birational to M'.*

The following condition on the function ring L of a scheme M in an Artin ring $\mathfrak{A}$ is sometimes a nice condition:

($\boldsymbol{Z}$) *An element of L is a zero-divisor in L if and only if it is a zero-divisor in $\mathfrak{A}e$, e being the identity of L.*

Proposition 2.8. *For a scheme M, the set of components of zero coincides with that of pseudo-components of zero if and only if the condition ($\boldsymbol{Z}$) is satisfied.*

The proof is straightforward.

3. Complete schemes and projective schemes

Let M be a set of local rings contained in an Artin ring $\mathfrak{A}$. Then the set of $(\mathfrak{p}, v) \in ZM(\mathfrak{A})$ which dominate some members of

M is denoted by $ZR(M;\mathfrak{A})$. Let S be another set of local rings contained in $\mathfrak{A}$. We say that M is *complete with respect to* S if $ZR(S;\mathfrak{A})\subseteq ZR(M;\mathfrak{A})$. Note that the definition of completeness does not depend on the particular choice of $\mathfrak{A}$.

When M is a scheme over a ground scheme S, then $ZR(S;\mathfrak{A})\subseteq ZR(M;\mathfrak{A})$ implies $ZR(S;\mathfrak{A})=ZR(M;\mathfrak{A})$. In this case, we say that M is a *complete scheme over* S.

Next we define the notion of a projective scheme over a ground scheme S. Let $\mathfrak{A}$ be an Artin ring containing the function ring K of S and let e be the identity of K.

We first consider the case where S is the affine scheme defined by a Noetherian ring $\mathfrak{o}$. Let $x_0, \cdots, x_n$ be elements of $\mathfrak{A}e$ such that the module $\sum x_i\mathfrak{o}$ is generated by $y_0, \cdots, y_m$ over $\mathfrak{o}$ so that every y_i is not a zero-divisor in $\mathfrak{A}e$. Then we consider the union M of affine schemes A_i $(i=0, \cdots, m)$ defined by $\mathfrak{o}[y_0/y_i, \cdots, y_m/y_i]$, where y_j/y_i are considered in the Artin ring $\mathfrak{A}e$. This M is really independent of the particular choice of the basis $y_0, \cdots, y_n$ and M is a scheme. This M is called the *projective scheme over* S *defined by homogeneous coordinates* $(x_0, \cdots, x_n)$.

Note that:

Lemma 3.1. *With the same notation as just above, let* U *be an affine scheme defined by a Noetherian ring* R *such that* $U\subseteq S$ *and let* e' *be the identity of* R. *Then* $T_{S\to M}(U)$ *is the projective scheme defined by* $(x_0e', \cdots, x_ne')$ *over* U.

This lemma enables us to define projective schemes in the general case as follows: A scheme M over a scheme S is called a *projective scheme* over S if for every affine scheme A contained in S, $T_{S\to M}(A)$ is a projective scheme over A (defined by a certain system of homogeneous coordinates).

Ppoposition 3.2. *A projective scheme* M *over a scheme* S *is a complete scheme over* S.

Proof is easy (cf. [2, I]).

Proposition 3.3. *If* M *and* M' *are projective schemes over a scheme* S, *then the join* $J(M, M')$ *of* M *and* M' *is a projective scheme over* S.

Proof is easy (cf. [2, I]).

We say that a scheme M is a *quasi-projective* scheme over a scheme S if M is an open subset of a projective scheme over S.

In closing this section, we shall state an easy result without proof:

Proposition 3.4. *If a scheme M is complete with respect to another scheme M', then for an arbitrary closed set F of M, $T_{M\to M'}(F)$ is a closed set of M'.*

4. Some remarks and formulation of main theorems

Let M be a scheme whose local rings are contained in an Artin ring $\mathfrak{A}$. We say that M is *normally imbedded* in $\mathfrak{A}$ if the function ring of M has common identity with $\mathfrak{A}$.

