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Higher composition products

By

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0. If W, X, Y, and Z are topological spaces with base point, and $\alpha\in\Pi(W,X)$, $\beta\in\Pi(X,Y)$, $\gamma\in\Pi(Y,Z)$ are homotopy classes of base point respecting maps such that the compositions $\gamma\circ\beta$ and $\beta\circ\alpha$ are zero in $\Pi(X,Z)$ and $\Pi(W,Y)$ respectively, Toda [11] defined the triple product $\{\gamma,\beta,\alpha\}\subseteq\Pi(SW,Z)$. He has since employed it to great advantage in studying the homotopy groups of spheres ([14], [16]). It has often been noted that the triple product bears a formal resemblance to the Massey triple product. In [7], Massey showed how to define longer products analogous to the Massey products, and in [10] Spanier showed how to define longer products analogous to the triple product. It is the object of this paper to give another definition of longer composition products analogous to the triple product, and to explore some of their properties. The advantage of this definition, as will be seen in Sections 5 and 6, is that it enables us to make certain computations in the homotopy groups of spheres. It is, unfortunately, a very cumbersome definition; it is hoped to give a more categorical approach to it in a later paper. These products seem related to those defined by D. M. Kahn (private communication). Their relation to those defined by Spanier is unclear.

1. In this paper all spaces will have the homotopy type of a

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countable CW complex, and will come equipped with a base point $*$. All maps and homotopies will preserve base point. We shall give two sets of definitions and propositions which will later be related to each other. The two sets are dual in the sense of Hilton-Eckmann [3]. While this duality is only heuristic, each of the proofs of our propositions may be straightforwardly translated into a proof of the dual proposition; hence we give only one proof here. Our numbered propositions and definitions will be expressed in terms of loop spaces, fibrations, etc.; the dual proposition or definition, expressed in terms of cofibrations, suspensions, etc., will bear the same number followed by the letter D.

Let X and Y be topological spaces, and let $f\colon X\to Y$ be a map.

Definition 1.1. The *fibration induced by* f is the fibration p: $P_f\to X$ induced from the path space over Y by f.

Definition 1.1D. The *cofibration induced by* f is the cofibration j_f: $Y\to C_f$ induced from the cofibration i: $X\to CX$, where CX is the cone over X.

We make certain conventions. The path space over a space Y is $PY=\{\lambda\,|\,\lambda\colon I\to Y, \lambda(1)=*\}$. In PY, $*(t)=*$, all t. Thus, $P_f=\{(x, \lambda)\in X\times PY\,|\,fx=\lambda(0)\}$. In P_f, $*=(*, *)$. The cone over X, CX, is $X\times I$ with $X\times\{0\}\cup *\times I$ identified to a point $*$. We denote a point in CX by $\langle x, t\rangle$, the image of (x, t) in $X\times I$. C_f is the space $Y\cup CX$, with $\langle x, 1\rangle$ identified to fx. $*$ is the image of $*$ under this identification. The suspension of X, SX, is CX with $\{\langle x, 1\rangle\}$ identified to $*$; we denote a point in SX by $[x, t]$, the image of $\langle x, t\rangle$.

Definition 1.2. A *tower of fibrations* $T=\{X_1, \cdots, X_n; P_1, \cdots, P_{n+1}; p_2, \cdots, p_{n+1}; f_1, \cdots, f_n\}$ *of height* n *over a space* P_1 is a collection of spaces $X_1, \cdots, X_n$, $P_1, \cdots, P_{n+1}$, and maps f_i: $P_i\to X_i$, $i=1, \cdots, n$, p_{i+1}: $P_{i+1}\to P_i$, $i=1, \cdots, n$ such that p_{i+1}: $P_{i+1}\to P_i$ is a fibration with fibre ΩX_i, and $P_{i+1}\xrightarrow{p_{i+1}}P_i\xrightarrow{f_i}X_i$ is an exact sequence of spaces in the sense of [9], i.e., the sequence

$\Pi(Y, P_{i+1}) \to \Pi(Y, P_i) \to \Pi(Y, X_i)$ is always exact.

Definition 1.2D. A *shaft of cofibrations* $S = \{X_1, \cdots, X_n;\ C_1, \cdots, C_{n+1};\ f_1, \cdots, f_n;\ j_1, \cdots, j_n\}$ *of depth n under* C_1 is a collection of spaces $X_1, \cdots, X_n$, $C_1, \cdots, C_{n+1}$ and maps $f_i : X_i \to C_i$, $j_i : C_i \to C_{i+1}$, $i = 1, \cdots, n$, such that j_i is a cofibration with cofiber SX_i, and the sequence $X_i \xrightarrow{f_i} C_i \xrightarrow{j_i} C_{i+1}$ is coexact in the sense of [9], i.e., $\Pi(C_{i+1}, Y) \to \Pi(C_i, Y) \to \Pi(X_i, Y)$ is always exact.

Definition 1.3. If T is a tower of fibrations of height n, and $f : Y \to P_1$ is a map, a map $\bar{f} : Y \to P_{n+1}$ is called a *lift* of f if $p_2 \circ \cdots \circ p_{n+1} \circ \bar{f} = f$. The corresponding lift to P_i, $\bar{f}_i$, is $p_{i+1} \circ \cdots \circ p_{n+1} \circ \bar{f}$.

Definition 1.3D. If S is a shaft of cofibrations of depth n, and $f : C_1 \to Y$ is a map, a *drop* of f is a map $\bar{f} : C_{n+1} \to Y$ such that $f = \bar{f} \circ j_n \circ \cdots \circ j_1$. The corresponding drop to C_i, $\bar{f}^i$, is $\bar{f} \circ j_n \circ \cdots \circ j_i$.

Given a tower of fibrations of height n over P_1, a map $f : Y \to P_1$, and a lift of f, $\bar{f} : Y \to P_{n+1}$, we define Q_i, $i = 1, \cdots, n+2$, as follows:

$$Q_{i+1} = P_{\bar{f}_i},\ i = 1, \cdots, n+1;\ Q_1 = Y.$$

We define $q_{i+1} : Q_{i+1} \to Q_i$ as the map induced by p_i, i.e.,

$$Q_{i+1} = \{(y, \lambda) \in Y \times PP_i \mid \lambda(0) = \bar{f}_i(y),\ \lambda(1) = *\}.$$

$$q_{i+1}(y, \lambda) = (y, p_i \lambda) \text{ if } i > 1;\ q_2(y, \lambda) = y.$$

We define maps $g_i : Q_{i+1} \to \Omega X_i$, $i = 1, \cdots, n$, $g_1 : Q_1 \to P_1$ as follows. $g_1 = f$. Let $\Pi_{i+1} : Q_{i+1} \to Y$, $i = 1, \cdots, n+1$ be defined by $\Pi_{i+1}(y, \lambda) = y$. Then $p_{i+1} \circ \bar{f}_{i+1} \circ \Pi_{i+1} = \bar{f}_i \circ \Pi_{i+1} \sim *$ by the homotopy $H((y, \lambda), t) = \lambda(t)$ so there is a map $g_i : Q_{i+1} \to \Omega X_i$ such that if $k : \Omega X_i \to P_{i+1}$ is inclusion, $k \circ g_i \sim \bar{f}_{i+1} \circ \Pi_{i+1}$. In general, there is no canonical way to pick g_i; the homotopy classes of any two choices of g_i differ by an element in the image of $\Pi(Q_{i+1}, \Omega P_i)$.

Similarly, given a shaft S of cofibrations of depth n, and a map $f : C_1 \to Y$ and a drop $\bar{f}$, we define $D_1 = Y$, D_{i+1} the cofibration induced by $\bar{f}_i$, $i = 1, \cdots, n+1$, $k_i : D_i \to D_{i+1}$, and maps $g_{i+1} : SX_i \to D_{i+1}$, $i = 1, \cdots, n$, $g_1 : C_1 \to Y$, $g_1 = f$.

Definition 1.4. Given a tower T of fibrations of height n, a map $f: Y \to P_1$, and a lift $\bar{f}: Y \to P_{n+1}$ of f, we define a *tower induced by* $\bar{f}$ as being $T\bar{f} = \{P_1, \Omega X_1, \cdots, \Omega X_n; Q_1, \cdots, Q_{n+2}; q_2, \cdots, q_{n+2}; g_1, \cdots, g_{n+1}\}$ for some choice of the g_i's.

Definition 1.4D. Given a shaft S of cofibrations of depth n, a map $f: C_1 \to Y$, and a drop $\bar{f}: C_{n+1} \to Y$ of f, we define a *shaft induced by* $\bar{f}$, as being $S\bar{f} = \{C_1, SX_1, \cdots, SX_n; D_1, \cdots, D_{n+2}; k_1, \cdots, k_{n+1}; g_1, \cdots, g_{n+1}\}$ for some choice of the g_i's.

Proposition 1.5. *The tower induced by $\bar{f}$ is a tower of fibrations.*

Proposition 1.5D. *The shaft induced by $\bar{f}$ is a shaft of cofibration if the X_i are simply connected.* (See the proof of 1.6 for an explanation of the last condition.)

Proof of 1.5. 1.5 follows immediately from the next lemma, setting $g = \bar{f}_i$ and $\bar{g} = \bar{f}_{i+1}$.

Lemma 1.6. *Suppose $\Pi: E \to B$ is a fibration, with fibre $i: F \to E$, and $g: X \to B$ is a map, $\bar{g}: X \to E$ a map such that $\Pi\bar{g} = g$. Define $\sigma: P_{\bar{g}} \to P_g$ by $\sigma(x, \mu) = (x, \Pi\mu)$. Then, σ is a fibration with fibre ΩF. Define $K: P_g \times I \to B$ by $K((x, \lambda), t) = \lambda(t)$. Then, K is a null homotopy of $g \circ p = \Pi \circ \bar{g} \circ p$* (here $p: P_g \to X$); *let $\bar{K}: P_g \times I \to E$ be a covering homotopy. Let $\bar{K}_1: P_g \to F$ be $\bar{K}((x, \lambda), 1)$. Let $\xi: P_{\bar{K}_1} \to P_g$ be the fibration induced by $\bar{K}_1$. Then, there is a singular homotopy equivalence $n: P_{\bar{K}_1} \to P_{\bar{g}}$ making the diagram*

$$\begin{tikzcd}[column sep=small] & P_{\bar{K}_1} \arrow[dr, "\xi"] \arrow[dd, "n"] & \\ \Omega F \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & P_g \\ & P_{\bar{g}} \arrow[ur, "\sigma"'] & \end{tikzcd}$$

homotopy commute.

Proof: We first show $\sigma: P_{\bar{g}} \to P_g$ is a fibration. Let Y be a space, and suppose $J: Y \times I \to P_g$ and $h: Y \to P_{\bar{g}}$ are maps such that $\sigma h = J(y, 0)$. Note that $J(y, s) = (x(y, s), \lambda(y, s))$, where $x(y, s) = pJ(y, s)$, and $\lambda(y, s): I \to B$ has $\lambda(y, s)(0) = gx(y, s)$, $\lambda(y, s)(1) = *$. Further, $h(y) = (x(y, 0), \mu(y))$, where $\mu(y): I \to E$, $\mu(y)(0) =$

$\bar{g}x(y,0)$, $\mu(y)(1)=*$, $\Pi\mu(y)(t)=\lambda(y,0)(t)$. Define $N: Y\times I\times I \to B$ by $N(y,s,t)=\lambda(y,s)(t)$. Then, $N(Y\times I\times\{1\})=*$, $N(y,s,0) =\lambda(y,s)(0)=gx(y,s)$, $N(y,0,t)=\lambda(y,0)(t)=\Pi\mu(y)(t)$.

Define $\bar{N}_0: Y\times(I\times\{0\}\cup\{0\}\times I\cup I\times\{1\})\to E$ by $\bar{N}_0(y,s,0)= \bar{g}x(y,s)$, $\bar{N}_0(y,0,t)=\mu(y)(t)$, $\bar{N}_0(y,s,1)=*$. Then, $\bar{N}_0$ is well-defined, and $\Pi\bar{N}_0=N|Y\times(I\times\{0\}\cup\{0\}\times I\cup I\times\{1\})$. Hence, $\bar{N}_0$ extends to $\bar{N}: Y\times I\times I\to E$, $\Pi\bar{N}=N$. Define $\mu(y,s)(t)=\bar{N}(y,s,t)$. Then, $\mu(y,s)(0)=gx(y,s)$, $\mu(y,s)(1)=*$. Define $\bar{J}: \bar{Y}\times I\to P_{\bar{g}}$ by $\bar{J}(y,s)=(x(y,s),\mu(y,s))$. Then, $\sigma\bar{J}=J$. Thus, σ is a fibration; the fibre is easily seen to be ΩF.

Next, $P_g=\{(x,\lambda)\mid\lambda: I\to B, \lambda(0)=g(x), \lambda(1)=*\}$, $P_{\bar{g}}=\{(x,\mu)\mid\mu: I\to E, \mu(0)=\bar{g}(x), \mu(1)=*\}$, and $P_{\bar{K}_1}=\{(x,\lambda,\nu)\mid(x,\lambda)\in P_g, \nu: I\to F, \nu(0)=\bar{K}_1(x,\lambda), \nu(1)=*\}$. Define $n: P_{\bar{K}_1}\to P_{\bar{g}}$ by $n(x,\lambda,\nu)=(x,\bar{K}_\lambda\nu)$, where $\bar{K}_\lambda\nu: I\to E$ is given by:

$$\bar{K}_\lambda\nu(t)=\begin{cases}\bar{K}((x,\lambda),2t), & 0\leq t\leq 1/2\\ \nu(2t-1), & 1/2\leq t\leq 1.\end{cases}$$

Then, $\sigma n(x,\lambda,\nu)=(x,\Pi\bar{K}_\lambda\nu)$.

$$\Pi\bar{K}_\lambda\nu(t)=\begin{cases}\lambda(2t), & 0\leq t\leq 1/2\\ *, & 1/2\leq t\leq 1.\end{cases}$$

Hence, $\sigma n\sim\xi$. If $\omega\in\Omega F$, $ni_1(\omega)=n(*,*,\omega)=(*,\bar{K}_*\omega)$.

$$\bar{K}_*\omega(t)=\begin{cases}*, & 0\leq t\leq 1/2\\ \omega(2t-1), & 1/2\leq t\leq 1.\end{cases}$$

$i_2(\omega)=(*,\omega)$. Hence, $i_2\sim ni_1$. That n is a singular homotopy equivalence now follows from the five-lemma applied to the homotopy sequences of the fibrations $\sigma: P_{\bar{g}}\to P_g$, $\nu: P_{\bar{K}_1}\to P_g$, the second being mapped into the first by n. In proving the lemma dual to 1.6, we only have homology isomorphisms. Hence, we require the spaces to be simply connected so that we may obtain singular homotopy equivalences. One can, in fact, by a much longer proof, prove that we actually have strong homotopy equivalences; since this is not needed in the sequel, we omit it.

