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Differentiable manifolds admitting complex distributions

By

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The present paper continues the study made in a recent series of papers [2] and [3][1], concerning with an analytic n-dimensional manifold admitting a complex r-dimensional distribution satisfying a certain condition. For brevity, such a structure of a manifold will be called a π^r-structure. In the first paper [2], the case where $n=2r+1$ was treated and some results were obtained in connection with an almost contact metric structure due to S. Sasaki. In the second paper [3] was presented a generalisation of these results to the case where $n \geqq 2r$ and it was found that there is a close relation between a π^r-structure and an f_r-structure due to K. Yano.

The purpose of the present paper is to show an existence of a certain f_r-structure and a symmetric real affine connection on a manifold with a $(\pi^r - \Gamma)$-structure such that f is covariant constant. To do this, we shall first make clear a more precise relation between a π^r-structure and an f_r-structure, and discuss an integrable π^r-structure. We shall express our main result in Theorem 9. It is remarked here that we assumed, in preceding papers, the manifold under consideration to be analytic, and, however, we shall treat, in the following, manifolds of class C^∞, unless otherwise provided.

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1) Numbers in brackets refer to the references at the end of the paper.

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§1. Historical remarks

In 1953, E. M. Patterson published a very interesting paper [4]. In his paper, he dealt, for the first time, with a field of complex planes (hereafter we call it *a complex null distribution*) and obtained a geometrical characterisation of Kähler manifolds. That is to say, he found several properties of complex null distributions and proved that a differentiable manifold M^{2r} of class C^2 admitting an r-dimensional complex distribution π^r which is null and parallel with respect to a given positive definite metric g on M^{2r} is a Kähler manifold whose Kähler metric is g. It is important, in the proof of this theorem, that the distribution π^r and its conjugate complex distribution $\bar{\pi}^r$ have only the zero vector in common. Though the above property is derived from the fact that g is positive definite, the existence of a metric g is not necessarily essential. From the above consideration he finally concluded that, if a differentiable manifold M^{2r} of class C^2 admits a complex distribution π^r, such that π^r and $\bar{\pi}^r$ at each point have only the zero vector in common, and a symmetric affine connection Γ with respect to which π^r is parallel, then the manifold M^{2r} is a complex analytic manifold.

In these theorems, Patterson treated only the case where the dimension n of the manifold is $2r$. It is obvious that, if there is such an r-dimensional distribution, the number n has to be equal or more than $2r$. Accordingly it will be interesting to study on a structure of a manifold such as $n>2r$.

The present author firstly treated the case $n=2r+1$ [2] and obtained the theorem that a manifold M^{2r+1} admitting a field π^r which satisfies the similar conditions as in Patterson's first theorem admits a (φ, ξ, η, g)-structure having the covariant constant φ-tensor. The notion of (φ, ξ, η, g)-structure in odd dimensional manifolds was introduced by S. Sasaki, which is equivalent to the almost contact metric structure.

Recently, K. Yano [12], [13] introduced the notion of an f_r-structure including an almost complex structure and an almost contact structure, and obtained a number of interesting results. The present author found, in his paper [3], the close relation between an f_r-structure satisfying a certain condition and the existence of a complex distribution π^r $(2r \leqq n)$ satisfying the similar conditions as in Patterson's second theorem.

On the other hand, A. G. Walker [9] and, at the same time, T. J. Willmore [10] studied on connections for integrable real distributions, and they succeeded in proving that for any system of integrable real distributions there exists an affine connection in the large with respect to which the distributions are parallel and which is symmetric. The present paper gives a condition of the existence of a real symmetric affine connection Γ with respect to which the complex distribution π^r is parallel, which is an extension of Walker and Willmore's result in a real distribution and also can be regarded as an analytic interpretation of the assumption in Patterson's theorem [4] and the present author's [3].

§2. Manifolds with π^r-structures

In this paper we treat n-dimensional differentiable manifolds of class C^∞. We shall begin with a definition of a π^r-structure.

Definition. *A manifold with a π^r-structure is a manifold admitting an r-dimensional complex distribution π^r satisfying the relation $\pi^r \cap \bar{\pi}^r = \{0\}$ at each point of the manifold where $\bar{\pi}^r$ means a conjugate complex distribution of π^r.*

As is well known, an f_r-structure defined by Yano [12] is an example of such a structure. That is to say, in a manifold admitting a real non-zero tensor field f of type (1, 1) such that

$$(2.1) \qquad f^3+f=0, \qquad rank \text{ of } (f)=2r,$$

r-dimensional eigen vector spaces f^r and $\bar{f}^r$, spanned by eigen vectors corresponding respectively to eigen value $-\sqrt{-1}$ and $\sqrt{-1}$, are defined globally. Since they are mutually disjoint and conjugate complex, they construct a π^r-structure. We call the distribution f^r the *complex f-distribution* hereafter.

Now we consider the converse of this fact and prove

Theorem 1. *A manifold M^n admits a π^r-structure if and only if M^n admits an f_r-structure.*

Proof. If M^n admits an f_r-structure, then it admits a π^r-structure as is shown above.