Proposition 4.1. *Let M be a scheme over a scheme S and let L and K be the function rings of M and S respectively. Let e', e be the identitihs of K, L respectively. Then,* (i) *in general, we have $ee'=e$ and* (ii) *if $K\subseteq L$ then K and L have the same identity.*

The proof if straightforward.

Corollary 4.2. *Let M be a scheme over a scheme S and assume that local rings of M and S are contained in an Artin ring $\mathfrak{A}$. If M is normally imbedded in $\mathfrak{A}$, then S is also normally imbedded in $\mathfrak{A}$.*

We shall state now our main theorems:

Theorem 1. *Let M be a scheme over a ground scheme S. If M and S are normally imbedded in an Artin ring $\mathfrak{A}$, then there is a complete scheme M' over S so that M' is strongly birational to M (with respect to $\mathfrak{A}$) and contains M as an open subset.*

Theorem 2. *Let M be a scheme over a ground scheme S, all being considered in an Artin ring $\mathfrak{A}$. Then there is a complete scheme M' over S such that M is an open subset of M'.*

These main theorems will be proved later.

We have to explain how Theorem 1 proves the imbedding problem of a scheme in a proper scheme.

Let M be a scheme over a scheme S and let K and L be the function rings of S and M respectively. Let e be the identity of L and consider $Se=\{Re \mid R\in S, Re\neq 0\}$. Then Se is a scheme (see Lemma 2.5) and M is a scheme over Se. Conversely, a scheme over Se is a scheme over S. On the other hand, one can see easily that M is a proper scheme over S if and only if M is a complete scheme over Se. Therefore the imbedding of a scheme in a proper scheme is an immediate consequence of Theorem 1, and we can assert that:

If M is a scheme over a scheme S, then there is a proper scheme M' over S so that (i) *M' contains M as an open subset and* (ii) *M' is birational to M.*

5. Dilatation by an ideal

Let M be a scheme whose local rings are contained in an Artin ring $\mathfrak{A}$. An *ideal* I of M is a map defined on M such that for each $R\in M$, $I(R)$ is an ideal of R and there is an open covering U_i of M so that all $I(R)$ $(R\in U_i)$ for each i have a common basis. $I(R)$ is called the *stalk* of I at R. *Closed sets defined by ideals, prime ideals* and *primary ideals* are defined as in [2, III].

Let I be an ideal of a scheme M. Assume that the closed set defined by I do not carry any pseudo-component of zero of M. If the condition ($\boldsymbol{Z}$) in §2 is satisfied, then the condition is equivalent to that every $I(R)$ $(R\in M)$ contains at least one element which is not a zero-divisor. In this case, we define a new scheme M' which is called the *dilatation* of M (defined) by I as follows:

By the definition of an ideal, we can choose affine schemes $M_1, \cdots, M_t$ which cover M such that $I(R)$ is generated by an ideal I_i of the affine ring $\mathfrak{o}_i$ of M_i for every $R\in M_i$ (for each i). Let e_i be the identity of $\mathfrak{o}_i$. By the assumption, I_i contains at least one non-zero-divisor in $\mathfrak{A}e_i$, whence we can choose a basis $(x_{i0}, \cdots, x_{in_i})$ of I_i consisting only of non-zero-divisors in $\mathfrak{A}e_i$. Then we can consider the projective scheme $M*_i$ over M_i defined by homogeneous coordinates $(x_{i0}, \cdots, x_{in_i})$. Then the union $M*$ of all $J(M_i, M*_i)$ is a scheme, complete over M and independent of particular choice of the basis $(x_{i0}, \cdots, x_{in_i})$. This $M*$ is called the dilatation of M defined by I. Verification of the above construction is substantially the same as the case of algebraic geometry over fields. As is obvious,

Proposition 5.1. *The dilation $M*$ is complete over M and strongly birational to M.*

When M is a projective scheme over an affine scheme S, then M has a homogeneous coordinate ring, say $\mathfrak{h}$, over the affine ring of S. (Homogeneous coordinate rings are defined as usual.) Then a primary ideal of M come from a homogeneous primary ideal of $\mathfrak{h}$. An ideal of M is the intersection of a finite number of primary ideals of M, whence every ideal of M come from a homogeneous ideal of $\mathfrak{h}$. Therefore the dilatation of M defined by an ideal of M (provided that it is well defined) is again a projective scheme. Therefore, by the definition of a projective scheme in the general case, we have