2. We first indicate roughly our definition of the $(n-1)$-fold composition product. Let $X_1, \cdots, X_n$ be spaces, and $f_i : X_i \to X_{i+1}$, $i=1, \cdots, n-1$, maps. If $f_{n-1} \circ f_{n-2}$ is null-homotopic there is a map $\overline{f_{n-2}}$: $X_{n-2} \to P_{f_{n-1}}$ which projects to f_{n-2}.

We construct the tower of height two over X_{n-2} induced by $\overline{f_{n-2}}$, and see if there is a lift of f_{n-3}. We proceed in this manner, if we can, until we have a tower of height $(n-2)$ over X_2. Let $P_{2,n-2}$ and $P_{2,n-3}$ be the two spaces at the top of this tower; we have a map $f_{2,n-3} : P_{2,n-3} \to \Omega^{n-3} X_n$. We find, if possible, a map $\overline{f_1} : X_1 \to P_{2,n-3}$ which is a lift of f_1. The element of $\Pi(X_1, \Omega^{n-3} X_n)$ represented by $f_{2,n-3} \circ \overline{f_1}$ is an element of the $(n-1)$-fold product $\{f_{n-1}, \cdots, f_1\}$.

Formally, our definitions follow; $X_1, \cdots, X_n$ are spaces, and f_i: $X_i \to X_{i+1}$, as above.

Definition 2.1. An *F-presentation* for the given sequence of spaces and maps is a collection of $(n-2)$-towers of fibrations, one each of height $(n-i)$ over the space X_i, $i=2, \cdots, n-1$, of the form $\{X_{i+1}, \Omega X_{i+2}, \cdots, \Omega^k X_{i+k+1}, \cdots, \Omega^{n-i-1} X_n; P_{i,0}=X_i, P_{i,1}, \cdots, P_{i,n-i}; f_{i,0}=f_i, f_{i,1}, \cdots, f_{i,n-i-1}; p_{i,1}, \cdots, p_{i,n-i}\}$ and maps $\overline{f_{i-1}}$: $X_{i-1} \to P_{i,n-i}$, $i>2$, $\overline{f_1} : X_1 \to P_{2,n-3}$, such that $\overline{f_{i-1}}$ is a lift of f_{i-1}, and, the tower over $X_{i-1} (i>2)$ is induced by $\overline{f_{i-1}}$.

Definition 2.1D. A *C-presentation* for the given set of spaces and maps is a collection of $(n-2)$-shafts of cofibrations, one each of depth $(i-1)$ under X_i, $i=2, \cdots, n-1$, of the form $\{X_{i-1}, SX_{i-2}, \cdots, S^k X_{i-k-1}, \cdots, S^{i-2} X_1; C_{i,0}=X_i, C_{i,1}, \cdots, C_{i,i-1}; g_{i,0}=f_{i-1}, g_{i,1}, \cdots, g_{i,i-2}; j_{i,0}, \cdots, j_{i,i-2}\}$, and maps $\overline{g_i} : C_{i,i-1} \to X_{i+1}$, $i<n-1$, $\overline{g_{n-1}} : C_{n-1,n-3} \to X_n$ so that each $\overline{g_i}$ is a drop of f_i, and the shaft under X_i is induced by $\overline{g_{i-1}}$. (In discussing C-presentations, we always assume the X_i simply connected.)

Definition 2.2. *The $(n-1)$-fold F-composition product* $\{f_{n-1}, \cdots, f_1\}_F$ exists if there exists an F-presentation for the given set of spaces and maps. It consists of the set of homotopy classes of all compositions $f_{2,n-3}\circ\overline{f_1}$ which occur in such an F-presentation. Thus, it is a subset of $\Pi(X_1, \Omega^{n-3}X_n)$.

We shall also refer to $\{f_{n-1}, \cdots, f_1\}_F$ for a particular F-presentation, by which we mean the set of homotopy classes of all $f_{2,n-3}\circ\overline{f_1}$, where we have fixed the tower over X_2, and allowed $\overline{f_1}$ to vary over all lifts of f_1 to $P_{2,n-3}$.

Definition 2.2D. The $(n-1)$-*fold C-composition product* $\{f_{n-1}, \cdots, f_1\}_C$ exists if there exists a C-presentation for the given set of spaces and maps. It consists of the set of all homotopy classes of compositions $\overline{g_{n-1}}\circ g_{n-1,n-3}$ which occur in such presentations, and is thus a subset of $\Pi(S^{n-3}X_1, X_n)$.

We shall also refer to $\{f_{n-1}, \cdots, f_1\}_C$ for a particular C-presentation, by which we mean the set of homotopy classes of all $\overline{g_{n-1}}\circ g_{n-1,n-3}$, where we have fixed the shaft under X_{n-1}, and allowed $\overline{g_{n-1}}$ to vary over all drops of f_{n-1}.

The construction of $\{f_{n-1}, \cdots, f_1\}_F$ for a given presentation may be looked at in at least two different ways. (A similar discussion holds for $\{f_{n-1}, \cdots, f_1\}_C$ and is omitted here.) In the first, the maps $f_{i,j}$ may almost be disregarded. Thus, we start with a lift of f_{n-2} to $P_{n-1,1}$ if one exists, and build $P_{n-2,2}$ and $P_{n-2,1}$. If there exists a lift of f_{n-3} to $P_{n-2,2}$, we can build the $P_{n-3,j}$'s. Continuing in this way, we need not consider, or bother to construct, a map $f_{i,j}$ until we construct $f_{2,n-3}$, having constructed $P_{2,n-3}$ by a choice of $\overline{f_3}$, over f_3. The choice of $f_{2,n-3}$ then fixes $\{f_{n-1}, \cdots, f_1\}_F$ for this presentation.

On the other hand, if, for example, the X_i are double suspensions, we may visualize the construction as one that consists of looking at many spectral sequences. To see this, note that a tower of fibrations determines an exact couple of spaces in the sense of [9]. The tower should be extended to infinity in both directions. To do this, using the notation of 1.2, we set $P_i = * = X_i$, $i \leq 0$, so that $P_1 \to P_0$ is a fibration with fibre P_1, and $P_1 \to P_0 \to X_0$ is exact. We set $X_{n+1} = P_{n+1}$, f_{n+1} the identity map, $P_{n+i} = PP_{n+1}$, $i \geq 2$, $p_{n+i} =$ identity, $i \geq 3$, and $X_{n+i} = *$, $i \geq 2$. Mapping a double suspension Y into this exact couple gives us an exact couple of groups, and hence a spectral sequence. The exact couple is displayed in diagram I, in which the groups A_i are simply $\Pi(Y, X_i)/im\Pi(Y, P_i)$, and are put in to extend the exact couple to the right.

$$
\begin{array}{ccccccccccccc}
 & \vdots & & & & \vdots & & & & \vdots & & & \\
 & \downarrow & & & & \downarrow & & & & \downarrow & & & \\
\longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \to & 0 & \to \\
 & \downarrow & & & & \downarrow & & & & \downarrow & & & \\
\to & \Pi(Y, \Omega P_{n+1}) & \xrightarrow{1} & \Pi(Y, \Omega P_{n+1}) & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \to & 0 & \to \\
 & \downarrow & & & & \downarrow & & & & \downarrow & & & \\
\to & \Pi(Y, \Omega P_n) & \longrightarrow & \Pi(Y, \Omega X_n) & \longrightarrow & \Pi(Y, P_{n+1}) & \xrightarrow{1} & \Pi(Y, P_{n+1}) & \to & 0 & \to & 0 & \to \\
 & \downarrow & & & & \downarrow & & & & \downarrow & & & \\
\to & \Pi(Y, \Omega P_{n-1}) & \to & \Pi(Y, \Omega X_{n-1}) & \to & \Pi(Y, P_n) & \longrightarrow & \Pi(Y, X_n) & \longrightarrow & A_n & \xrightarrow{1} & A_n & \to \\
 & \downarrow & & & & \downarrow & & & & \downarrow{\scriptstyle 0} & & & \\
 & \vdots & & & & \vdots & & & & \vdots & & & \\
 & \downarrow & & & & \downarrow & & & & \downarrow{\scriptstyle 0} & & & \\
\to & \Pi(Y, \Omega P_2) & \longrightarrow & \Pi(Y, \Omega X_2) & \longrightarrow & \Pi(Y, P_3) & \longrightarrow & \Pi(Y, X_3) & \longrightarrow & A_3 & \xrightarrow{1} & A_3 & \to \\
 & \downarrow & & & & \downarrow & & & & \downarrow{\scriptstyle 0} & & & \\
\to & \Pi(Y, \Omega P_1) & \longrightarrow & \Pi(Y, \Omega X_1) & \longrightarrow & \Pi(Y, P_2) & \longrightarrow & \Pi(Y, X_2) & \longrightarrow & A_2 & \xrightarrow{1} & A_2 & \to \\
 & \downarrow & & & & \downarrow & & & & \downarrow{\scriptstyle 0} & & & \\
\longrightarrow & 0 & \longrightarrow & \Pi(Y, P_1) & \xrightarrow{1} & \Pi(Y, P_1) & \longrightarrow & \Pi(Y, X_1) & \longrightarrow & A_1 & \xrightarrow{1} & A_1 & \to \\
 & \downarrow & & & & \downarrow & & & & \downarrow & & & \\
\longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \to & 0 & \to \\
 & \downarrow & & & & \downarrow & & & & \downarrow & & & \\
 & \vdots & & & & \vdots & & & & \vdots & & &
\end{array}
$$

Diagram I

In constructing $\{f_{n-1}, \cdots, f_1\}_F$, one may construct the exact couple obtained from mapping X_{i-1} into the tower over X_i. One then considers the class of f_{i-1} in $\Pi(X_{i-1}, X_i)$, a group occurring in the E_1 term. If all the differentials through the $(n-i)$-th vanish on this class, there is a lift of f_{i-1} to $P_{i, n-i}$; one chooses such a lift, constructs the tower over X_{i-1}, and continues in this way until the tower over X_2 is constructed. $\{f_{n-1}, \cdots, f_1\}_F$ for this presentation is then the image of the class of f_1 under the $(n-2)$-th differential in the spectral sequence obtained from mapping X_1 into the exact couple of spaces derived from the tower over X_2.

Questions about the meaning of other differentials in such spectral sequences, and about these products for maps of spectra may be answered easily by considering what happens when you take the loop space of all the spaces occurring in an F-presentation, and the loops of all the maps, or the suspensions of all the spaces and maps occurring in a C-presentation. This is done in lemmas 2.3 and 2.3D below, in which it is shown that, up to sign, a presentation for $\{\Omega f_{n-1}, \cdots, \Omega f_1\}_F$ or $\{Sf_{n-1}, \cdots, Sf_1\}_C$ is obtained. With the appropriate sign conventions, we could define these products for maps of spectra; however, our examples, in the homotopy groups of spheres, will simply be computed by taking a sufficiently high dimensional sphere.

Lemma 2.3. *Suppose* $\{f_{n-1}, \cdots, f_1\}_F$ *exists. Then* $\{\Omega f_{n-1}, \cdots, \Omega f_1\}$ *exists, and* $\Omega\{f_{n-1}, \cdots, f_1\}_F \subseteq (-1)^n \{\Omega f_{n-1}, \cdots, \Omega f_1\}_F$.

Lemma 2.3D. *Suppose* $\{f_{n-1}, \cdots, f_1\}_C$ *exists. Then,* $\{Sf_{n-1}, \cdots, Sf_1\}_C$ *exists, and* $S\{f_{n-1}, \cdots, f_1\}_C \subseteq (-1)^n \{Sf_{n-1}, \cdots, Sf_1\}_C$.

Proof of 2.3. This is an easy induction on n; the basic observation is of the changes in sign that occur when one takes loops in the situation of 1.6. This observation is used to prove the following statement, inductively.

(2.3) Let $\{P_{i,j}, f_{i,j}, p_{i,j}, \bar{f}_i\}$ be an F-presentation for $\{f_{n-1}, \cdots, f_1\}$. Then, there is an F-presentation $\{Q_{i,j}, g_{i,j}, q_{i,j}, \bar{g}_i\}$ for $\{\Omega f_{n-1}, \cdots, \Omega f_1\}$ and homeomorphisms $\rho_{i,j} : Q_{i,j} \to \Omega P_{i,j}$

such that the diagrams
$$\begin{tikzcd} & \Omega P_{i+1,n-i-1} \arrow[d, "\rho_{i+1,n-i-1}"] \\ \Omega X_i \arrow[ur, "\Omega \bar{f}_i"] \arrow[r, "\bar{g}_i"'] & Q_{i+1,n-i-1} \end{tikzcd}$$
commute,

the diagrams
$$\begin{tikzcd} \Omega P_{i,j} \arrow[d, "\Omega p_{i,j}"] & Q_{i,j} \arrow[l, "\rho_{i,j}"'] \arrow[d, "q_{i,j}"] \\ \Omega P_{i,j-1} & Q_{i,j-1} \arrow[l, "\rho_{i-1,j}"] \end{tikzcd}$$
commute,

and the diagrams
$$\begin{tikzcd} Q_{i,j} \arrow[d, "\rho_{i,j}"'] \arrow[dr, "g_{i,j}"] & \\ \Omega P_{i,j} \arrow[r, "(-1)^{n-i}\Omega f_{i,j}"'] & \Omega^{i+1}X_{i+j+1} \end{tikzcd}$$
commute,

where $(-1)^{n-i}\Omega f_{i,j}=\Omega f_{i,j}$ if $n-i$ is even, and equals $\sigma\circ\Omega f_{i,j}$ if $(n-i)$ is odd, where, if λ is a loop, $\sigma(\lambda)(t)=\lambda(1-t)$.

Lemma 2.4. *Given an F-presentation for $\{f_{n-1},\cdots,f_1\}$ as in (2.3), let $k:\Omega P_{i+1,j-1}\to P_{i,j}$ be the inclusion of $\Omega P_{i+1,j-1}$ as the fibre of $P_{i,j}$ over X_i. Then, $f_{i,j}\circ k$ is, up to homotopy, $\Omega f_{i+1,j-1}$.*

Lemma 2.4D. *Given a C-presentation for $\{f_{n-1},\cdots,f_1\}$ let $k:C_{i,j}\to SC_{i-1,j-1}$ be the projection of the cofibration $X_i\to C_{i,j}$. Then, $k\circ g_{i,j}$ is, up to homotopy, $Sg_{i-1,j-1}$.*

Proof of 2.4. 2.4 again follows from observing the situation of 1.6.