Conversely, let M^n admit a π^r-structure, then at each point of M^n, the relation

$$\pi^r \cap \bar{\pi}^r = \{0\} \tag{2.2}$$

holds good. Then the direct sum $\pi^r \oplus \bar{\pi}^r$ constructs a complex $2r$-dimensional distribution and has a real basis, that is, $\pi^r \oplus \bar{\pi}^r$ contains a real $2r$-dimensional distribution $L^{2r} = \mathfrak{Re}[\pi^r \oplus \bar{\pi}^r]$, which does not depend on the choice of basis of π^r. In tangent space at each point of M^n, we take a real complementary distribution of L^{2r}, which we denote by M^{n-2r}. Let $\lambda_{(\alpha)}$ and $N_{(A)}$[2] be basic contravariant vectors of π^r and M^{n-2r} respectively, then $\bar{\lambda}_{(\alpha)}$ are basic vectors of $\bar{\pi}^r$, and the determinant $|A| = |\lambda_{(\alpha)}, \bar{\lambda}_{(\alpha)}, N_{(A)}|$ does not vanish. Hence we construct the inverse (μ) of the matrix (A), which is expressed by $(\mu_j^{(a)}, \mu_j^{(A)})$ in terms of local coordinate (x^i, U). Then we have directly $\mu^{(\bar{\alpha})} = \bar{\mu}^{(\alpha)}$, $\bar{\mu}^{(B)} = \mu^{(B)}$, $(\bar{\alpha} = \alpha + r)$.

Next, let us define a complex tensor field φ of type (1, 1) by

$$\varphi = \sum_{\alpha} \lambda_{(\alpha)} \mu^{(\alpha)}; \tag{2.3}$$

the tensor φ does not depend on the choice of the basis of π^r, and

2) In this paper the indices a, b, c, d, e run over the range $1, \ldots, 2r$; $h, i, j, \ldots, r, s, t$ the range $1, \ldots, n$; A, B, C, D, E the range $2r+1, \ldots, n$; $\alpha, \beta, \gamma, \delta$ the range $1, \ldots, r$; $\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}$ the range $r+1, \ldots, 2r$; and for example, $\bar{\alpha}$ means $\alpha + r$.

it is easy to show that the following equations are satisfied:

$$(2\cdot4)\qquad \varphi^2=\varphi,\ \varphi\lambda_{(\alpha)}=\lambda_{(\alpha)},\ \varphi\bar{\lambda}_{(\alpha)}=0,\ \varphi N_{(A)}=0,$$

where φ^2 and $\varphi\lambda_{(\alpha)}$ mean, in terms of (x^i, U), $\varphi^i_l\varphi^l_j$ and $\varphi^i_l\lambda^l_{(\alpha)}$ respectively, from which it follows that φ is a projection tensor of π^r. It is obvious from (2.4) that the projection tensor ψ of $\bar{\pi}^r$ is given by $\psi=\sum_\alpha\bar{\lambda}_{(\alpha)}\bar{\mu}^{(\alpha)}$, and satisfies

$$(2.5)\qquad \psi=\bar{\varphi},\ \varphi\psi=0,\ \psi\varphi=0,\ \psi^2=\psi\,.$$

Now, making use of the tensor φ as above obtained, we shall construct an f_r-structure by putting

$$(2.6)\qquad f=-\sqrt{-1}(\varphi-\bar{\varphi}).$$

The tensor f constructs a real tensor field of type (1, 1) and satisfies the relation $f^3+f=0$ by virtue of the relations (2.4) and (2.5). The tensor f also satisfies

$$(2.7)\qquad f\lambda_{(\alpha)}=-\sqrt{-1}\,\lambda_{(\alpha)},\ \ f\bar{\lambda}_{(\alpha)}=\sqrt{-1}\,\bar{\lambda}_{(\alpha)},\ \ fN_{(A)}=0,$$

from which the rank of (f) is equal to $2r$. Q.E.D.

Remark. In the proof of the above theorem, we showed an existence of a special f_r-structure in M^n admitting a π^r-structure. Though we could construct the f_r-structure by (2.6), it is clear that the tensor f depends on the choice of the complementary distribution M^{n-2r}. But, the f_r-structure defined by (2.6) has *an interesting property that f^r coincides with π^r*. Therefore we shall call an f_r-structure defined by (2.6) *an induced one from π^r* (corresponding to the complementary M^{n-2r}), and the above L^{2r} *a real f-distribution* hereafter.

We shall now examine a change of a induced f_r-structure when a complementary distribution M^{n-2r} is replaced by an another one M'^{n-2r}.

Theorem 2. *In a manifold with a π^r-structure, induced f_r-structures f and f' corresponding to the complementary distri-*

butions M^{n-2r} and M'^{n-2r} of the real f-distribution L^{2r} satisfy equations $hf=0$, $-f^2h=h$, where $h=f'-f$.

Proof. Let us denote basic vectors of M'^{n-2r} by $N'_{(A)}$ and the inverse matrix of $(\lambda_{(\alpha)}, \bar{\lambda}_{(\alpha)}, N'_{(A)})$ by (μ'), then we have

$$h=-\sqrt{-1}\sum_{\alpha}[\lambda_{(\alpha)}\mu'^{(\alpha)}-\lambda_{(\alpha)}\mu^{(\alpha)}-\bar{\lambda}_{(\alpha)}\bar{\mu}'^{(\alpha)}+\bar{\lambda}_{(\alpha)}\bar{\mu}^{(\alpha)}],$$

from which we find

$$hf=-\sum_{\alpha}[\lambda_{(\alpha)}\mu^{(\alpha)}-\lambda_{(\alpha)}\mu^{(\alpha)}+\bar{\lambda}_{(\alpha)}\bar{\mu}^{(\alpha)}-\bar{\lambda}_{(\alpha)}\bar{\mu}^{\alpha}]=0.$$