Proposition 5.2. *If I is an ideal of a projective scheme M over a scheme S and if the dilatation $M*$ of M by I is well defined, then $M*$ is a projective scheme over S.*

Corollary 5.3. *Let M be a scheme and let I be an ideal of M such that I defines a dilatation $M*$ of M. If a subset U of M is a projective scheme (or a quasi-projective scheme) over a scheme S', then $T_{M\to M*}(U)$ is a projective scheme (or a quasi-projective scheme, respectively) over S'.*

6. Some preliminary results

Let M be a scheme whose function ring is contained in an Artin ring $\mathfrak{A}$. We introduce on $ZR(M; \mathfrak{A})$ a topology which is also called *Zariski topology* as follows:

Let $\mathrm{M} = \{M_\lambda\}$ be the set of schemes over M which are strongly birational to M with respect to $\mathfrak{A}$ and complete over M. For each closed subset $F_{\lambda\mu}$ of M_λ, let $F*_{\lambda\mu}$ be the set of $(\mathfrak{p}, v) \in ZR(M; \mathfrak{A})$ which dominate some local rings in $F_{\lambda\mu}$. The set of all $F*_{\lambda\mu}$ is defined to be a base of closed sets of $ZR(M; \mathfrak{A})$. If we define an equivalence relation $\sim$ in $ZR(M; \mathfrak{A})$ by : $(\mathfrak{p}, v) \sim (\mathfrak{p}', v')$ if and only if $(\mathfrak{p}, v)$ and $(\mathfrak{p}', v')$ dominate the same local ring on every M_λ. Then we see that the equivalence classes form naturally a space which is homeomorphic to the limit of the inverse system $\{M_\lambda\}$; the limit is denoted by $ZR(M)$. Thus we see that

Lemma 6.1. *$ZR(M)$ and $ZR(M; \mathfrak{A})$ are compact (non-Hausdorf except for a very special case where M consists only of a finite number of local rings).*

Lemma 6.2. *Let M and M' be schemes over an affine scheme S. Assume that M and S are normally imbedded in an Artin ring $\mathfrak{A}$. If M is strongly birational to M' (with respect to $\mathfrak{A}$) and if $(\mathfrak{p}, v) \in ZR(j(M, M'); \mathfrak{A})$, then there is a scheme $M*$ which is strongly birational to M so that* (1) *$M*$ is complete over M,* (2) $M \frown M' \lesssim M*$, (3) *for every quasi-projective subset U of M over a scheme S', the scheme $T_{M \to M*}(U)$ is a quasi-projective scheme over S' and* (4) *if $P*$ $(\in M*)$ and P' $(\in M')$ are dominated by $(\mathfrak{p}, v)$ then $P*$ dominates P' weakly.*

The proof is the same as that of Lemma 3.1 in [4] if we note the following obvious fact:

Lemma 6.3. *If M and M' are schemes which are strongly birational to each other with respect to the Artin ring $\mathfrak{A}$ and if a pseudo-component of zero of M is a specialization of $P \in M$, then $T_{M \to M'}$ is biregular at P.*

Therefore as in [4], we have

Theorem 6.4. *Let M and M' be schemes over a scheme S. If M is strongly birational to M' with respect to the Artin ring $\mathfrak{A}$, then there is a scheme $M*$ over S such that* (1) *$M*$ is complete over M,* (2) $M \frown M' \lesssim M*$, (3) *for every quasi-projective subset U of M over a scheme S', $T_{M \to M*}(U)$ is quasi-projective over S' and* (4) *for each $R \in M*$, either $T_{M* \to M'}(R)$ is empty or $T_{M* \to M'}$ is regular at R.*