The next proposition follows immediately from (2.3) and 2.4.

Proposition 2.5. *If $\{f_{n-1},\cdots,f_1\}_F$ exists, for a given F-presentation $\{f_{n-1},\cdots,f_1\}_F$ fills up a coset of the subgroup $(-1)^{n-1}\{\Omega f_{n-1},\cdots,\Omega f_3,(X_1,\Omega X_3)\}_F$, where this symbol means all elements of $\Pi(X_1,\Omega^{n-3}X_n)$ which occur in $(-1)^{n-1}\{\Omega f_{n-1},\cdots,\Omega f_3,g\}_F$, where $g:X_1\to\Omega X_3$ is some map such that $\{\Omega f_{n-1},\cdots,\Omega f_3,g\}_F$ exists for the presentation of* (2.3), *and where this symbol means $\{\Omega f_{n-1},\cdots,\Omega f_3,g\}_F$ for that presentation.* (If $n=4$, this is a right coset).

Proposition 2.5D. *If $\{f_{n-1},\cdots,f_1\}_C$ exists, for a given C-presentation $\{f_{n-1},\cdots,f_1\}_C$ fills up a coset of $(-1)^{n-1}\{(SX_{n-2},X_n),Sf_{n-3},\cdots,Sf_1\}_C$, where the definition of this object is dual to that of 2.5.* (If $n=4$, this is a left coset.)

3. In this section we prove two results: first, that if $\{f_{n-1},\cdots,f_1\}_F$ exists, so does $\{f_{n-1},\cdots,f_1\}_C$ (and vice-versa), and that if one contains zero, so does the other, and second that $\{f_{n-1},\cdots,f_1\}_F$ or $\{f_{n-1},\cdots,f_1\}_C$ is independent of the f_i up to homotopy.

Proposition 3. 1. *If* $\{f_{n-1}, \cdots, f_1\}_F \subseteq \Pi(X_1, \Omega^{n-3}X_n)$ *exists, and if the* X_i *are simply connected, then* $\{f_{n-1}, \cdots, f_1\}_C$ *exists. If* $\{f_{n-1}, \cdots, f_1\}_F$ *contains zero, so does* $\{f_{n-1}, \cdots, f_1\}_C$.

Proof: This follows immediately from lemma 3. 3 below.

The restriction to simply connected X_i is actually unnecessary, but the proof in that case is exceedingly long and technically complicated (though conceptually not difficult), and, since the condition on the X_i will be satisfied in our examples, we shall assume it here.

Proposition 3. 1D. *If* $\{f_{n-1}, \cdots, f_1\}_C \subseteq \Pi(S^{n-3}X_1, X_n)$ *exists, then so does* $\{f_{n-1}, \cdots, f_1\}_F$, *and if* $\{f_{n-1}, \cdots, f_1\}_C$ *contains zero, so does* $\{f_{n-1}, \cdots, f_1\}_F$.

Theorem 3. 2. *If the* X_i *are simply connected, then* $\{f_{n-1}, \cdots, f_1\}_F$ *exists if and only if* $\{f_{n-1}, \cdots, f_1\}_C$ *does, and one contains zero if and only if the other does.*

Lemma 3. 3. *Given an F-presentation* $\{P_{i,j}, f_{i,j}, p_{i,j}, \overline{f}_i\}$ *for the sequence* $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n$, *there exists a C-presentation* $\{C_{i,k}, g_{i,k}, j_{i,k}, \overline{g}_i\}$ *for this sequence and maps* $\theta_{i,k} : S^k P_{i,k} \to C_{i+k,k}$, $2 \leq i \leq n-1$, $0 \leq k \leq n-i-1$, *such that* $\theta_{i,0} : X_i = P_{i,0} \to X_i = C_{i,0}$ *is the identity, and such that if we denote by* $\overline{f}_{i,j} : X_i \to P_{i+1,j}$ *the composition* $p_{i+1,j+1} \circ p_{i+1,j+2} \circ \cdots \circ p_{i+1,n-(i+1)} \circ \overline{f}_i$, *and by* $\overline{g}_{i,k} : C_{i,k} \to X_{i+1}$ *the composition* $\overline{g}_i \circ j_{i,i-2} \circ \cdots \circ j_{i,k}$, *then the diagrams*

$$\begin{tikzcd} & S^kP_{i,k} \arrow[d, "\theta_{i,k}"] \\ S^kX_{i-1} \arrow[ur, "S^k\overline{f}_{i-1,k}"] \arrow[dr, "g_{i+k,k}"'] & \\ & C_{i+k,k} \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} S^kP_{i,k} \arrow[d, "\theta_{i,k}"] \arrow[dr, "Tf'_{i,k}"] & \\ & X_{i+k+1} \\ C_{i+k,k} \arrow[ur, "\overline{g}_{i+k,k}"'] & \end{tikzcd}$$

homotopy commute for $2 \leq i \leq n-2$, $1 \leq k \leq n-i-1$, *where* $f'_{i,k}$ *is (possibly) another choice of* $f_{i,k}$ (see the discussion in section 2). *Further,* $\{f_{n-1}, \cdots, f_1\}_C$ *for this presentation contains zero if* $\{f_{n-1}, \cdots, f_1\}_F$ *for the given presentation does.*

Lemma 3.3D. *If $\{C_{i,k}, g_{i,k}, j_{i,k}, \bar{g}_i\}$ is a C-presentation for a sequence of spaces and maps $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n$, then there exists an F-presentation $\{P_{i,k}, f_{i,k}, p_i, \bar{f}_i\}$ for this sequence and maps $\xi_{i,k}$: $P_{i-k,k} \to \Omega^k C_{i,k}$, $2 \leq i \leq n-1$, $0 \leq k \leq i-2$, such that $\xi_{i,0}$ is the identity and such that the diagrams*

$$\begin{tikzcd} & P_{i-k,k} \arrow[d, "\xi_{i,k}"] \\ X_{i-k-1} \arrow[ur, "\bar{f}_{i-k-1,k}"] \arrow[r, "T^{-1}g'_{i,k}"'] & \Omega^k C_{i,k} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} P_{i-k,k} \arrow[d, "\xi_{i,k}"'] \arrow[dr, "f_{i-k,k}"] & \\ \Omega^k C_{i,k} \arrow[r, "\Omega^k \bar{g}_{i,k}"'] & \Omega^k X_{i+1} \end{tikzcd}$$

homotopy commute, $3 \leq i \leq n-1$, $1 \leq k \leq i-2$, where the $g'_{i,k}$ are (possibly) other choices for the $g_{i,k}$. Further, $\{f_{n-1}, \cdots, f_1\}_F$ for this presentation contains zero if $\{f_{n-1}, \cdots, f_1\}_C$ for the given presentation does.

Proof of 3.3. The proof proceeds by induction on n, starting with $n=4$. The induction hypotheses follow, both on the assumption that we are given an F-presentation for $\{f_{n-1}, \cdots, f_1\}_F$.

A) There exists a C-presentation for $\{f_{n-1}, \cdots, f_1\}_C$ and maps $\theta_{i,k}$ as in the hypothesis of the lemma. In addition, the diagrams below homotopy commute,

$$\text{(A)} \qquad \begin{tikzcd} S^k P_{i,k} \arrow[d, "S^k p_{i,k}"'] \arrow[dr, "\theta_{i,k}"] & \\ S^k X_i & C_{i+k,k} \arrow[l, "\eta_{i+k,k}"] \end{tikzcd}$$

where $p_{i,k}$ is the projection of $P_{i,k}$ on X_i, and $\eta_{i+k,k}$ is the map of $C_{i+k,k}$ on the cofibre.

B) The $P_{i,k}, f_{i,k}$ and $p_{i,k}$ for $3 \leq i \leq n$, and the $\bar{f}_i$ for $2 \leq i \leq n-1$, form an F-presentation for the sequence $X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} X_n$. Suppose that we are given a C-presentation $\{C_{i,k}, g_{i,k}, j_{i,k}, \bar{g}_i\}$ for this sequence, and maps $\theta_{i,k}: S^k P_{i,k} \to C_{i+k,k}$ satisfying A). Then, there is a C presentation for $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n$, and maps $\theta_{i,k}: S^k P_{i,k} \to C_{i+k,k}$, satisfying A), and coinciding with the given ones when both are defined. (Essentially, condition B) says that a C-presentation for $X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n$ satisfying A) for the given F-presentation restricted to this sequence extends to a C-presentation for $X_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} X_n$ satisfying A)).

The case $n=4$ is easy. Here, B) is vacuous, since $\theta_{i,k}$ is only defined if $2 \leq i \leq n-1$, $0 \leq k \leq n-i-1$, and if $n=3$ we get $i=2$, $k=0$ in which case $\theta_{2,0}$ is the identity and there is nothing to extend.

To prove A) for $n=4$, we observe that $P_{2,1}$ is the fibre of the map $f_2: X_2 \to X_3$ and that $C_{3,1}$ is the cofibre of the same map. We choose for $\theta_{2,1}: SP_{2,1} \to C_{3,1}$ the usual map of the suspension of the fibre into the cofibre [5]. Thus, $P_{2,1}=\{(x, \lambda) \mid \lambda: I \to X_3, x \in X_2, \lambda(0)=f_2(x), \lambda(1)=*\}$, and $C_{3,1}=X_3 \cup_{f_2} CX_2$.

$$\theta_{2,1}[(x, \lambda), t] = \begin{cases} \langle x, 2t \rangle, & 0 \leq t \leq 1/2 \\ \lambda(2t-1), & 1/2 \leq t \leq 1. \end{cases}$$

The map $f_{2,1}: P_{2,1} \to \Omega X_4$ is given (at least up to homotopy) by some null-homotopy J of $f_3 f_2$ as

$$(f_{2,1}(x, \lambda))(t) = \begin{cases} J(x, 1-2t), & 0 \leq t \leq 1/2 \\ f_3 \lambda(2t-1), & 1/2 \leq t \leq 1. \end{cases}$$

Thus,
$$Tf_{2,1}[(x, \lambda), t] = \begin{cases} J(x, 1-2t), & 0 \leq t \leq 1/2 \\ f_3 \lambda(2t-1), & 1/2 \leq t \leq 1. \end{cases}$$

We use the same J in constructing $\bar{g}_3 = \bar{g}_{3,1}: C_{3,1} \to X_4$; thus, $g_{3,1}(x) = f_3(x)$, $x \in X_3$; $\bar{g}_{3,1}\langle x, t \rangle = J(x, 1-t)$, $\langle x, t \rangle \in CX_2$.

Then,
$$\bar{g}_{3,1} \circ \theta_{2,1}[(x, \lambda), t] = \begin{cases} J(x, 1-2t), & 0 \leq t \leq 1/2 \\ f_3 \lambda(2t-1), & 1/2 \leq t \leq 1, \end{cases}$$

so that $\bar{g}_{3,1} \circ \theta_{2,1} = Tf_{2,1}$ (again, at least up to homotopy). $\bar{f}_1: X_1 \to P_{2,1}$ is obtained from a null homotopy H of $f_2 f_1$; $\bar{f}_{1,1} = \bar{f}_1$, and $\bar{f}_1(x) = (f_1(x), H_x)$, where $H_x(t) = H(x, t)$. We use the same homotopy to construct $g_{3,1}: SX_1 \to C_{3,1}$. Thus,

$$g_{3,1}[x, t] = \begin{cases} \langle f_1(x), 2t \rangle, & 0 \leq t \leq 1/2 \\ H(x, 2t-1), & 1/2 \leq t \leq 1. \end{cases}$$

Then, $\theta_{2,1}\circ S\overline{f_{1}}_{,1}[x,t]=\theta_{2,1}[(f_{1}(x),H_{x}),t]=g_{3,1}[x,t]$. The homotopy commutativity of (A) follows easily.

Thus, the lemma is true for $n=4$. Assume, then that A) and B) are true for the case $(n-1)$, $(n-1)\geq 4$, and that we are given an F-presentation for $X_{1}\xrightarrow{f_{1}}X_{2}\xrightarrow{f_{2}}\cdots\xrightarrow{f_{n-1}}X_{n}$. The $P_{i,k}$ and appropriate maps for $i\geq 3$ form an F-presentation for $X_{2}\xrightarrow{f_{2}}\cdots\xrightarrow{f_{n-1}}X_{n-1}$. (Note that this includes $\overline{f}_{2,n-4}:X_{2}\to P_{2,n-4}$, which serves as the $\overline{f_{1}}$ of this presentation.) By induction hypothesis A), we may construct a C-presentation and maps satisfying A) for this F-presentation. We consider any such C-presentation and shall extend it (in the sense of hypothesis B)) to a C-presentation for $X_{1}\xrightarrow{f_{1}}\cdots\xrightarrow{f_{n-1}}X_{n}$ satisfying A). The remainder of the proof may thus be thought of as a proof of induction hypothesis B).

The F-presentation for $X_{1}\to\cdots\to X_{n}$ also gives F-presentations for $X_{1}\to\cdots\to X_{n-1}$ and $X_{2}\to\cdots\to X_{n-1}$. The C-presentation for $X_{2}\to\cdots\to X_{n}$ gives a C-presentation for $X_{2}\to\cdots\to X_{n-1}$, and appropriate maps which satisfy A) for the F-presentation for $X_{2}\to\cdots\to X_{n-1}$. Then, by B), this C-presentation extends to a C-presentation for $X_{1}\to\cdots\to X_{n-1}$.

The C-presentations for $X_{2}\to\cdots\to X_{n}$ and $X_{1}\to\cdots\to X_{n-1}$ combine to give us almost all of a C-presentation for $X_{1}\to\cdots\to X_{n}$; in fact, we now have all the $C_{i,k}$ except for $(i,k)=(n-1,n-3)$; we have maps $g_{i,k}:S^{k}X_{i-k-1}\to C_{i,k}$, maps $\overline{g}_{i,k}:C_{i,k}\to X_{i+1}$ for $i<n-1$, and maps $\theta_{i-k,k}:S^{k}P_{i-k,k}\to C_{i,k}$ making the appropriate diagrams homotopy commute. The choice of the space $C_{n-1,n-3}$ is now determined; it is the cofibre of the map $\overline{g}_{n-2,n-4}:C_{n-2,n-4}\to X_{n-1}$. We first construct $\theta_{2,n-3}$. Since the maps $f'_{i,j}$ and $g_{i,j}$ are only defined up to homotopy, we may, by changing our choice of them, assume that the diagrams of the lemma's conclusion actually commute.