In the same way, we have $hf'=0$, which gives us $ff'=(f'-h)f'=f'^2$. These results, therefore, lead us to

$$-f^2h=-f^2(f'-f)=-f'^3+f^3=f'-f=h. \qquad \text{Q.E.D.}$$

Now, let us consider the converse of Theorem 2. For this purpose, we have first

Lemma 1. *In a manifold with an f_r-structure, if there exists a tensor field h of type* (1, 1) *satisfying $hf=0$, the vectors $N'_{(A)}=N_{(A)}+fhN_{(A)}$ are linearly independent where $N_{(A)}$ are basic vectors of a distribution M defined by a projection tensor $m=f^2+I$.*

Proof. It is obvious that the tensor m is a projection operator and the distribution M defined by m is $(n-2r)$-dimensional [12]. Then there exist $n-2r$ linearly independent vectors $N_{(A)}$ in M. Now the relation $hf=0$ shows that the determinant $|I+fh|$ does not vanish. Thus the vectors $N'_{(A)}=(I+fh)N_{(A)}$ are linearly independent.

Theorem 3. *In a manifold M^n with an f_r-structure, if there exists a tensor field h of type* (1, 1) *satisfying*

$$hf=0, \quad -f^2h=h, \tag{2.8}$$

then a tensor field $f'=f+h$ constructs an f_r-structure in M^n, whose complex f-distribution coincides with that of f.

Proof. The conditions (2.8) give us directly $h^2=0$. From this and (2.1), we can easily verify the relation $f'^3+f'=0$. The definition of the complex f-distribution f^r gives us $\lambda_{(\alpha)}=\sqrt{-1}f\lambda_{(\alpha)}$, which leads us to

$$(f'+\sqrt{-1}I)\lambda_{(\alpha)}=f\lambda_{(\alpha)}+h\lambda_{(\alpha)}+\sqrt{-1}\lambda_{(\alpha)}=\sqrt{-1}hf\lambda_{(\alpha)}=0.$$

Then we see that the rank of $(f')\geqq 2r$. On the other hand, from the result of Lemma 1, the vectors $N'_{(A)}=(I+fh)N_{(A)}$ construct an $(n-2r)$-dimensional real distribution. Since the definition of the vectors $N_{(A)}$ given in Lemma 1 shows $fN_{(A)}=0$, we find $f'N'_{(A)}=f^2hN_{(A)}+hN_{(A)}=0$. Thus the rank of $(f')\leqq 2r$ must hold. Consequently the rank of $(f)=2r$.

§3. An f_r-structure whose complex f-distribution is integrable

In a manifold with a π^r-structure, if we take a complementary distribution of the real distribution L^{2r}, we can define an induced f_r-structure by (2.6). Moreover Remark of Theorem 1 shows that the complex f-distribution f^r corresponding to the above induced f_r-structure coincide with the given distribution π^r. Therefore in order to consider a manifold with a π^r-structure whose distribution π^r is integrable, it is convenient to treat a manifold with an f_r-structure whose complex f-distribution is integrable. Hence in this section we treat a manifold with an f_r-structure.

Now, the projection tensor φ of the complex f-distribution f^r is given by

$$\varphi=\frac{1}{2}(-f^2+\sqrt{-1}f). \tag{3.1}$$

Then, the relation (2.4) holds good.

We shall now extend the results given by Walker [9] in the case of real distributions to that of the complex distributions. In the first place, we have

Lemma 2. *If the complex f-distribution f^r is integrable, the equation*

$$(3.2)\qquad (\varphi^i_{p,q}-\varphi^i_{q,p})\varphi^p_j\varphi^q_k=0$$

holds good, where commas denote partial differentiation.

Proof. For the differential operator $X_\alpha=\lambda^i_{(\alpha)}\partial_i$, the condition for f^r to be integrable can be written in the form $(X_\alpha X_\beta-X_\beta X_\alpha)=\phi^\gamma_{\alpha\beta}X_\gamma$. These are also equivalent to

$$\lambda^q_{(\beta),p}\lambda^p_{(\alpha)}-\lambda^q_{(\alpha),p}\lambda^p_{(\beta)}=\phi^\gamma_{\alpha\beta}\lambda^q_{(\gamma)}\,.$$

Multiplying by $\mu^{(\beta)}_k\mu^{(\alpha)}_j(\delta^i_q-\varphi^i_q)$ and summing for α, β and q, it can be verified that these equations are equivalent to the condition (3.2).

Lemma 3. *A necessary and sufficient condition for the complex f-distribution f^r to be parallel with respect to a given affine connection Γ is that the projection tensor φ satisfies*

$$(3.3)\qquad \nabla\varphi\varphi=0\,,$$

where, expressed in terms of local coordinate (x^i, U), $\nabla\varphi\varphi$ means $(\nabla_k\varphi^i_l)\varphi^l_j$.

Proof. If the distribution f^r is parallel with respect to the connection Γ, then there exist r^2 local covariant vector fields $\mu^{(\beta)}_{(\alpha)k}$ with respect to which the relation

$$(3.4)\qquad \nabla_k\lambda^i_{(\alpha)}=\sum_\beta\mu^{(\beta)}_{(\alpha)k}\lambda^i_{(\beta)}$$

holds good. From the relations (3.4) and $(2.4)_2$, it follows that $\varphi\nabla\lambda_{(\alpha)}=\nabla\lambda_{(\alpha)}$. Differentiating $(2.4)_2$ covariantly and using the above result, we have $\nabla\varphi\varphi\lambda_{(\alpha)}=0$. On the other hand, the relations $(2.4)_3$ and $(2.4)_4$ give $\nabla\varphi\varphi\bar{\lambda}_{(\alpha)}=0$ and $\nabla\varphi\varphi N_{(A)}=0$. Thus we obtain the relation $\nabla\varphi\varphi=0$.