Let M be a scheme over a scheme S and let M' be a projective scheme over Se, e being an idempotent of the Artin ring of consideration. For each $R \in M$, let A be an affine open set of S containing $T_{M \to S}(R)$ and consider the projective scheme $T_{S \to M'}(A)$ over Ae. Let $x_0, \cdots, x_n$ be a system of homogeneous coordinates which defines $T_{S\to M'}(A)$ over Ae. Then the set of elements $x'_0, \cdots, x'_n$ of R such that $x'_0e=ax_0, \cdots, x'_ne=ax_n$ (considering all possible a) generates an ideal $I(R)$ of R. Then $I(R)$ is independent of the particular choice of A and also of x_i. Hence the set of $I(R)$ defines an ideal I of M. This ideal is called the *ideal* of M *associated with the transformation* $T_{M\to M'}$.

Theorem 6.5. *With the same notation as above, for a* $P\in M$, *P is in the closed set defined by I if and only if $T_{M\to M'}$ is not regular at P and $T_{M\to M'}(P)$ is not empty. If the ideal I defines dilatation $M*$ of M, then $T_{M'\to M*}(M')$ dominates M' weakly.*

The proof is the same as that of Theorem 3.3 in [4].

Proposition 6.6. *Under the notation in Theorem* 6.5, *if $T_{M\to M'}$ is strongly quasi-rational, then $M*$ is well defined.*

The proof is straightforward.

7. Proof of Theorem 1.

Here we shall prove Theorem 1, which was stated in §4.

First we shall show that if Theorem 1 is true for affine S, then it is true in the general case. Indeed, assume that Theorem 1 is true if S is affine and let $S_1, \cdots, S_n$ be affine open subsets of S which cover S. We shall use induction argument on n. Set $S_0=\bigcup_{i<n}S_i$ and $M_0=T_{S\to M}(S_0)$. Then M_0 is a scheme over S_0. Let K, L, K_0, L_0 be function rings of S, M, S_0, M_0 respectively. By our assumption, K and L have the same identity, say 1, with $\mathfrak{A}$. Let e and e' be the identities of K_0 and L_0 respectively. Since M_0 is a scheme over S_0, we have $ee'=e'$. Set $e''=e-e'$. We want to show that $e''=0$. Assume the contrary. Then e'' is an idempotent of L, hence there is a local ring R in M which is a pseudo-component of zero of M and is dominated by $(\mathfrak{p}, L/\mathfrak{p})$, where $\mathfrak{p}$ is a maximal ideal of L containing $1-e''$, by Proposition 2.3. Since $ee''=e''$, $(\mathfrak{p}, L/\mathfrak{p})$ dominates a local ring in S_0, hence R must be in M_0. This is a contradiction, and $e''=0$. Thus we see that M_0 and S_0 are normally imbedded in $\mathfrak{A}e$. Therefore, by our induction assumption, there is a scheme $M*_0$ such that it is complete over S_0,

contains M_0 as an open set and such that it is strongly birational to M_0 with respect to $\mathfrak{A}e$. Then $M*_1=M*_0\bigcup M$ is a scheme over S and strongly birational to M with respect to $\mathfrak{A}$. Next we apply the same to S_n. Namely, set $M_n=T_{S\to M*_1}(S_n)$. Let $M*_n$ be a complete scheme over S_n such that $M*_n$ is strongly birational to M_n and contains M_n as an open set. The $M*=M*_1\bigcup M*_n$ is the desired scheme.

Next step is to show that

Lemma 7.1. *Let M be a scheme over an affine scheme S. If M and S are normally imbedded in an Artin ring $\mathfrak{A}$, then for any given member $(\mathfrak{p}, v)$ of $ZR(S;\mathfrak{A})$, there is an affine scheme M' which contains a local ring dominated by $(\mathfrak{p}, v)$ such that $M\cup M'$ is a scheme over S and strongly birational to M with respect to $\mathfrak{A}$.*

The proof is substantially the same as that of Lemman 4.1 in our paper [4] by virtue of Proposition 2.3.