Consider the diagram II. The top line

$$
\begin{tikzcd}
 & \Pi(P_{2,n-3},\ P_{2,n-4}) \arrow[r, "f'_{2,n-4*}"] \arrow[d, "S^{n-4}"] & \Pi(P_{2,n-3},\ \Omega^{n-4}X_{n-1}) \arrow[d, "T"] \\
\Pi(S^{n-4}P_{2,n-4},\ Tf'_{2,n-4}) \arrow[r, "\partial_*"] \arrow[d, "\theta_{2,n-4*}"] & \Pi(S^{n-4}P_{2,n-3},\ S^{n-4}P_{2,n-4}) \arrow[r, "Tf'_{2,n-4*}"] \arrow[d, "\theta_{2,n-4*}"] & \Pi(S^{n-4}P_{2,n-3},\ X_{n-1}) \\
\Pi(S^{n-4}P_{2,n-3},\ \overline{g}_{n-2,n-4}) \arrow[r, "\partial_*"] & \Pi(S^{n-4}P_{2,n-3},\ C_{n-2,n-4}) \arrow[ur, "\overline{g}_{n-2,n-4*}"'] &
\end{tikzcd}
$$

Diagram II

is self-evident; the second line is the mapping sequence of the map $Tf'_{2,n-4}$ with "contraefficients" in $P_{2,n-3}$ [3], and the third line is the mapping sequence of the map $\overline{g}_{n-2,n-4}$ with "contraefficients" in $P_{2,n-3}$. We start with the homotopy class of $p_{2,n-3}: P_{2,n-3} \to P_{2,n-4}$ in $\Pi(P_{2,n-3}, P_{2,n-4})$. Let this class be α. $f'_{2,n-4} \circ p_{2,n-3}$ is null-homotopic, so that $f'_{2,n-4*}\alpha = 0$, and $Tf'_{2,n-4*}S^{n-4}\alpha = 0$. Hence, $Tf'_{2,n-4*}S^{n-4}\alpha$ is in the image of $\Pi(S^{n-4}P_{2,n-4}, Tf'_{2,n-4})$. Let β be an element such that $\partial_*\beta = S^{n-4}\alpha$. Consider $\theta_{2,n-4*}\beta \in \Pi(S^{n-4}P_{2,n-3}, \overline{g}_{n-2,n-4})$. If η_*: $\Pi(S^{n-4}P_{2,n-3}, g_{n-2,n-4}) \to \Pi(S^{n-3}P_{2,n-3}, C_{n-1,n-3})$ associates to each map the corresponding map of cofibres, we choose $\theta_{2,n-3}$ as a representative of $\eta_*\beta$. Thus, we have many choices of $\theta_{2,n-3}$; we shall construct appropriate maps corresponding to each choice.

It is clear from the construction of $\theta_{2,n-3}$ that the diagram (A) homotopy commutes.

Next, we construct $\overline{g}_{n-1,n-3}$. Consider Diagram III. In this diagram, rectangle I represents $\theta_{2,n-4*}\beta$, and hence commutes; rectangle VI constructs

$$
\begin{tikzcd}
S^{n-4}P_{2,n-3} \arrow[r, "S^{n-4}p_{2,n-3}"] \arrow[d, "i"'] \arrow[dr, phantom, "\mathrm{I}"] & S^{n-4}P_{2,n-4} \arrow[r, "\theta_{2,n-4}"] \arrow[d, "Tf'_{2,n-4}"', "\mathrm{II}"] & C_{n-2,n-4} \arrow[r, "\eta_{n-2,n-4}"] \arrow[dl, "\overline{g}_{n-2,n-4}"'] \arrow[dr, phantom, "\mathrm{III}"] & S^{n-4}X_2 \arrow[r, "S^{n-4}\overline{f}_{2,n-4}"] \arrow[d, "g_{n-1,n-4}"'] \arrow[dr, phantom, "\mathrm{IV}"] & S^{n-4}P_{3,n-4} \arrow[d, "Tf'_{3,n-4}", "\mathrm{V}"'] \arrow[dl, "\theta_{3,n-4}"'] \\
CS^{n-4}P_{2,n-3} \arrow[r, "b"] \arrow[d, "\eta"'] \arrow[dr, phantom, "\mathrm{VI}"] & X_{n-1} \arrow[rr, "j"] \arrow[d, "j"', "\mathrm{VII}"] & & C_{n-1,n-4} \arrow[r, "\overline{g}_{n-1,n-4}"] \arrow[dll, "j_{n-1,n-4}"] & X_n \\
S^{n-3}P_{2,n-3} \arrow[r, "\theta_{2,n-3}"] & C_{n-1,n-3} & & &
\end{tikzcd}
$$

Diagram III

$\theta_{2,n-3}$ from $\theta_{2,n-4*}\beta$, and hence commutes; triangles II, IV, and V commute by the induction hypothesis and the remark above; triangle VII commutes by definition, and square III homotopy commutes by the construction of a C-presentation. By the homotopy commutativity of (A), the composition $\eta_{n-2,n-4}\circ\theta_{2,n-4}\circ S^{n-4}p_{2,n-3}$ is homotopic to the $(n-4)$-fold suspension of the projection $p: P_{2,n-3}\to X_2$. The map $Tf'_{3,n-4}\circ\bar{f}_{2,n-4}$ is null-homotopic since the F-presentation is an F-presentation of $\{f_{n-1},\cdots,f_1\}$. We may use a null homotopy of this composition to extend the map $\bar{g}_{n-1,n-4}\circ j$ to the cofibre of $\bar{g}_{n-2,n-4}$, i.e., $C_{n-1,n-3}$, and to extend the map $\bar{g}_{n-1,n-4}\circ j\circ b$ to the cofibre of i, i.e., $S^{n-3}P_{2,n-3}$. It follows from the observation above that $\eta_{n-2,n-4}\circ\theta_{2,n-4}\circ S^{n-4}p_{2,n-3}$ is homotopic to $S^{n-4}p$ and from the construction of an F-presentation that the extension of $\bar{g}_{n-1,n-4}\circ j\circ b$ is the adjoint of a map $f'_{2,n-3}: P_{2,n-3}\to\Omega^{n-3}X_n$ which may occur in an F-presentation. The commutativity of the diagram

$$\begin{tikzcd}
S^{n-3}P_{2,n-3} \arrow[r, "Tf'_{2,n-3}"] \arrow[d, "\theta_{2,n-3}"] & X_n \\
C_{n-1,n-3} \arrow[ur, "\bar{g}_{n-1,n-3}"'] &
\end{tikzcd}$$

where $\bar{g}_{n-1,n-3}$ is the extension of $\bar{g}_{n-1,n-4}\circ j$, now follows immediately.

In order to construct $g_{n-1,n-3}: S^{n-3}X_1\to C_{n-1,n-3}$ consider Diagram IV. To construct a

$$\begin{tikzcd}
\Pi(S^{n-4}X_1,\bar{g}_{n-2,n-4}) \arrow[r] \arrow[d, "\eta'_*"] & \Pi(S^{n-4}X_1, C_{n-2,n-4}) \arrow[r, "\bar{g}_{n-2,n-4*}"] & \Pi(S^{n-4}X_1, X_{n-1}) \\
\Pi(S^{n-3}X_1, C_{n-1,n-3}) & &
\end{tikzcd}$$

Diagram IV

$g_{n-1,n-3}$ we start with the class of $g_{n-2,n-4}$ in $\Pi(S^{n-4}X_1, C_{n-2,n-4})$, see if it goes into zero under $\bar{g}_{n-2,n-4*}$, pull it back to $\Pi(S^{n-4}X_1,\bar{g}_{n-2,n-4})$ if it does, and project to a map of cofibres. However, we can map the bottom sequence of Diagram II into the top line of Diagram IV by $\bar{f}^*_{1,n-3}$. Under this map, the element $\theta_{2,n-4*}S^{n-4}\alpha$ goes into the class of $g_{n-2,n-4}$ by our induction hypothesis. Thus, $f^*_{1,n-3}\theta_{2,n-4*}\beta$ is a pull-back of the class of $g_{n-2,n-4}$, and its projection into a map of cofibres gives the desired $g_{n-1,n-3}$.

If $\{f_{n-1},\cdots,f_1\}_F$ contains zero, we may choose an F-presentation such that $f_{2,n-3}\circ\overline{f}_{1,n-3}$ is null-homotopic for any choice of $f_{2,n-3}$ (see the discussion of section 2). Then, in the C-presentation constructed to satisfy our induction hypotheses, $\overline{g}_{n-1,n-3}\circ g_{n-1,n-3}\sim Tf'_{2,n-3}\circ S^{n-3}\overline{f}_{1,n-3}$, which is null-homotopic. This concludes the proof of 3. 3.

Next, we prove that $\{f_{n-1},\cdots,f_1\}$ is independent of the choice of the f_i up to homotopy.

Theorem 3. 4. *Suppose* $\{f_{n-1},\cdots,f_1\}$ *exists, and suppose* $g_i\sim f_i$, $i=1,\cdots,n-1$. *Then,* $\{g_{n-1},\cdots,g_1\}$ *exists, and is equal to* $\{f_{n-1},\cdots, f_1\}$.

The proof in the case of $\{f_{n-1},\cdots,f_1\}_F$ follows immediately from 3. 5; similar arguments hold for $\{f_{n-1},\cdots,f_1\}_C$ when simple connectivity assumptions hold.

Lemma 3.5. *Suppose* $\{P_{i,j}, f_{i,j}, p_{i,j}, \overline{f}_i\}$ *is an F-presentation for* $\{f_{n-1},\cdots,f_1\}_F$, *and* $g_i\sim f_i$. *Then, there is an F-presentation* $\{Q_{i,j}, g_{i,j}, q_{i,j}, \overline{g}_i\}$ *for* $\{g_{n-1},\cdots,g_1\}_F$ *and singular homotopy equivalences* $\rho_{i,j}:Q_{i,j}\to P_{i,j}$ *such that the diagrams below homotopy commute.*

$$\text{(A)}\quad \begin{array}{ccc} P_{i,j} & \xleftarrow{\rho_{i,j}} & Q_{i,j} \\ \downarrow{\scriptstyle p_{i,j}} & & \downarrow{\scriptstyle q_{i,j}} \\ P_{i,j-1} & \xleftarrow{\rho_{i,j-1}} & Q_{i,j-1} \end{array} \qquad \text{(B)}\quad \begin{tikzcd} Q_{i,j} \arrow[dr, "g_{i,j}"] \arrow[d, "\rho_{i,j}"'] & \\ P_{i,j} \arrow[r, "f_{i,j}"'] & \Omega^i X_{i+j+1}, \end{tikzcd}$$

$$\text{(C)}\quad \begin{tikzcd} & P_{i+1,j} \\ X_i \arrow[ur, "\overline{f}_{i,j}"] \arrow[dr, "\overline{g}_{i,j}"'] & \\ & Q_{i+1,j} \arrow[uu, "\rho_{i+1,j}"'] \end{tikzcd} .$$

Proof: Suppose X, B, C are topological spaces, $f:X\to B$, $\rho: B\to C$, $g:X\to C$ are maps, and $H:\rho f\sim g$. We define $H\rho:P_f\to P_g$ by

$$H\rho(x,\lambda)=(x,\mu),\quad \mu(t)=\begin{cases} H(x,1-2t), & 0\leq t\leq 1/2 \\ \rho\lambda(2t-1), & 1/2\leq t\leq 1. \end{cases}$$

Then, if $\Pi:P_f\to X$, $\theta:P_g\to X$ are the projections, $\theta(H\rho)=\Pi$; if $i:\Omega B\to P_f$ and $j:\Omega C\to P_g$ are the inclusions of the fibers, we see immediately that $j(\Omega\rho)\sim(H\rho)i$. If ρ was a singular homotopy equivalence it follows that (H_ρ) was a singular homotopy equivalence.

Using the facts adduced in the last paragraph, it is then an easy task to build the $\rho_{i,j}$ by induction. The homotopy commutativity of A) follows easily; the choice of $g_{i,j}$ in B) is made by taking $g_{i,j}=f_{i,j}\circ\rho_{i,j}$, and checking that $g_{i,j}$ satisfies the conditions necessary in an F-presentation. $\overline{g}_{i,j}$ may be chosen as a map representing an element of $\Pi(X_i, Q_{i+1,j})$ corresponding to that represented by $\overline{f}_{i,j}$ under $\rho_{i+1,j*}$. (Of course, one should choose $\overline{g}_i$ thus, and define $\overline{g}_{i,j}$ by composition with appropriate $q_{i+1,j}$'s.)

3. 4. allows us to make the following definition.

Definition 3. 6. Suppose $\alpha_i\in\Pi(X_i, X_{i+1})$, $i=1, \cdots, n-1$. Then, $\{\alpha_{n-1}, \cdots, \alpha_1\}$ exists if $\{f_{n-1}, \cdots, f_1\}$ exists for some choice of f_i representing α_i, and is equal to $\{f_{n-1}, \cdots, f_1\}$ for such a choice. $\{\alpha_{n-1}, \cdots, \alpha_1\}_F$ and $\{\alpha_{n-1}, \cdots, \alpha_1\}_C$ have the appropriate meanings.

4. In this section we prove an easy result which will be useful in section 6.

Proposition 4.1. *Suppose* $X_1\xrightarrow{f_1}X_2\xrightarrow{f_2}\cdots\xrightarrow{f_{n-1}}X_n$ *is a set of spaces and maps such that* $\{f_{n-1}, \cdots, f_1\}_F$ *exists. Suppose that* $g: Y\to X_1$ *is a map such that* $f_1\circ g$ *is null homotopic. Then, there are maps* $h: Y\to\Omega X_3$, *representing elements of* $\{f_2, f_1, g\}$ *such that the* $(n-1)$-*fold product* $\{\Omega f_{n-1}, \cdots, \Omega f_3, h\}$ *exists. If* $k: X_1\to\Omega^{n-3}X_n$ *represents an element of* $\{f_{n-1}, \cdots, f_1\}_F$ *then* $k\circ g$ *represents an element of* $(-1)^{n-1}\{\Omega f_{n-1}, \cdots, \Omega f_3, h\}_F$ *for some* h *representing an element of* $\{f_2, f_1, g\}$. *This result may be expresed crudely by saying that* $\{\{f_{n-1}, \cdots, f_1\}_F, g\}_F\subseteq(-1)^{n-1}\{\Omega f_{n-1}, \cdots, \Omega f_3, \{f_2, f_1, g\}_F\}_F$.