Conversely, if the condition $\nabla\varphi\varphi=0$ is satisfied, $(2.4)_2$ gives $\nabla\lambda_{(\alpha)}=\varphi\nabla\lambda_{(\alpha)}$, which shows us that the distribution f^r is parallel.

Q.E.D.

Remark. In this Lemma it is to be noted that the affine connection Γ is not necessarily real one.

Now let $L=(L^i_{jk})$ be an arbitrary symmetric affine connection defined over the M^n, and denote by | covariant differentiation with respect to L, then the quantity Γ defined by

$$\Gamma=L+T \tag{3.5}$$

gives an affine connection over M^n where T is a tensor field whose local components have the following form:

$$T^i_{jk}=-\varphi^i_{p|j}\varphi^p_k-\varphi^i_{p|k}\varphi^p_j+\varphi^i_{p|q}\varphi^p_k\varphi^q_j\,. \tag{3.6}$$

Then we get

Lemma 4. *The complex f-distribution f^r is parallel with respect to the affine connection Γ defined by (3.5) and (3.6) where L is an arbitrary symmetric affine connection.*

Proof. Denoting by ∇ the covariant differentiation with respect to the Γ, we find, using the relations (3.5), (3.6) and $\varphi^2=\varphi$, $\nabla_k\varphi^i_l\varphi^l_j=\varphi^i_p\varphi^p_{q|k}\varphi^q_j$. On the other hand, the relation $\varphi^2=\varphi$ gives $\varphi^i_p\varphi^p_{q|k}\varphi^q_j=0$. Consequently we obtain $\nabla\varphi\varphi=0$. Thus Lemma 4 follows at once from Lemma 3.

Theorem 4. *If the complex f-distribution f^r is integrable, there exists a symmetric affine connection Γ with respect to which f^r is parallel.*

Proof. From the result of Lemma 4, the distribution f^r is parallel with respect to Γ given by the relations (3.5) and (3.6). Hence it remains for us to prove that the Γ is symmetric. To do this, using the relation $\varphi^2=\varphi$ and (3.2), we find

$$\begin{aligned}\Gamma^i_{jk}-\Gamma^i_{kj}&=T^i_{jk}-T^i_{kj}\\&=[\varphi^i_{p,q}+L^i_{lq}\varphi^l_p-\varphi^i_{q,p}-L^i_{lp}\varphi^l_q]\,\varphi^p_k\varphi^q_j\\&=(\varphi^i_{p,q}-\varphi^i_{q,p})\varphi^p_k\varphi^q_j\\&=0\,.\end{aligned}$$

Theorem 5. *In a manifold with an f_r-structure whose complex f-distribution f^r and real f-distribution L^{2r} are both integrable, there exists a real symmetric affine connection $\overset{*}{\Gamma}$ with respect to which f^r is parallel.*

Proof. It follows, from the definition of the real f-distribution L^{2r}, that the tensor $l=-f^2$ is a real projection operator of L^{2r}. As the distribution L^{2r} is integrable, from the well known Walker's Theorem [9], [10] there exists a real symmetric affine connection $\overset{\circ}{\Gamma}$ satisfying the equation $\overset{\circ}{\nabla} ll=0$. Hence, on taking $\overset{\circ}{\Gamma}$ instead of L in Lemma 4, the tensor T is determined by (3.6). Now putting

$$\overset{*}{\Gamma}=\overset{\circ}{\Gamma}+T+\overline{T}, \tag{3.7}$$

we get a real affine connection $\overset{*}{\Gamma}$. Then the proof of Theorem 4 shows us $\overset{*}{\Gamma}$ is symmetric. To complete the proof of this theorem, it is sufficient to show $\overset{*}{\nabla}\varphi\varphi=0$. However, from Theorem 4, relation $\nabla\varphi\varphi=0$ holds good with respect to the affine connection $\Gamma=\overset{\circ}{\Gamma}+T$. Hence we have $\overset{*}{\nabla}_k\varphi^i_l\varphi^l_j=(\overline{T}^i_{mk}\varphi^m_l-\varphi_m{}^i\overline{T}^m_{lk})\varphi^j_l$. Substituting (3.6) into the right hand member of the last equation, and taking account of (2.5), we have

$$\overset{*}{\nabla}_k\varphi^i_l\varphi^l_j=-\overline{\varphi}^i_{p|m}\varphi^m_j\overline{\varphi}^p_k-\varphi^i_{m|l}\varphi^i_j\overline{\varphi}^m_k .$$

Moreover we substitute (3.1) into the right hand member of this equation, and making use of the relations $lf=fl=f$, $\overset{\circ}{\nabla} ll=0$ and $\overset{\circ}{\nabla} lf=0$, we find

$$\begin{aligned}\overset{*}{\nabla}_k\varphi^i_l\varphi^l_j=&-\frac{1}{8}[-\sqrt{-1}\,f^i_{p|m}(l^p_kl^m_j+f^m_jf^p_k+\sqrt{-1}\,l^p_kf^m_j-\sqrt{-1}\,l^m_jf^p_k)\\&+\sqrt{-1}\,f^i_{m|l}(l^m_kl^l_j+f^m_kf^l_j-\sqrt{-1}\,f^m_kl^l_j+\sqrt{-1}\,l^m_kf^l_j)]\\=&\,0 .\end{aligned}$$