Thus, as in our paper [4], we have only to show the following lemma:

Lemma 7.2. *Let M_1 and M_2 be strongly birationally equivalent schemes over an affine schemes S with respect to an Artin ring $\mathfrak{A}$. Assume that these schemes are normally imbedded in $\mathfrak{A}$. Set $M=M_1\cap M_2$ and assume that M_1-M is contained in a projective scheme $M*$ over $Se*$, $e*$ being an idempotent of $\mathfrak{A}$. Then there is a scheme M_3 which contains M such that M_3 is strongly birational to M with respect to $\mathfrak{A}$ and such that $ZR(M_3;\mathfrak{A})=ZR(M_1;\mathfrak{A})\cup ZR(M_2;\mathfrak{A})$.*

Proof. To begin with, we may assume that $T_{M_2\to M_1}$ is regular at every local ring of $T_{M_1\to M_2}(M_1)$ by Theorem 6.3. Set $F=M-(M*\cap M)$ and $F*=M*-(M*\cap M_1)$. $F*$ is a closed set of $M*$, $M_1=M\cup(M*-F*)$ and F is the union of $T_{M*\to M}(F*)$ and the set F_0 of local rings in M which do not correspond to any local ring of $M*$. Since $M*$ is projective over $Se*$, we have $F_0=\{R\mid R\in M,\ e*R=0\}$. Set $H=M_2-T_{M_1\to M_2}(M_1)$, $G=T_{M*\to M_2}(M_1-M)$ and let F_2 be the closure of F in M_2. We want to show that $F_2-F\subseteq H$.

$F=T_{M*\to M}(F*)\cup F_0$ and $T_{M*\to M_2}(F*)$ is closed in M_2. Let $\bar{F}_0$ be the closure of F_0 in M_2 and let R be a member of F_2-F. Then either (i) $R\in T_{M*\to M_2}(\bar{F}*)-T_{M*\to M}(F*)$ or (ii) $R\in \bar{F}_0-F_0$. Assume that R corresponds to a local ring R_1 in M_1, and we went to show a contradiction. By our assumption on the regularity of $T_{M_2\to M_1}$, we see that R dominates R_1 weakly. Case (i): Let $R*$ be a local ring in $F*$ which corresponds to R. Since R dominates R_1 weakely, we see that R_1 corresponds to $R*$. If $R_1\in M$, then $R_1\in F\subseteq M_2$, whence $F\not\ni R=R_1\in F$, which is a contradiction. Hence $R_1\notin M$, whence $R_1\in M*$, and $R_1=R*\in F*$, which is a contradiction to our assumption that $R_1\in M_1$. Thus in Case (i), we have $R\in H$. Case (ii): If $R_1\in M$, then $R_1=R$ and we have a contradiction. Therefore $R_1\notin M$, whence $R_1\in M_1-M\subseteq M*$. Let e and e_1 be the identities of R and R_1 respectively. Since $R_1\in M*$, we have $e_1e*=e_1$. Since R dominates R_1 weakly, we have $ee_1=e$. Since $e*F_0=0$, we have $(1-e*)e\neq 0$. Therefore $0=0e=e_1(1-e*)ee_1=(1-e*)e\neq 0$, which is a contradiction. Thus we have shown that $F_2-F\subseteq H$. This being settled, we can adapt the proof of Lemma 4.2 in our paper [4] and we prove our lemma.

Thus Theorem 1 is proved.

8. Proof of Theorem 2.

By Proposition 4.1, we may assume that S is normally imbedded in the Artin ring $\mathfrak{A}$. Let 1 and e be the identities of $\mathfrak{A}$ and the function ring L of M respectively. If $e=1$, then Theorem 1 shows the result, and we assume that $e\neq 1$. Se and $S(1-e)$ are schemes over S, and furthermore $Se\cup S(1-e)$ is a complete scheme over S. $M\cup S(1-e)$ is a scheme over $Se\cup S(1-e)$, whence $M\cup S(1-e)$ is a scheme over S. $M\cup S(1-e)$ is normally imbedded in $\mathfrak{A}$, and we prove the assertion by Theorem 1.

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