We use the rough notation above to write 4. 1D.

Proposition 4. 1D. *Suppose* $\{f_{n-1}, \cdots, f_1\}_C$ *exists, and* $g: X_n\to Y$ *is a map such that* $\{g, f_{n-1}\}_C=0$. *Then,* $\{\{g, f_{n-1}, f_{n-2}\}_C, Sf_{n-3}, \cdots, Sf_1\}_C$ *exists, and* $\{g, \{f_{n-1}, \cdots, f_1\}_C\}_C\subseteq(-1)^{n-1}\{\{g, f_{n-1}, f_{n-2}\}_C, Sf_{n-3}, \cdots, Sf_1\}_C$.

Proof of 4. 1. Suppose $\{P_{i,j}, p_{i,j}, f_{i,j}, \overline{f}_i\}$ is an F-presentation for

$\{f_{n-1}, \cdots, f_1\}$, and $k=f_{2,n-3}\circ\bar{f}_1$ represents the corresponding element of $\{f_{n-1}, \cdots, f_1\}_F$. Let $p_{1,1}: P_{1,1}\to X_1$ be the fibration induced by f_1. Then, $p_{2,1}\circ p_{2,2}\circ\cdots p_{2,n-3}\circ\bar{f}_1\circ p_{1,1}=f_1\circ p_{1,1}$ which is null-homotopic. The fiber of $p_{2,1}\circ p_{2,2}\circ\cdots\circ p_{2,n-3}$ is, by definition of $P_{2,n-3}$, $\Omega P_{3,n-4}$. There is thus a map $\bar{b}_1: P_{1,1}\to\Omega P_{3,n-4}$ such that, if $q_i:\Omega P_{3,i-1}\to P_{2,i}$ is the inclusion of the fiber over X_2, $q_{n-3}\circ\bar{b}_1\sim\bar{f}_1\circ p_{1,1}$. Since $f_1\circ g$ is null-homotopic, there is a map $\bar{g}: Y\to P_{1,1}$ such that $p_{1,1}\circ\bar{g}=g$. Then $k\circ g=f_{2,n-3}\circ\bar{f}_1\circ g=f_{2,n-3}\circ\bar{f}_1\circ p_{1,1}\circ\bar{g}\sim f_{2,n-3}\circ q_{n-3}\circ\bar{b}_1\circ\bar{g}$. Now, $p_{2,2}\circ p_{2,3}\circ\cdots\circ p_{2,n-3}\circ q_{n-3}\circ\bar{b}_1: P_{1,1}\to\Omega X_3$ (since this map factors through q_1) is a map that occurs in an F-presentation for $\{f_2, f_1, g\}_F$. Thus, $h=p_{2,2}\circ p_{2,3}\circ\cdots\circ p_{2,n-3}\circ q_{n-3}\circ\bar{b}_1\circ\bar{g}$ represents an element of $\{f_2, f_1, g\}_F$. The result now follows from this observation and (2.3).

For our application of 4.1 in section 6 it is convenient to rephrase it. The composite map $\Omega P_{3,i-1}\to P_{2,i}\to\Omega^i X_{i+3}$ is, as we have seen, $\Omega f_{3,i-1}$ up to sign. If we change the sign where necessary, we get what is essentially an F-presentation for testing products $\{\Omega f_{n-1}, \cdots, \Omega f_3, a\}_F$. If, now, we look at the adjoint of h, $\bar{h}: SX_1\to X_3$, and de-loop the maps $\Omega f_{3,i-1}$, we get an F-presentation for $\{f_{n-1}, \cdots, f_3, \bar{h}\}_F$. Thus we obtain 4.2.

Proposition 4.2. *Uuder the conditions of* 4.1, *there are maps* $\bar{h}: SY\to X_3$, *adjoint to representatives of* $\{f_2, f_1, g\}_F$, *such that* $\{f_{n-1}, \cdots, f_3, \bar{h}\}_F$ *exists. If* $k: X_1\to\Omega^{n-3}X_n$ *represents an element of* $\{f_{n-1}, \cdots, f_1\}_F$, *then the adjoint of* $k\circ g$, *mapping* $SY\to\Omega^{n-4}X_n$ *is an element of* $(-1)^{n-1}\{f_{n-1}, \cdots, f_3, \bar{h}\}_F$ *for some such* $\bar{h}$. *In the presentation for this product, the tower of fibrations over* X_3 *may be taken to be that occurring in the original presentation for* $\{f_{n-1}, \cdots, f_1\}_F$.

5. In this section we apply the techniques developed above to the study of the p-components of the stable homotopy groups of spheres, for an odd prime p. While we do not compute any new groups, we are able to show the existence of some previously unknown regularity in the known groups.

We begin by recalling some of the results of [13]. Let p be an odd prime. We denote by $\Pi_k(S, p)$ the p-primary component of the k-th stable homotopy group of spheres. $\Pi_0(S, p)$ is infinite cyclic with generator ι, the unit of the ring $\Pi_*(S, p)=\sum_{k\geq 0}\Pi_k(S, p)$. There is an element $\alpha_1\in\Pi_{2p-3}(S, p)$ of order p, and elements $\alpha_s\in\Pi_{2s(p-1)-1}(S, p)$, $p\alpha_s=0$, $\alpha_s\in\{\alpha_{s-1}, p\iota, \alpha_1\}$. In [14] this is proved for $s<p^2$; in [1] and [15] it is proved for all s. If $p^r|s$, α_s is also divisible by p^r; this is proved in [14] for $s<p^2$; Adams is believed to have supplied a proof for all s in his study of the J-homomorphism. Further, there are elements $\beta_s\in\Pi_{2(sp+s-1)(p-1)-2}(S, p)$, $1\leq s\leq p-1$, of order p. If we set α'_{kp} equal to an appropriate element such that $p\alpha'_{kp}=\alpha_{kp}$ then it is shown in [14] that the α_s, α'_s, β_s generate $\Pi_*(S, p)$ multiplicatively in degrees less than $2p^2(p-1)-3$. We shall show that β_1 arises from α_1 as a p-fold product, and that the β_s arise from α_1 and β_1 in a manner analogous to that by which the α_s are constructed from α_1.

Lemma 5.1. *The p-fold product $\{\alpha_1, \cdots, \alpha_1\}_C$ exists and does not contain zero. It does contain $x\beta_1$ for some $x\in Z_p$, $x\neq 0$.*

Proof: This is just lemma 4.10 (ii) of [14], translated into our terminology. The proof given in [14] is exactly the construction of a C-presentation for this product.

Proposition 5.2. *If $2\leq s\leq p-1$, the five-fold product $\{\beta_{s-1}, p\iota, \beta_1, p\iota, \alpha_1\}$ exists and does not contain zero. It does contain $x\beta_s$ for some $x\in Z_p$, $x\neq 0$.*

This proposition suggests at least two questions.

A) Does this sequence of β_s continue indefinitely? That is, does there exist β_s, of order p, in degree $2(sp+s-1)(p-1)-2$, $\beta_s\in\{\beta_{s-1}, p\iota, \beta_1, p\iota, \alpha_1\}_C$, for all $s>1$?

B) If A) is true, do there exist more, similar, families? In particular, it is easy to see that the p-fold product $\{\alpha'_p, \cdots, \alpha'_p\}_C$ exists. This lies in degree $2p^2(p-1)-2$. $\Pi_{2p^2(p-1)-2}(S, p)$ is either

Z_p or 0 [13]. Is this product non-zero? If it is, suppose γ_1 is a non-zero element in this product. Can we form a sequence of non-zero elements γ_s, $\gamma_s \in \{\gamma_{s-1}, p\iota, \gamma_1, p\iota, \beta_1, p\iota, \alpha_1\}_C$? γ_s would have degree $2(sp^2+(s-1)(p+1))(p-1)-2$.

Before proving 5.2, we need a lemma.

Lemma 5.3. *Suppose $\gamma \in \Pi_{n+k}(S^n)$ is an element of order p. Then, the triple product $\{p\iota_n, \gamma, p\iota_{n+k}\}_C$ is defined and contains zero.*

Proof: We use the definition of the triple product as a functional homotopy operation [10]. Thus, if $p_1: S^n \to S^n$ is a map of degree p, and $p_2: S^{n+k} \to S^{n+k}$ is the k-th suspension of p_1, we form the following coexact sequences of spaces in the sense of [9]: $S^{n+k} \xrightarrow{p_2} S^{n+k} \to C_p \to S^{n+k+1} \to S^{n+k+1}$, where C_p is the mapping cone of p_2.

We then form the diagram

$$\begin{array}{ccccccccc}
\Pi_{n+k+1}(S^n) & \xrightarrow{\times p} & \Pi_{n+k+1}(S^n) & \xrightarrow{i} & \Pi(C_p, S^n) & \xrightarrow{j} & \Pi_{n+k}(S^n) & \xrightarrow{\times p} & \Pi_{n+k}(S^n) \\
\downarrow{\times p} & & \downarrow{\times p} & & \downarrow{\times p} & & \downarrow{\times p} & & \downarrow{\times p} \\
\Pi_{n+k+1}(S^n) & \xrightarrow{\times p} & \Pi_{n+k+1}(S^n) & \xrightarrow{i} & \Pi(C_p, S^n) & \xrightarrow{j} & \Pi_{n+k}(S^n) & \xrightarrow{\times p} & \Pi_{n+k}(S^n),
\end{array}$$

in which the horizontal rows are obtained by mapping the sequence of spaces into S^n, and the vertical arrows are induced by p_1 and hence are multiplication by p. (Note that $\Pi(C_p, S^n)$ is an abelian group since C_p is a double suspension because $k>2$; in fact, $k \geq 2p-3$.) $\{p\iota_n, \gamma, p\iota_{n+k}\}_C$ is obtained by considering $\gamma \in \Pi_{n+k}(S^n)$, observing that $p\gamma=0$, taking $j^{-1}(\gamma)$ in $\Pi(C_p, S^n)$, multiplying by p, and pulling back to $\Pi_{n+k+1}(S^n)$. Suppose $j\bar{\gamma}=\gamma$. Since C_p is the Moore space $H(Z_p, n+k)$, $\Pi(C_p, S^n)$ is a cohomotopy group with coefficients in Z_p for an odd prime p. By Theorem 3.6 of [8], $\Pi(C_p, S^n)$ is a vector space over Z_p. Hence, $p\bar{\gamma}=0$, so $p\bar{\gamma}$ pulls back to zero.

Proof of 5.2. The proof falls naturally into two parts: existence of the product, and a proof that it is non-zero.

We first prove that the product exists. We shall consider the elements β_s, α_1, etc. as being in the homotopy groups of appropriately high dimensional spheres. Recall (see section 2) that the products may be considered as the images of elements under the differential in a spectral sequence that comes from the C-presentation. $p\alpha_1=0$, so one may look at $\{\beta_1, p\iota, \alpha_1\}_C=d_2(\beta_1)$ in an appropriate spectral sequence. This is a coset of some subgroup of $\Pi_{2(p+1)(p-1)-2}(S)$. This group has no p-component [13]; since β_1 is in the p-component, $d_2(\beta_1)=0$. Hence, we may construct a shaft of cofibrations on which to test the existence of $\{p\iota, \beta_1, p\iota, \alpha_1\}_C$. We display the relevant part of the exact couple on which this is computed; here, n is a sufficiently large integer, $a=2p(p-1)-2$, $b=2(p-1)-1$.

$$\begin{array}{ccccc}
\Pi_{n+a+2}(S^n) & \to & \Pi(C_{4,2}, S^n) & \to & \Pi_{n+a+b+1}(S^n) \\
 & & \downarrow & & \\
\Pi_{n+a+1}(S^n) & \to & \Pi(C_{4,1}, S^n) & \to & \Pi_{n+a+1}(S^n) \\
 & & \downarrow & & \\
\Pi_n(S^n) & \longrightarrow & \Pi_n(S^n) & \xrightarrow{\beta_1^*} & \Pi_{n+a}(S^n)
\end{array}$$

We start with $p\iota$ in the $\Pi_n(S^n)$ on the left. $d_1(p\iota)=\beta_1^* p\iota=p\beta_1=0$, so we have an element $[p\iota]_2$ in E_2. $\Pi_{n+a+1}(S^n, p)=Z_{p^2}(\alpha'_p)$. Two copies of $\Pi_{n+a+1}(S^n)$ occur in the exact couple; call the left hand one G' and the right hand one G''. It follows from the discussion in section 2 that $d_1(G'')=0$, and that $d_1: G'\to G''$ is multiplication by p. Therefore, (recalling that $\alpha_p=p\alpha'_p$), the E_2 term in the position of G'' is G''/pG'', a group of order p. Now, $d_2[p\iota]_2=\{p\iota, \beta_1, p\iota\}_C$, which, by 5.3, is zero mod pG''. Hence, $d_2[p\iota]_2=0$, so that $\{p\iota, \beta_1, p\iota, \alpha_1\}_C$ exists. The image of d_1 in G' is the image of multiplication by β_1, and hence is zero. Thus, E_2 in the position of G' is $Z_p(\alpha_p)$. $d_2(\alpha_p)$ is a representative of $\{\alpha_p, p\iota, \alpha_1\}_C$ which, by Theorem 4.14 of [14] contains a generator of $\Pi_{n+a+b+1}(S^n, p)$. Thus, E_3 in the position of $\Pi_{n+a+b+1}(S^n, p)$ has no p-component. Thus, for any C-presentation, $\{p\iota, \beta_1, p\iota, \alpha_1\}_C$, as an element of E_3, is in a group with no p-component. By 3.2, there exists an F-presentation for $\{p\iota, \beta_1, p\iota, \alpha_1\}_F$. It is easily seen that $\Pi_{n+a+b}(P_{2,2})$ is finite in any such presentation. Hence, one may choose a lift of a representative of α_1 that represents an element of the p-component of $\Pi_{n+a+b}(P_{2,2})$. It then follows from 3.3 that there exists a C-presentation for $\{p\iota, \beta_1, p\iota, \alpha_1\}_C$ which has in it an element in the p-component of $\Pi_{n+a+b+1}(S^n)$. Hence, by the remark above, in this C-presentation $\{p\iota, \beta_1, p\iota, \alpha_1\}_C=0$ in E_3. Hence, we may construct a shaft of cofibrations on which to test the existence of $\{\beta_{s-1}, p\iota, \beta_1, p\iota, \alpha_1\}_C$. In the corresponding spectral sequence, $d_1(\beta_{s-1})=p\beta_{s-1}=0$; $d_2[\beta_{s-1}]_2=\{\beta_{s-1}, p\iota, \beta_1\}_C\subseteq\Pi_{2(sp+s-2)(p-1)-3+n}(S^n)$, which has no p-component [13]; since $[\beta_{s-1}]_2$ is in the p-component, $d_2[\beta_{s-1}]_2=0$. Hence, $[\beta_{s-1}]_3$ exists; again, $d_3[\beta_{s-1}]_3=\{\beta_{s-1}, p\iota, \beta_1, p\iota\}_c\subseteq\Pi_{2(sp+s-2)(p-1)-2}(S)$, which again has no p-component [13], so that $d_3[\beta_{s-1}]_3=0$, so that $[\beta_{s-1}]_4$ exists. Thus, $d_4[\beta_{s-1}]_4=\{\beta_{s-1}, p\iota, \beta_1, p\iota, \alpha_1\}_C$ exists.