Thus the proof is completed. Moreover, considering the Theorem 1 and Theorem 5, we obtain

Corollary. *In a manifold with a π^r-structure whose complex distribution π^r and real distribution $L^{2r}=\mathfrak{Re}[\pi^r\oplus\bar{\pi}^r]$ are both integrable, there exists a real symmetric affine connection $\overset{*}{\Gamma}$ with respect to which the distribution π^r is parallel.*

§4. A manifold with a $(\pi^r-\Gamma)$-structure

Let us consider a manifold M^n with a π^r-structure. In such a manifold we defined a $(\pi^r-\Gamma)$-structure in the paper [3] as follows:

Definition. *A manifold M^n with a $(\pi^r-\Gamma)$-structure is a manifold M^n admitting a π^r-structure and a real symmetric affine connection Γ with respect to which the distribution π^r is parallel.*

Now we shall prove

Theorem 6. *A necessary and sufficient condition for a manifold M^n to admit a $(\pi^r-\Gamma)$-structure is that the M^n admit a π^r-structure whose complex distribution π^r and real distribution $L^{2r}=\mathfrak{Re}[\pi^r\oplus\bar{\pi}^r]$ are both integrable.*

Proof. The conditions are sufficient because of Corollary of Theorem 5. To prove the necessity, we first write the basic vectors $\lambda_{(\alpha)}$ of π^r in the form

$$\lambda_{(\alpha)}=a_{(\alpha)}+\sqrt{-1}\,b_{(\alpha)}\,, \tag{4.1}$$

where $a_{(\alpha)}$ and $b_{(\alpha)}$ are both real vectors. When we use the notations

$$c_{(\alpha)}=a_{(\alpha)},\quad c_{(\bar{\alpha})}=b_{(\alpha)} \tag{4.2}$$

the $2r$ vectors $c_{(a)}$ are real basic vectors of the real f-distribution L^{2r}. Since the manifold M^n admits a $(\pi^r-\Gamma)$-structure, the relations

$$\nabla_k\lambda^i_{(\alpha)}=\sum_{\beta}\mu^{(\beta)}_{(\alpha)k}\lambda^i_{(\beta)}\,, \tag{4.3}$$

$$\nabla_k c^i_{(a)}=\sum_{b}A^{(b)}_{(a)k}c^i_{(b)} \tag{4.4}$$

hold good for suitable covariant vectors $\mu^{(\beta)}_{(\alpha)k}$ and $A^{(b)}_{(a)k}$. Now we put

(4. 5) $\lambda^i_{(\alpha)}\partial_i = X_\alpha\ ,\ c^i_{(a)}\partial_i = X_a\ ,$

then, by making use of (4. 3) and (4. 4), we have

(4. 6) $(X_\alpha X_\beta - X_\beta X_\alpha) = \sum_\gamma \Phi^\gamma_{\alpha\beta} X_\gamma\ ,$

(4. 7) $(X_a X_b - X_b X_a) = \sum_c h^c_{ab} X_c\ ,$

where we put $\lambda^j_{(\alpha)}\mu^{(\gamma)}_{(\beta);j} - \lambda^j_{(\beta)}\mu^{(\gamma)}_{(\alpha);j} = \Phi^\gamma_{\alpha\beta}$ and $C^j_{(a)}A^{(c)}_{(b);j} - C^j_{(b)}A^{(c)}_{(a);j} = h^c_{ab}$. That is to say, the distribution π^r and L^{2r} are both integrable.

Q.E.D.

Next, we shall give a simple proof of a theorem in a preceding paper [3], i.e.

Theorem 7. *If a manifold M^n admits a $(\pi^r - \Gamma)$-structure, then an f_r-structure induced from the π^r-structure satisfies the relation*

(4. 8) $\nabla ff = 0\ .$

Proof. By virtue of Theorem 1 and its Remark, we know that, for an f_r-structure induced from the given π^r-structure, the complex f-distribution f^r coincides with the distribution π^r. Therefore the distribution f^r is assumed to be parallel with respect to the given connection Γ. Then Lemma 3 shows us that $\nabla\varphi\varphi = 0$, which can be rewritten by means of (3. 1) as

$$(-f\nabla ff^2 + 2\nabla ff) + \sqrt{-1}(2\nabla ff^2 + f\nabla ff) = 0\ .$$

Since f and Γ are both real, the real part and imagirary part of the last equation are equal to zero. Then straightforward calculation gives (4. 8). Q.E.D.

Now, in order to prove the following theorem, let us recollect the some results obtained in the preceding paper [3]. Consider a manifold M^n with a $(\pi^r - \Gamma)$-structure, then the distribution L^{2r} and π^r are both integrable and the equations (4. 6) and (4. 7) are satisfied. Hence a system of differential equations

(4. 9) $X_a f = 0 \qquad (a = 1, 2, \cdots, 2r)$

is completely integrable and has $n-2r$ independent solutions, saying $w^{2r+1}, w^{2r+2}, \cdots, w^n$, which are C^∞-functions of x^i in each local coordinate neighbourhood (x^i, U). Similarly the equations (4.6) shows that a system of complex partial differential equations

$$(4.10) \qquad X_\alpha f=0 \qquad (\alpha=1, 2, \cdots, r)$$

is also completely integrable and has $n-r$ independent solutions. However $w^A (A=2r+1, \cdots, n.)$ are also the solutions of (4.10), and real independent solutions of (4.10) are only w^A and their functions, because of the definition of $C_{(a)}$. Now let $z^\alpha=u^\alpha+\sqrt{-1}v^{\bar{\alpha}}(\bar{\alpha}=\alpha+r)$ be the other r complex independent solutions of (4.10). Then $(z^1, \cdots, z^r, w^{2r+1}, \cdots, w^n)$ can be regarded as fundamental solutions of (4.10), and z^α are necessarily complex valued functions and of differentiability class C^∞.