Next, we show that it does not contain zero. (It must contain elements of the p-component since β_{s-1} is in the p-component). We use a method introduced by Toda [14], namely, constructing maps of the $C_{5,3}$ that occurs in a C-presentation into appropriate elements of the Postnikov system of a sphere, and observing what happens in the mod p cohomology.

As above, $a=2p(p-1)-2$, $b=2(p-1)-1$. Let $c=2((s-1)p+s-2)(p-1)-2$. Again, n is sufficiently large. Let $Y=C_{5,3}$. Y has cells in dimensions 0, $n+c$, $n+c+1$, $n+a+c+2$, $n+a+c+3$. The $(n+c+1)$-cell is glued to S^{n+c} by a map of degree p; the $(n+a+c+2)$-cell is glued to the $(n+c+1)$-skeleton by a map b_1 which represents β_1 when the $(n+c)$-skeleton is identified to a point; and the $(n+a+c+3)$-cell is glued to the $(n+a+c+2)$-skeleton by a map which is a map of degree p when the $(n+a+c+1)$-skeleton is identified to a point. It follows that $H^*(Y, Z_p)$ is zero except in degrees 0, $n+c$, $n+c+1$, $n+a+c+2$, $n+a+c+3$, in each of which degrees it is Z_p. Further, if we pick generators σ of $H^{n+c}(Y, Z_p)$ and τ of $H^{n+a+c+2}(Y, Z_p)$, then $\delta\sigma$ generates $H^{n+c+1}(Y, Z_p)$ and $\delta\tau$ generates $H^{n+a+c+3}(Y, Z_p)$, where δ is the mod p Bockstein.

There are maps $f: Y\rightarrow S^n$, $f\,|\,S^{n+c}$ represents β_{s-1}, and $g: S^{n+a+b+c+3}$

$\rightarrow Y$ such that if $\eta: Y\rightarrow S^{n+a+c+3}$ collapses the $(n+a+c+2)$-skeleton to a point, ηg represents α_1. The composition fg represents $\{\beta_{s-1}, p\iota, \beta_1, p\iota, \alpha_1\}_C$.

Next. we show that $P^p\delta\sigma\neq 0$. If W is Y with the $(n+c)$-skeleton identified to a point, $H^*(W, Z_p)$ is non-zero in dimensions $n+c+1$, $n+a+c+2$, $n+a+c+3$. Let σ' be a generator in dimension $n+c+1$, τ in dimension $n+a+c+2$, and $\delta\tau$ in dimension $n+a+c+3$. It is sufficient to show that $P^p\sigma'\neq 0$. Let K be that element of the Postnikov system for S^{n+c+1} such that if $k: S^{n+c+1}\rightarrow K$ is the map occurring in the Postnikov system, $k_*: \Pi_i(S^{n+c+1})\rightarrow\Pi_i(K)$ is an isomorphism for $i<n+a+c+1=n+c+1+2p(p-1)-2$, and $\Pi_i(K)=0$ for $i\geq n+a+c+1$. Then, k extends to $\bar{k}: W\rightarrow K$, since there is no obstruction to extension. If $i: S^{n+c+1}\rightarrow W$ is inclusion, it is clear that $i_*: \Pi_j(S^{n+c+1})\rightarrow\Pi_j(W)$ is an isomorphism for $j<n+a+c+1$. β_1 is a generator of $\Pi_{n+a+c+1}(S^{n+c+1})$; since the $(n+a+c+2)$-cell of W is attached by a map representing β_1, $\Pi_{n+a+c+1}(W, p)=0$. Then, if C is the class of finite abelian groups of order prime to p, $\bar{k}_*: \Pi_j(W)\rightarrow\Pi_j(K)$ is a C-isomorphism for $j\leq n+a+c+1$, and a C-epimorphism for $j=n+a+c+2$. Hence, $\bar{k}^*: H^j(K, Z_p)\rightarrow H^j(W, Z_p)$ is an isomorphism for $j\leq n+a+c+1$ and a monomorphism for $j=n+a+c+2$. According to statement (3.12) on p. 203 of [13], $H^{n+c+1}(K, Z_p)=Z_p(a_0)$, $H^{n+a+c+2}(K, Z_p)=Z_p(b_1)$, where a_0 and b_1 are generators of these groups, and $P^p a_0=\delta b_1$. Now, $\bar{k}^*(a_0)\neq 0$, and $P^p\bar{k}^*(a_0)=\bar{k}^*(P^p a_0)=\bar{k}^*(\delta b_1)=\delta\bar{k}^*(b_1)$. Since $\bar{k}^* b_1\neq 0$, and δ is non-zero on $H^{n+a+c+2}(W, Z_p)$, $\delta k^*(b_1)\neq 0$. Hence, P^p is non-zero on $H^{n+c+1}(W, Z_p)$, and hence on $H^{n+c+1}(Y, Z_p)$. Thus, to sum up, $H^*(Y, Z_p)$ has generators 1 in H^0, σ in H^{n+c}, $\delta\sigma$ in H^{n+c+1}, τ in $H^{n+a+c+2}$, $\delta\tau$ in $H^{n+a+c+3}$, and $P^p\delta\sigma\neq 0$, so $P^p\delta\sigma=x\delta\tau$, for some $x\in Z_p$, $x\neq 0$.

Now let Z be the complex obtained from Y by attaching a cell by the map g. The mod p cohomology of Z is the same as that of Y, with an additional generator in degree $n+a+b+c+4$. We denote the generators in Z by the same symbols as those in Y, and claim that $P^1\delta\tau \neq 0$. This follows from the well-known [2] fact that α_1 is an element of mod p Hopf invariant one, and the fact (noted above) that ηg represents α_1.

Let L be that member of the Postnikov system of S^n such that if $l: S^n \to L$ is the map occurring in the Postnikov system then $l_*: \Pi_i(S^n) \to \Pi_i(L)$ is an isomorphism for $i < n+c$, and $\Pi_i(L) = 0$, $i \geq n+c$. Then, $l \circ f$ is null-homotopic, since there is no obstruction to a null-homotopy. Let $m: M \to S^n$ be the fibration induced by l. Since $l \circ f$ is null-homotopic, there exists a map $\bar{f}: Y \to M$, $m \circ \bar{f} = f$. The p-component of $\Pi_{n+c}(S^n)$ is generated by β_{s-1}. Since $f | S^{n+c}$ represents β_{s-1}, and since $\Pi_{n+c}(Y) = Z_p$, and is generated by the inclusion of S^{n+c} into Y, it follows that $f_*: \Pi_{n+c}(Y) \to \Pi_{n+c}(S^n)$, and hence $\bar{f}_*: \Pi_{n+c}(Y) \to \Pi_{n+c}(M)$ is a C-isomorphism. Since both Y and M are $(n+c-1)$-connected, $\bar{f}^*: H^{n+c}(M, Z_p) \to H^{n+c}(Y, Z_p)$ is an isomorphism. It follows from theorem 3.10 of [13] and the discussion on p. 310 of [14] that $H^{n+c}(M, Z_p)$ is generated by an element $b_{s-1}^{(s-2)}$, and that, in $H^*(M, Z_p)$, $W_{s-1} b_{s-1}^{(s-2)} = 0$, where $W_{s-1} = sP^pP^1\delta - (s-1)P^{p+1}\delta + (s-2)\delta P^{p+1}$.

If $f \circ g$ is null-homotopic, so is $\bar{f} \circ g$, so that $\bar{f}$ extends to $\hat{f}: Z \to M$, where $\hat{f}^*: H^{n+c}(M, Z_p) \to H^{n+c}(Z, Z_p)$ is an isomorphism. Hence, $\hat{f}^*(b_{s-1}^{(s-2)}) = z\sigma$, $z \in Z_p$, $z \neq 0$. Hence $W_{s-1}\sigma = 0$ in $H^*(Z, Z_p)$.

Now, $P^1\delta\sigma = 0$ for dimensional reasons. Hence, $W_{s-1}\sigma = -(s-1) P^{p+1}\delta\sigma + (s-2)\delta P^{p+1}\sigma$. By the Adem relations, $P^1P^p = P^{p+1}$. Hence, $P^{p+1}\sigma = P^1P^p\sigma = wP^1\tau$ for some $w \in Z_p$. But $P^1\tau = 0$ for dimensional reasons. Hence, $W_{s-1}\sigma = -(s-1)P^{p+1}\delta\sigma = -(s-1)P^1P^p\delta\sigma$. But $s > 1$, and $P^1P^p\delta\sigma \neq 0$. Thus, if $f \circ g$ is null-homotopic, we contradict $W_{s-1} b_{s-1}^{(s-2)} = 0$. Hence, $f \circ g$ is essential, and $\{\beta_{s-1}, p\iota, \beta_1, p\iota, \alpha_1\}_C$ (and hence, by 3.3, $\{\beta_{s-1}, p\iota, \beta_1, p\iota, \alpha_1\}_F$) does not contain zero. As we have seen, it does contain an element of the p-component, which proves 5.2.

6. In this section, we turn to the unstable p-components. It is first necessary to review the results of [16].

Since p is odd, it is sufficient to confine oneself to odd-dimensional spheres (Theorem 13. 12 of [6]). We shall be interested in the p-components of $\Pi_{2k+1+n}(S^{2k+1})$, which we denote by $G(n, k)$, for $n\leq 2p(p-1)-1$. Recall that $S^2: G(n, k)\rightarrow G(n, k+1)$ is an isomorphism if $n<2(k+1)(p-1)-2$, and an epimorphism if $n=2(k+1)(p-1)-2$ (Theorem XI, 8. 3 of [6]). Toda has shown [16] that if $n<2p(p-1)-2$ then $G(n, k)=0$ unless n is of the form $2s(p-1)-1$, $1\leq s<p$, or $2(r+k)(p-1)-2$, $1\leq r<p-k$. If n has either of these forms, $G(n, k)=Z_p$; if $n=2s(p-1)-1$ then $G(n, k)$ is generated by an element $\alpha_s(2k+1)$ which suspends into the stable element α_s [16]. In addition, $G(2p(p-1)-1, 1)=Z_p$ on a generator $\alpha_p(3)$, and $G(2p(p-1)-2, 1)=Z_p$ [16].

Additional information about the $2p(p-1)-1$ and $2p(p-1)-2$ stems may be gained from [14] and [16]. The main tool of [16] is the exact sequence (A), where $\Pi_{2k+n-1}(\Omega^2 S^{2k+1}, S^{2k-1}; p)$ is the p-component of the homotopy group.

$$\text{(A)}\quad \rightarrow G(n, k-1)\xrightarrow{S^2} G(n, k)\xrightarrow{j_*}\Pi_{2k+n-1}(\Omega^2 S^{2k+1}, S^{2k-1}; p)\rightarrow$$
$$\rightarrow G(n-1, k-1)\rightarrow\cdots$$

Formulas 13. 6 of [16] give values for $\Pi_{2k+n-1}(\Omega^2 S^{2k+1}, S^{2k-1}; p)$ that allow us to write the exact sequences (a), (b), (c) below, where $N=2p(p-1)-1$.

$$\text{(a)}\quad 0\rightarrow G(N, k-1)\xrightarrow{S^2} G(N, k)\xrightarrow{j_*} Z_p\rightarrow G(N-1, k-1)\xrightarrow{S^2} G(N-1, k)$$
$$\rightarrow Z_p\rightarrow 0,\quad 2\leq k<p$$

$$\text{(b)}\quad 0\rightarrow G(N, p-1)\xrightarrow{S^2} G(N, p)\xrightarrow{j_*} Z_p\rightarrow G(N-1, p-1)\xrightarrow{S^2}$$
$$\xrightarrow{S^2} G(N-1, p)\rightarrow 0$$

$$\text{(c)}\quad 0\rightarrow G(N, p)\xrightarrow{S^2} G(N, p+1)\rightarrow 0.$$

Note that $G(N, p+1)$ is stable, and hence is $Z_{p^2}(\alpha'_p(2p+3))$. Also, $G(N-1, p)$ is stable, and hence is $Z_p(\beta_1(2p+1))$. Thus, (c) implies that $G(N, p)=Z_{p^2}(\alpha'_p(2p+1))$, and (b) reduces to (b′).

(b′) $0 \to G(N, p-1) \xrightarrow{S^2} Z_{p^2}(\alpha'_p(2p+1)) \xrightarrow{j_*} Z_p \to$

$\to G(N-1, p-1) \xrightarrow{S^2} Z_p(\beta_1(2p+1)) \to 0.$

From (a) and (b′) it is evident that the iterated suspension is a monomorphism of $G(N, k)$ into $Z_{p^2}(\alpha'_p(2p+1))$, $1 \leq k \leq p$. Hence, $G(N, k)$ is Z_p or Z_{p^2} for $1 < k < p$.

The following lemma is due to Hardie [4].

Lemma 6.1. $G(N, 2) = Z_{p^2}(\alpha'_p(5))$.

Proof: Either $G(N, 2) = Z_{p^2}(\alpha'_p(5))$ or $Z_p(\alpha_p(5))$. In [16] it is shown that $G(N-1, 1)$ is generated by $\alpha_1(3) \circ \alpha_{p-1}(2p)$. If $G(N, 2) = Z_p$, it follows from the exact sequence (a), with $k=2$, and recalling that $G(N-1, 1) = Z_p$, $G(N, 1) = Z_p$, that $S^2(\alpha_1(3) \circ \alpha_{p-1}(2p)) = 0 = \alpha_1(5) \circ \alpha_{p-1}(2p+2)$. It follows that $\{\alpha_1(5), \alpha_{p-1}(2p+2), p\iota_{2p^2-2p+3}\}$ exists in $G(N, 2)$. Let γ be any element in this triple product, and suspend it until the stable range is reached, giving $S^m\gamma \in \{\alpha_1, \alpha_{p-1}, p\iota\}$. It is shown in [14] that this triple product contains an element of order p^2, and that the indeterminary consists of elements of order p. Hence, $S^m\gamma$ is of order p^2. Hence γ is of order p^r, $r \geq 2$. But $\gamma \in G(N, 2)$ which was assumed to be Z_p. Henc $G(N, 2) = Z_{p^2}(\alpha'_p(5))$.

Corollary 6.2. $G(N, k) = Z_{p^2}(\alpha'_p(2k+1))$, $k > 1$.

From 6.2 and the sequences (a) and (b′) we obtain the sequences (a′), (a″), (b″).

(a′) $0 \to Z_p \to G(N-1, k-1) \xrightarrow{S^2} G(N-1, k) \xrightarrow{j_*} Z_p \to 0, \quad 3 \leq k < p$

(a″) $0 \to G(N-1, 1) \to G(N-1, 2) \xrightarrow{j_*} Z_p \to 0$

(b″) $0 \to Z_p \to G(N-1, p-1) \xrightarrow{S^2} Z_p(\beta_1(2p+1)) \to 0.$

We conclude that $G(N-1, k)$ is $Z_p \oplus Z_p$ or Z_{p^2}, $2 \leq k < p$; it is shown below that $G(N-1, k)$ is Z_{p^2}.

For convenience in notation we introduce the following convention. In writing a composition product of elements which suspend to stable elements, we only write the dimensions in which they occur once; the other dimensions are then determined by the stems of the elements involved. Thus, the composition of $\alpha_1(3) \in \Pi_{2p}(S^3)$ and $\alpha_1(2p)$ in $\Pi_{4p-3}(S^{2p})$ is written $\{\alpha_1(3), \alpha_1\}$; the triple product of $\alpha_1(5) \in \Pi_{2p+2}(S^5)$, $\alpha_1(2p+2) \in \Pi_{4p-1}(S^{2p+2})$ and $\alpha_2(4p-1) \in \Pi_{8p-6}(S^{4p-1})$ (assuming it exists) is written $\{\alpha_1(5), \alpha_1, \alpha_2\}$.

Our next proposition is an unstable version of 5.1. It shows that the unstable element of lowest degree in the range in which we are interested may be viewed as an attempt to approximate β_1.

Proposition 6.3. *For* $1 \leq k \leq p-1$ *the* $(k+1)$*-fold product* $\{\alpha_1(2k+1), \alpha_1, \cdots, \alpha_1\}_C \subseteq G(2(k+1)(p-1)-2, k)$ *exists and does not contain zero.*

Proof: The proof is an unstable version of Toda's proof of 5.1. The product exists since it is seen that the degrees in which all the shorter C-products occur are in the stable range, and contain zero p-component. Further, each such product has a representative in the p-component, and thus is zero. In view of the fact that α_1 is an element of mod p Hopf invariant one, the space $C_{k+1,k-1}$ in a C-presentation for the $(k+1)$-fold product of the proposition is a cell complex with one zero cell and one cell in dimension $2k+2i \times (p-1)$, $1 \leq i \leq k$, with the property that P^1 is non-zero on $H^{2k+2i(p-1)}(C_{k+1,k-1}, Z_p)$, $1 \leq i < k$. There are maps $f: C_{k+1,k-1} \to S^{2k+1}$, extending a map representing $\alpha_1(2k+1)$ on the $2k+2(p-1)$ skeleton, and $g: S^{2(k+1)p-3} \to C_{k+1,k-1}$ such that if $\eta: C_{k+1,k-1} \to S^{2kp}$ collapses the $(2kp-1)$ skeleton to a point, $\eta \circ g$ represents $\alpha_1(2kp)$. Let $Y = C_{k+1,k-1} \cup_g e^{2(k+1)p-2}$. Then, P^1 is non-zero on $H^{2kp}(Y; Z_p)$. If $f \circ g$ is null-homotopic, f extends to $\bar{f}: Y \to S^{2k+1}$. Suppose $f \circ g$ is null-homotopic (i.e., the product contains zero). Let $Z = S^{2k+1} \cup_{\bar{f}} CY$. Then, Z has cells in dimensions zero and $2k+2i(p-1)+1$, $0 \leq i \leq k+1$, and P^1 is non-zero on $H^{2k+2i(p-1)+1}(Z, Z_p)$, $0 \leq i \leq k+1$. Hence, $(P^1)^{k+1}$ is non-zero on $H^{2k+1}(Z, Z_p)$. If $k < p-1$, $(P^1)^{k+1} = xP^{k+1}$, $x \neq 0$, $x \in Z_p$, so that P^{k+1} is non-zero on an element of degree $2k+1$, which is im-possible. If $k=p-1$, we conclude that $(P^1)^p \neq 0$, which is also false. Thus, if the product contains zero, we get a contradiction which proves the proposition.

The remainder of our discussion will show that the other unstable elements in the range in which we are interested arise in a manner similar to those in $G(2(k+1)(p-1)-2, k)$ and will settle the structure of $G(N-1, k)$.

It is shown in [16] that the groups $G(2(r+1)(p-1)-2, 1)$ are isomorphic to Z_p with generators $\alpha_1(3)\circ\alpha_r(2p)=\{\alpha_1(3), \alpha_r\}$. We generalize this result below; we shall treat $G(n, k)$ with $p-1 \geq k \geq 2$. The case $p=3$ is treated in [16] using triple products; our results extend these proofs to higher p using longer products.

If $2\leq k\leq p-1$ and $1\leq s\leq p-k$ the k-fold products $\{\alpha_1(2k+2(p-1)), \alpha_1, \cdots, \alpha_1, \alpha_s\}_C$ exist. To see this, it is only necessary to observe that, in constructing a C-presentation all the groups that occur have stable p-components which are zero. This product is contained in $\Pi_{2k+2(p-1)+2(k+s-1)(p-1)-2}(S^{2k+2(p-1)}; p)$, which is zero unless $k=p-1$, $s=1$.

Similarly, if $2\leq k\leq p-1$ and $1\leq s\leq p-k$, the k-fold products $\{\alpha_1(2k+2(p-1)-2), f_1, \cdots, \alpha_1, \alpha_s\}_C$ exist; this product is contained in $\Pi_{2(k-1)+2(p-1)+2(k+s-1)(p-1)-2}(S^{2(k-1)+2(p-1)}, p)$, which is zero. Thus, by taking the double suspension (see 2.3D) we see that the k-fold product $\{\alpha_1(2k+2(p-1)), \alpha_1, \cdots, \alpha_s\}_C$ always exists and contains zero. Thus, we may construct shafts of cofibrations on which to test the existence of the $(k+1)$-fold products $\{\rho, \alpha_1, \cdots, \alpha_1, \alpha_s\}_C$ where $\rho\in \Pi_{2k+2(p-1)}(X)$ or $\Pi_{2k+2(p-1)-2}(X)$, any space X. The space $C_{k+1, k-1}$ in a shaft for testing the existence of the second product is a finite cell complex of dimension $2kp-2$, with one cell in dimension zero, and one cell in each dimension $2k+2i(p-1)-2$, $1\leq i\leq k$. If η: $C_{k+1, k-1}\to S^{2kp-2}$ is the map collapsing the $2kp-3$ skeleton to a point, the map $g_s=g_{k+1, k-1}: S^{2kp+2s(p-1)-3}\to C_{k+1, k-1}$ has the property that $\eta\circ g_s$ represents $\alpha_s(2kp-2)$. Further, $C_{k+1, k-2}$ of this shaft is simply the $2kp-3$ skeleton of $C_{k+1, k-1}$; if $\zeta: C_{k+1, k-2}\to S^{2p(k-1)}$ collapses the

$2p(k-1)-1$ skeleton to a point, then $g_{k+1,k-2}=h$ has the property that $\zeta\circ h$ represents $\alpha_1(2p(k-1))$. The double suspensions of $C_{k+1,k-1}$ and $C_{k+1,k-2}$ and of the maps g_s and h give parts of the shaft for testing the existence of the first product (at least up to sign).

We note that $C_{k+1,k-2}$ is a space of the type discussed in the proof of 6.3; it occurs in a shaft of cofibrations in a C-presentation for the k-fold product $\{\alpha_1(2k-1), \alpha_1, \cdots, \alpha_1\}_C$. Let $m: S^{2k+2(p-1)-2} \to C_{k+1,k-2}$ be the inclusion of the $2k+2(p-1)-2$ skeleton. If $a: S^{2k+2(p-1)-2} \to S^{2k-1}$ represents $\alpha_1(2k-1)$, then (see the proof of 6.3) a extends to $a: C_{k+1,k-2} \to S^{2k-1}$, $\bar{a}m=a$. Then, $\bar{a}\circ h$ represents a non-zero element of $G(2((k-1)+1)(p-1)-2, k-1)$, as in 6.3. Let $i: S^{2k-1} \to \Omega^2 S^{2k+1}$ be the usual inclusion. Since $G(2k(p-1)-2, k)=0$, $i\circ h\circ\bar{a}$ is null-homotopic. Hence, $i\circ\bar{a}$ may be extended to $b: C_{k+1,k-1} \to \Omega^2 S^{2k+1}$. We wish to show that b may be constructed so as to factor through $q: \Omega S^{2k}_{p-1} \to \Omega^2 S^{2k+1}$, where S^{2k}_{p-1} is the $(p-1)$-fold reduced product of S^{2k}.

The non-zero elements in $G(2(r+k)(p-1)-2, k)$, $2\leq k\leq p-1$, $1\leq r<p-k$ arise because in that degree the map j_* of sequence A) is an isomorphism; the elements of $G(N-1, k)$ which are not images of the double suspension are mapped into non-zero elements by the map j_*. In these degrees, the group $\Pi_i(\Omega^2 S^{2k+1}, S^{2k-1}; p)$ is Z_p. Further, the map $q_*: \Pi_i(\Omega S^{2k}_{p-1}, S^{2k-1}; p) \to \Pi_i(\Omega^2 S^{2k+1}, S^{2k-1}; p)$ is an epimorphism for these i, i.e., $i=2(r+k)p-2r-3$, $2\leq k\leq p-1$, $1\leq r\leq p-k$. These facts are to be found in [12]. Toda also shows there that there is a map $\theta: S^{2kp-3} \to P(\Omega S^{2k}_{p-1}, S^{2k-1}, *)$, the space of paths in ΩS^{2k}_{p-1} beginning in S^{2k-1} and ending in base point, such that θ induces a C-isomorphism of homotopy groups (where again C is the category of all finite abelian groups of order prime to p) in a range of dimensions that includes all the groups which interest us here.

Hence, $\Pi_{2kp-3}(P(\Omega S^{2k}_{p-1}, S^{2k-1}, *))=Z\oplus T$, where T is a finite abelian group of order prime to p. It is also shown in [16] and

[12] that $\Pi_{2kp-2}(\Omega^2S^{2k+1}, S^{2k-1}; p)=Z_p$, and that the map $\partial:\Pi_{2kp-2}(\Omega^2S^{2k+1}, S^{2k-1}; p)\to\Pi_{2kp-3}(S^{2k-1}; p)$ is an isomorphism, and that $q_*: \Pi_{2kp-2}(\Omega S^{2k}_{p-1}, S^{2k-1})\to\Pi_{2kp-2}(\Omega^2S^{2k+1}, S^{2k+1}; p)$ is an epimorphism. Hence, $\partial:\Pi_{2kp-2}(\Omega S^{2k}_{p-1}, S^{2k-1})\to\Pi_{2kp-3}(S^{2k-1}; p)$ is an epimorphism of the free part onto Z_p. Let $p: P(\Omega S^{2k}_{p-1}; S^{2k-1}, *)\to S^{2k-1}$ be the projection of paths onto their initial points. It follows that $p\theta$ represents a generator of $\Pi_{2kp-3}(S^{2k-1}; p)$. Thus the diagram (B) may be assumed to homotopy commute. (We may have to take $\bar{a}h$ to represent $\{x\alpha_1(2k-1), \alpha_1, \cdots, \alpha_1\}$ for some $x\in Z_p$, $x\neq 0$, instead of $\{\alpha_1(2k-1), \alpha_1, \cdots, \alpha_1\}$, but this does not essentially change our argument and we disregard it.)

$$\text{(B)}\qquad \begin{array}{ccc} S^{2kp-3} & \xrightarrow{\theta} & P(\Omega S^{2k}_{p-1}; S^{2k-1}, *) \\ \downarrow h & & \downarrow p \\ C_{k+1, k-2} & \xrightarrow{\bar{a}} & S^{2k-1} \end{array}$$

θ defines a map $\bar{\theta}: (D^{2kp-2}, S^{2kp-3})\to(\Omega S^{2k}_{p-1}, S^{2k-1})$ so that the diagram (C) commutes.

$$\text{(C)}\qquad \begin{array}{ccc} \Pi_i(D^{2kp-2}, S^{2kp-3}) & \xrightarrow{\partial} & \Pi_{i-1}(S^{2kp-3}) \\ \downarrow \bar{\theta}_* & & \downarrow (p\theta)_* \\ \Pi_i(\Omega S^{2k}_{p-1}, S^{2k-1}) & \xrightarrow{\partial} & \Pi_{i-1}(S^{2k-1}) \\ \downarrow q_* & \nearrow \partial & \\ \Pi_i(\Omega^2S^{2k+1}, S^{2k-1}) & & \end{array}$$

The map h is exactly the attaching map of the $(2kp-2)$-cell of $C_{k+1, k+1}$ to $C_{k+1, k-2}$. Hence, $\bar{a}$ extends to $C_{k+1, k-1}$ in such a manner that, if $\mu: (D^{2kp-2}, S^{2kp-3})\to(C_{k+1, k-1}, C_{k+1, k-2})$ is the characteristic map of the top-dimensional cell, then the diagram (D) homotopy commutes,

$$\text{(D)}\qquad \begin{array}{ccc} (D^{2kp-2}, S^{2kp-3}) & \overset{\bar{\theta}}{\searrow} & \\ \downarrow \mu & & (\Omega S^{2k}_{p-2}, S^{2k-1}) \\ (C_{k+1, k-1}, C_{k+1, k-2}) & \underset{\sigma}{\nearrow} & \end{array}$$

where σ is the map extending $\bar{a}$. Note that $q\sigma$ extends $\bar{a}$ to a map of $C_{k+1, k-2}$ to Ω^2S^{2k+1}.