Although a system of complex partial differential equations

$$(4.11) \qquad \bar{\lambda}^i_{(\alpha)}\partial_i f=0 \qquad (\alpha=1, 2, \cdots, r)$$

is also completely integrable, its fundamental solutions are given by $(\bar{z}^1, \cdots, \bar{z}^r, w^{2r+1}, \cdots, w^n)$. That is to say, the relations

$$(4.12) \qquad \bar{\lambda}^i_{(\alpha)}\partial_i \bar{z}^\beta=0, \quad \bar{\lambda}^i_{(\alpha)}\partial_i w^A=0$$

hold good. Since the $n-r$ functions $(\bar{z}^\alpha, w^A)$ are also independent functions, the rank of the matrix $(\partial_i \bar{z}^\alpha, \partial_i w^A)$ is $n-r$ over the M^n. From this fact and the condition (2.2) of a π^r-structure, it follows that the determinant $|\mu^\alpha_\beta|=|\lambda^i_{(\beta)}\partial_i \bar{z}^{(\alpha)}|$ does not vanish. Thus we can take a matrix (σ^α_β) over the M^n as the inverse of the matrix (μ^α_β).

On the other hand, it is clear that there is no functional relationship of the form $F(z^1, \cdots, z^r, \bar{z}^1, \cdots, \bar{z}^r, w^{2r+1}, \cdots, w^n)\equiv 0$, and hence we can take $(u^1, \cdots, u^r, v^{\bar{1}}, \cdots, v^{\bar{r}}, w^{2r+1}, \cdots, w^n)$ as a coordinate system in (x^i, U), which was called *a canonical coordinate system* in the paper [3].

By means of these results, we have obtained, in the paper [3], the following theorem.

Theorem 8. *In a manifold with a $(\pi^r - \Gamma)$-structure, integral submanifolds of $L^{2r} = \mathfrak{Re}[\pi^r \oplus \bar{\pi}^r]$ have a complex analytic structure.*

For the proof in detail, refer to the paper [3]. Here we shall sketch a brief proof, which we need in the proof of the next Theorem 9.

Integral manifolds of L^{2r} can be represented by the equations $w^A = \text{const.}(A = 2r+1, \cdots, n)$ in terms of the canonical coordinate system. If we take an arbitrary pair U, U' of intersecting neighbourhood, admitting canonical coordinate systems $(u^\alpha, v^{\bar{\alpha}}, w'^A)$, and $(u'^\alpha, v'^{\bar{\alpha}}, w'^A)$, then $(u^\alpha, v^{\bar{\alpha}})$ and $(u'^\alpha, v'^{\bar{\alpha}})$ can be regarded as coordinate systems of an integral manifold R of L^{2r} in U and U' respectively, while w^A and w'^A are solutions of (4. 9) then, on an integral manifold R, w'^A can be represented by C^∞-functions of w^{2r+1}, $\cdots$, w^n.

By calculating the components of $\lambda^i_{(\alpha)}$ in a canonical coordinate systems, we have the relation $\lambda^\beta_{(\alpha)} + \sqrt{-1}\,\lambda^{\bar{\beta}}_{(\alpha)} = 0$. In the integral manifold R in $U \cap U'$, the last equation leads us to the well known Cauchy-Riemann's differential equations:

$$\frac{\partial u'^\beta}{\partial u^\gamma} = \frac{\partial v'^{\bar{\beta}}}{\partial v^{\bar{\gamma}}}, \quad \frac{\partial u'^\beta}{\partial v^{\bar{\gamma}}} = -\frac{\partial v'^{\bar{\beta}}}{\partial u^\gamma}. \tag{4. 13}$$

Consequently, z'^α are of the form $z'^\alpha = \Phi^\alpha(z^1, \cdots, z^\gamma)$ and Φ^α are complex analytic. Thus the integral manifold R has a complex analytic structure.

Now we are in a position to show the main theorem of the present paper:

Theorem 9. *A manifold with a $(\pi^r - \Gamma)$-structure admits an f_r-structure f^* induced from π^r and a real symmetric affine connection $\overset{*}{\Gamma}$ with respect to which the relation $\overset{*}{\nabla} f^* = 0$ holds good.*

Proof. By making use of the inverse matrix (σ^β_α) defined in the above, let us take a new basis $\Lambda_{(\alpha)}$ of π^r, which is given by

$$\Lambda_{(\alpha)} = \lambda_{(\beta)} \sigma^\beta_\alpha. \tag{4. 14}$$

It is easy to see that $A^i_{(\alpha)}\partial_i z^\beta=0$, $A^i_{(\alpha)}\partial_i\bar{z}^\beta=\delta^\beta_\alpha$. Using the notations $\partial_i z^\alpha=z^\alpha_i$, $\partial_i w^A=w^A_i$ we have a regular matrix $(z)={}^t(\bar{z}^\alpha_i, z^\alpha_i, w^A_i)$. Then, construct the inverse of the matrix (z), which can be represented by taking account of (4. 14), as the form $(A^i_{(\alpha)}, \bar{A}^i_{(\alpha)}, M^i_{(A)})$. Since $n-2r$ vectors $M^i_{(A)}$ are real and pseudo-normal to the distribution $\pi^r\oplus\bar{\pi}^r$, they construct a real $n-2r$ dimensional distribution M^{*n-2r}. It is easily verified that the distribution M^{*n-2r} is a global one. Thus a special induced f-structure f^* is obtained from the distributions π^r and M^{*n-2r} by the process which was used in the proof of Theorem 1. Of course, from Theorem 7, we have:

$$\nabla f^* f^*=0\,. \tag{4.15}$$

Now, refering to a canonical coordinate system, the matrices (z) and $(z)^{-1}$ have the following components

$$(z)=\begin{pmatrix} E^r, & -\sqrt{-1}\,E^r, & 0 \\ E^r, & \sqrt{-1}\,E^r, & 0 \\ 0\ , & 0\ , & E^{n-2r} \end{pmatrix},$$

$$(z)^{-1}=\begin{pmatrix} \frac{1}{2}E^r, & \frac{1}{2}E^r, & 0 \\ \frac{\sqrt{-1}}{2}E^r, & -\frac{\sqrt{-1}}{2}E^r, & 0 \\ 0\ , & 0\ , & E^{n-2r} \end{pmatrix}.$$

Then, from (2. 3) and (2. 6), the tensors φ^* and f^* are written down in the form

$$\varphi^*=\begin{pmatrix} \frac{1}{2}E^r, & -\frac{\sqrt{-1}}{2}E^r, & 0 \\ \frac{\sqrt{-1}}{2}E^r, & \frac{1}{2}E^r, & 0 \\ 0\ , & 0\ , & 0 \end{pmatrix},$$

$$f^*=\begin{pmatrix} 0\ , & -E^r, & 0 \\ E^r, & 0\ , & 0 \\ 0\ , & 0\ , & 0 \end{pmatrix}.$$

Hence the Nijenhuis tensor N^* corresponding to the tensor f^*, namely,

$$N^{*i}_{jk}=f^{*l}_{k}\partial_l f^{*i}_{j}-f^{*l}_{j}\partial_l f^{*i}_{k}+f^{*i}_{l}\partial_j f^{*l}_{k}-f^{*i}_{l}\partial_k f^{*l}_{j}$$

is identically zero. Since the affine connection Γ is assumed to be symmetric, $N^{*i}_{jk}=0$ is reducible to

$$f^{*l}_{k}\nabla_l f^{*m}_{j}-f^{*l}_{j}\nabla_l f^{*m}_{k}=f^{*m}_{l}\nabla_k f^{*l}_{j}-f^{*m}_{l}\nabla_j f^{*l}_{k}. \tag{4.16}$$

Therefore, if we put $T^{*i}_{jk}=-f^{*m}_{k}\nabla_m f^{*i}_{j}-f^{*i}_{m}\nabla_j f^{*m}_{k}$, and consider an affine connection $\overset{*}{\Gamma}$ given by $\overset{*}{\Gamma}=\Gamma+T^*$, then the connection $\overset{*}{\Gamma}$ is real. Covariant derivative $\overset{*}{\nabla}_k f^{*i}_{j}$ of f^{*i}_{j} with respect to $\overset{*}{\Gamma}$ are written as

$$\overset{*}{\nabla}_k f^{*i}_{j}=\nabla_k f^{*i}_{j}-f^{*m}_{k}\nabla_m f^{*i}_{l}f^{*l}_{j}-f^{*i}_{m}\nabla_l f^{*m}_{k}f^{*l}_{j}$$
$$+f^{*i}_{l}f^{*m}_{k}\nabla_m f^{*l}_{j}+f^{*i}_{l}f^{*l}_{m}\nabla_j f^{*m}_{k},$$

and, from the equations $f^{*3}+f^*=0$, (4.15) and (4.16), we find

$$\overset{*}{\nabla}_k f^{*i}_{j}=\nabla_k f^{*i}_{j}+f^{*i}_{m}[f^{*l}_{k}\nabla_l f^{*m}_{j}-f^{*l}_{j}\nabla_l f^{*m}_{k}]-\nabla_j f^{*i}_{k}$$
$$=0.$$

The symmetricity of $\overset{*}{\Gamma}$ is verified as follows:

$$\overset{*}{\Gamma}{}^{i}_{jk}-\overset{*}{\Gamma}{}^{i}_{kj}=T^{*i}_{jk}-T^{*i}_{kj}=-N^{*i}_{jk}=0.$$

§5. Riemannian manifolds admitting null distributions

We finally turn to a consideration of an n-dimensional Riemannian manifold V^n with a positive definite Riemannian metric g. In the first place, it is assumed that V^n admits a null r-dimensional distribution π^r, i.e. a field of null r planes π^r. The basic vectors $\lambda_{(\alpha)}$ of π^r must satisfy the equations

$$^t\lambda_{(\alpha)}g\lambda_{(\beta)}=0. \qquad (\alpha, \beta=1, 2, \cdots, r) \tag{5.1}$$

Since g is positive definite, the vectors $\lambda_{(\alpha)}$ are complex ones, and the relation $\pi^r\cap\bar{\pi}^r=\{0\}$ holds good automatically in each point of the manifold V^n. Thus the null distribution π^r constructs a π^r-structure.

Now we take a real $(n-2r)$-dimensional distribution M^{n-2r} which is orthogonal to the real $2r$-dimensional distribution $L^{2r}=\mathfrak{Re}[\pi^r\oplus\bar{\pi}^r]$ with respect to the given metric g. Hence the distribution π^r and M^{n-2r} construct an f_r-structure. Thus the manifold V^n admits a positive definite metric g and an f_r-structure f.

As is shown in the equation (2. 7), the distributions L^{2r} and M^{n-2r} are respectively given by the projection tensors

$$l=-f^2 \quad \text{and} \quad m=f^2+I. \tag{5. 2}$$

Definition. *A manifold with an (f_r-g)-structure* (cf. [3].) *is a Riemannian manifold with a positive definite Riemannian metric g and an f_r-structure f satisfying*

$$g=gm+{}'fgf\,. \tag{5. 3}$$

Theorem 10. *In order that a Riemannian manifold V^n admits an (f_r-g)-structure, it is necessary and sufficient that the manifold V^n admits a null r-dimensional distribution with respect to a given Riemannian metric.*

Proof. For the necessity refer to Proposition 2 in the paper [3]. That is to say, the basic vectors $\lambda_{(\alpha)}$ of a complex f-distribution f^r satisfy relations $f\lambda_{(\alpha)}=-\sqrt{-1}\,\lambda_{(\alpha)}$ and $m\lambda_{(\alpha)}=0$. These relations and (5. 3) lead us to ${}'\lambda_{(\alpha)}g\lambda_{(\beta)}={}'\lambda_{(\alpha)}gm\lambda_{(\beta)}+{}'(f\lambda_{(\alpha)})g(f\lambda_{(\beta)})=-{}'\lambda_{(\alpha)}g\lambda_{(\beta)}$. Then ${}'\lambda_{(\alpha)}g\lambda_{(\beta)}=0$ holds good and the distribution f^r is a null r-dimensional distribution.

Conversely, suppose that the manifold V^n admits a null r-dimensional distribution π^r with respect to a given Riemannian metric g, then V^n admits an f_r-structure according to the method described at the beginning of this section. Then it remains for us to prove (5. 3). To do this, we notice that the rank of the matrix $(\sigma_{\alpha\beta})$ defined by

$$\sigma_{\alpha\beta}={}'\lambda_{(\alpha)}g\bar{\lambda}_{(\beta)} \tag{5. 4}$$

is r, at each point of V^n, because of the condition $\pi^r \cap \bar{\pi}^r = \{0\}$.

Therefore we construct the inverse of the matrix $(\sigma_{\alpha\beta})$, which is denoted by $(\tau^{\alpha\beta})$. Hence the projection tensor φ of the distribution π^r has the form $\varphi = \sum_{\alpha} \lambda_{(\alpha)} (\sum_{\beta} g \bar{\lambda}_{(\beta)} \tau^{\beta\alpha})$ by means of (2. 3). Then (2. 6) and (5. 2) lead us to

$$gm + {}'fgf = g - g\varphi - g\bar{\varphi} - {}'\varphi g\varphi + {}'\varphi g\bar{\varphi} + {}'\bar{\varphi} g\varphi - {}'\bar{\varphi} g\bar{\varphi} .$$

Since π^r is null with respect to g, (5. 1) gives ${}'\varphi g\varphi = 0$ and ${}'\bar{\varphi} g\bar{\varphi} = 0$. Moreover, (5. 4) gives ${}'\varphi g\bar{\varphi} = g\bar{\varphi}$. Thus (5. 3) is proved.

Definition. *A manifold with a $(\pi^r - g)$-structure is a Riemannian manifold admitting an r-dimensional distribution which is null and parallel with respect to a given positive definite Riemannian metric g.* (cf. [3].)

We have already established, in the paper [3], the relation of $(f_r - g)$-structure and $(\pi^r - g)$-structure as follows:

Theorem 11. *In order that a manifold V^n admits a $(\pi^r - g)$-structure, it is necessary and sufficient that V^n admits an $(f_r - g)$-structure satisfying the condition $\nabla f = 0$ where ∇ denotes a covariant differentiation with respect to the given metric g.* (cf. [3].)

In the following we shall give a simple proof of this theorem.

Suppose the manifold V^n admits a $(\pi^r - g)$-structure, then it follows, from Theorem 10, that V^n admits a $(f_r - g)$-structure. Since the distribution π^r coincides with the complex f-distribution and is assumed to be parallel with respect to g, Theorem 7 shows

$$\nabla ff = 0 . \tag{5. 5}$$

On the other hand, for an $(f_r - g)$-structure, we have the relation $gf + {}'fg = 0$, (cf. [12].) and hence $f = -g^{-1}{}'fg$. These equations give $\nabla ff = -g^{-1}{}'(f\nabla f)g$, and (5. 5) is rewritten as $f\nabla f = 0$. Then we obtain $\nabla f = 0$, by virtue of the relation $f^3 + f = 0$.

Conversely, suppose V^n admits an $(f_r - g)$-structure satisfying the condition $\nabla f=0$, then V^n admits a null r-dimensional distribution f^r by virtue of Theorem 10. Next, the condition $\nabla f=0$ gives $\nabla\varphi=0$. Hence, Lemma 3 shows us that the null r-dimensional distribution f^r is parallel with respect to g.

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