Now let $Q=2s(p-1)+2kp-3$. $g_s: S^Q \to C_{k+1,k-1}$ represents an element $\gamma_s \in \Pi_Q(C_{k+1,k-1})$. It follows by an easy induction, using the definition of a C-presentation in which all the spaces X_i are spheres, that, if $j: C_{k+1,k-1} \to (C_{k+1,k-1}, C_{k+1,k-2})$, then $j_*\gamma_s$ is in the image of μ_*. Let $\nu: (D^{2kk-2}, S^{2kp-3}) \to (S^{2kp-2}, *)$ be the map identifying S^{2kp-3} to a point. Then, since we are in the p-stable range, $\nu_*: \Pi_Q(D^{2kp-2}, S^{2kp-3}, p) \to \Pi_Q(S^{2kp-2}; p)$ is an isomorphism. Hence, there is a unique element $\bar{\alpha}_s \in \Pi_Q(D^{2kp-2}, S^{2kp-3}; p)$ such that $\nu_*(\bar{\alpha}_s) = \alpha_s(2kp-2)$.

(E).
$$\begin{tikzcd}
\Pi_Q(D^{2kp-2}, S^{2kp-3}; p) \arrow[r, "\nu_*"] \arrow[d, "\mu_*"] & \Pi_Q(S^{2kp-2}; p) \\
\Pi_Q(C_{k+1,k-1}, C_{k+1,k-2}; p) \arrow[ur, "\eta_*"] & \Pi_Q(C_{k+1,k-1}; p) \arrow[l, "j_*"] \arrow[u, "\eta_*"']
\end{tikzcd}$$

Consider the diagram (E). The relative η_* is also a C-isomorphism in this degree range. Hence, $\mu_*(\bar{\alpha}_s) = j_*(\gamma_s)$. Now consider the diagram (F), where i' is inclusion. It is clearly commutative. We have noted that $\bar{\theta}_*$ and q_* are isomorphisms in this dimension and that i'_* is an isomorphism if $s<p-k$ and an epimorphism if $s=p-k$. We know also that $j_*(\gamma_s) = \mu_*(\bar{\alpha}_s)$. Hence, $(q\sigma)_* j_*(\gamma_s)$ is a generator of $\Pi_Q(\Omega^2 S^{2k+1}, S^{2k-1}; p)$.

(F)
$$\begin{tikzcd}
 & \Pi_Q(D^{2kp-2}, S^{2kp-3}; p) \arrow[r, "\bar{\theta}_*"] \arrow[d, "\mu_*"] & \Pi_Q(\Omega S^{2k}_{p-1}, S^{2k-1}; p) \arrow[ddl, "q_*"] \\
\Pi_Q(C_{k+1,k-1}; p) \arrow[r, "j_*"] \arrow[d, "(q\sigma)_*"] & \Pi_Q(C_{k+1,k-1}, C_{k+1,k-2}; p) \arrow[ur, "\sigma_*"] \arrow[d, "(q\sigma)_*"] & \\
\Pi_Q(\Omega^2 S^{2k+1}; p) \arrow[r, "i'_*"] & \Pi_Q(\Omega^2 S^{2k+1}, S^{2k-1}; p) &
\end{tikzcd}$$

It then follows that $(q\sigma)_*(\gamma_s)$ is a generator of $\Pi_Q(\Omega^2 S^{2k+1}; p)$ if $s<p-k$, and an element which maps into a generator of $\Pi_Q(\Omega^2 S^{2k+1}, S^{2k-1}; p)$ if $s=p-k$.

We now replace the maps $S^Q \xrightarrow{g_s} C_{k+1,k-1} \xrightarrow{q\sigma} \Omega^2 S^{2k+1}$ by the adjoint maps $S^{Q+2} \xrightarrow{S^2 g_s} S^2 C_{k+1,k-1} \xrightarrow{(q\sigma)'} S^{2k+1}$. It follows from 2.3D that this composition represents an element of $\{\alpha_1(2k+1), \alpha_1, \cdots, \alpha_s\}_C$ (the $(k+1)$-fold product). It follows from the remarks above that this element is in the p-component, is non-zero, and is not in the kernel of the map j_* of sequences (A) above. Hence, if $s<p-k$, this element generates $G(2(k+s)(p-1)-2, k)$, and if $s=p-k$, it maps into a generator of Z_p under the map j_* of sequences a$'$) or a$''$), whichever is appropriate.

This proves that there are C-presentations for the $(k+1)$-fold products $\{\alpha_1(2k+1), \alpha_1, \cdots, \alpha_1, \alpha_s\}_C$. $1\leq s\leq p-k$, which contain non-zero elements as discussed above. However, we have more control over the product if we rephrase this result as follows. Let $\tilde{\alpha}_1(2k+1) \in \Pi_{2(p+k-2)}(\Omega^2 S^{2k+1}; p)$ be the element corresponding to $\alpha_1(2k+1)$. We have constructed a C-presentation for the $(k+1)$-fold product $\{\tilde{\alpha}_1(2k+1), \alpha_1, \cdots, \alpha_1, \alpha_s\}_C$ with the property that the map of $C_{k+1, k-2}$ into $\Omega^2 S^{2k+1}$ factors through S^{2k-1}. By 3.3D we may construct an F-presentation for this product so that the map $f_{3, k-2} : P_{3, k-2} \rightarrow \Omega^k S^{2k+1}$ factors through $\Omega^{k-2} S^{2k-1}$. We may have had to change the maps $g'_{m, n}$ of the C-presentation (see 3.3D) but it is easily seen, using the arguments given above, that, looked at in $\Pi_Q(S^{2k+1})$, this product is still non-zero and maps right under the maps j_* of sequences (A), a$'$), a$''$) above. Further, any two elements of $\{\tilde{\alpha}_1(2k+1), \alpha_1, \cdots, \alpha_s\}_F$ for this presentation differ by a map into $\Omega P_{3, k-2}$ composed with $\Omega f_{3, k-2}$, and hence, looked at in $\Pi_Q(S^{2k+1})$, by an element in the image of the double suspension. This proves the following theorem.

Theorem 6.4. *If* $2\leq k\leq p-1$, $1\leq s\leq p-k$, *then there exist F-presentations for the* $(k+1)$*-fold product* $\{\tilde{\alpha}_1(2k+1), \alpha_1 \cdots, \alpha_1, \alpha_s\}_F$ *such that:* i) *if* ρ, σ *are in the product then* $\rho-\sigma$ *is in the image of the double suspension when the product is viewed as being in* $\Pi_{2(k+s)(p-1)+2k-1}(S^{2k+1})$. ii) *the product contains elements which are in the p-component, and which generate the p-component if* $s<p-k$, *and map into non-zero elements under the maps* j_* *of sequences* a$'$), a$''$) *if* $s=p-k$.

Corollary 6.5. *If* $2\leq k\leq p-1$, $1\leq s<p-k$ *then for an F-pre-*

sentation as in 6.4, *if* $\{\tilde{\alpha}_1(2k+1), \alpha_1, \cdots, \alpha_1, \alpha_s\}_F$ *is viewed as being in* $\Pi_{2(k+s)(p-1)+2k-1}(S^{2k+1})$, $\{\tilde{\alpha}_1(2k+1), \alpha_1, \cdots, \alpha_1, \alpha_s\}_F \cap G(2(k+s)(p-1)-2, k)$ *contains one element which generates* $G(2(k+s)(p-1)-2, k)$.

Proof: This follows from 6.4 and the fact (see pp. 177–178 of [16]) that the image of the double suspension in this group is zero.

Recall that we have set $N=2p(p-1)-1$. We now examine the unstable $(N-1)$-stem. First note that by 6.4 there is an element $\delta_2 \in G(N-1, 2)$ such that $j_* \delta_2 \neq 0$, $\delta_2 \in \{\tilde{\alpha}_1(5), \alpha_1, \alpha_{p-2}\}_F$. According to Proposition 4.17, ii) of [14], $\{\alpha_1, \alpha_s, p\iota\} = m_s \alpha_{s+1}$, where $1 \leq s < p-1$, and $m_s \equiv \dfrac{1}{s+1} \pmod{p}$. (The indeterminary of the product need not concern us here; it is all of orders prime to p, and may be gotten rid of by multiplying by an appropriate integer prime to p. In any case, $m_s \alpha_{s+1}$ is the only element in the p-component in this product.)

Then, $p\delta_2 \in \{\{\tilde{\alpha}_1(5), \alpha_1, \alpha_{p-2}\}, p\iota\}_F$, which, by 4.2, is, up to sign, $\{\tilde{\alpha}_1(5), m_{p-2}\, \alpha_{p-1}\}$, which, viewed in the homotopy of S^5 rather than in that of $\Omega^2 S^5$, is the double suspension of $m_{p-2}\alpha_1(3) \circ \alpha_{p-1}$. But $m_{p-2}\alpha_1(3) \circ \alpha_{p-1}$ is a generator of $G(N-1, 1)$. If follows from the exact sequence a″) that $G(N-1, 2) \simeq Z_{p^2}$ with generator δ_2.

To prove that $G(N-1,k) \simeq Z_{p^2}$, $2 \leq k \leq p-1$, we make the induction hypothesis that if $\delta_k \in G(N-1, k)$ occurs in the $(k+1)$-fold product $\{\alpha_1(2k+1), \alpha_1, \cdots, \alpha_1, \alpha_{p-k}\}_F$ as in 6.4, then δ_k is of order p^2. It follows from sequence a′) that if this statement is true, then $S^2 \delta_k$ is of order p, and $S^4 \delta_{k-1} = 0$. We have proved the statement for $k=2$.

In order to make the inductive step, we must modify the definition of δ_k slightly. Let $\hat{\alpha}_1(2k+1) \in \Pi_{2(k+p-3)}(\Omega^4 S^{2k+1})$ be the element corresponding to $\alpha_1(2k+1) \in \Pi_{2(k+p-1)}(S^{2k+1})$. As in the proof of 6.4 we may construct a C-presentation for the $(k+1)$-fold product $\{\hat{\alpha}_1(2k+1), \alpha_1, \cdots, \alpha_1, \alpha_{p-k}\}_C$ in such a way that the map of $C_{k+1, k-3}$ into $\Omega^4 S^{2k+1}$ factors through S^{2k-3} and the map of $C_{k+1, k-2}$ into $\Omega^4 S^{2k+1}$ factors through $\Omega^2 S^{2k-1}$. These maps may be so chosen that the two diagrams analogous to diagram (D) in the proof of 6.4 (one diagram for the extension of the map of $C_{k+1,k-3}$ into S^{2k-3} to a map of $C_{k+1,k-2}$ into Ω^2S^{2k-1}, and the other for the extension of the map of $C_{k+1,k-2}$ into Ω^2S^{2k-1} to a map of $C_{k+1,k-1}$ into Ω^4S^{2k+1}) homotopy commute.

The product we obtain is in the homotopy of Ω^4S^{2k+1}. If we look at this in the homotopy of Ω^2S^{2k+1}, taking adjoints and double suspensions of the presentation as in the proof of 6.4, we see that we get a C-presentation like that in the proof of 6.4, and hence that we get an element like that in the C-presentation of the proof of 6.4. If we take a corresponding F-presentation, using 3.3D, we obtain an F-presentation in which $f_{4,k-3}: P_{4,k-3}\to\Omega^{k+1}S^{2k+1}$ factors through $\Omega^{k-3}S^{2k-3}$, and $f_{3,k-2}: P_{3,k-2}\to\Omega^{k+1}S^{2k+1}$ factors through $\Omega^{k-1}S^{2k-1}$. If we choose an element in $\{\hat{\alpha}_1(2k+1), \alpha_1, \cdots, \alpha_1, \alpha_{p-k}\}_F$ for this product, then, looked at in $\Pi_{2k+N}(S^{2k+1})$, it maps into a non-zero element under the map j_* of sequence a''). Let δ_k' be such an element. Assuming the induction hypothesis for $k-1$, it suffices to prove that $p\delta_k'\neq 0$, since this will prove that $G(N-1, k)$ is Z_{p^2}, and hence that it is generated by any element δ such that $j_*\delta$ is non-zero.

Now, $\delta_k'\in\{\hat{\alpha}_1(2k+1), \alpha_1, \cdots, \alpha_1, \alpha_{p-k}\}_F$, a $(k+1)$-fold product. Using 4.2, $p\delta_k'\in\{\{\hat{\alpha}_1(2k+1), \alpha_1, \cdots, \alpha_1, \alpha_{p-k}\}_F, p\iota\}\subseteq\pm\{\hat{\alpha}_1(2k+1), \cdots, \alpha_1, \{\alpha_1, \alpha_{p-k}, p\iota\}\}_F$. Any element in $\{\alpha_1, \alpha_{p-k}, p\iota\}$ is of the form $m_{p-k}\alpha_{p-(k-1)}+\sigma$, where the order of σ is prime to p. Since this product is additive in the last variable, and since $\{\hat{\alpha}_1(2k+1), \cdots, \alpha_1, \sigma\}_F$ will be zero since σ is prime to p, we conclude that $p\delta_k'\in\pm m_{p-k}\{\hat{\alpha}_1(2k+1), \alpha_1, \cdots, \alpha_1, \alpha_{p-(k-1)}\}_F$, a k-fold product. Note that the towers of fibrations that occur in this presentation are exactly those that occur in the presentation for δ_k'. They are in fact the $P_{i,j}$ for that presentation with $i\geq 3$; the $f_{i,j}$ are the same as in that presentation, and the only new thing is the map of $S^{N+2k-4}\to P_{3,k-2}$. In using 3.3 to construct a C-presentation for this product, we observe that we almost have this already in the C-presentation corresponding to the F-presentation for δ'_k. In fact, we already have maps $\theta_{i,j}$: $S^j P_{i,j} \to C_{i+j,j}$ for all the (i, j) in this presentation, making the diagrams (see 3.3)

$$\begin{tikzcd} S^j P_{i,j} \arrow[d] \arrow[dr] & \\ & X_{i+j+1} \\ C_{i+j,j} \arrow[ur] & \end{tikzcd}$$

homotopy commute for all the (i, j) involved, and the diagrams

$$\begin{tikzcd} & S^j P_{i,j} \arrow[d] \\ S^j X_i \arrow[ur] \arrow[dr] & \\ & C_{i+j,j} \end{tikzcd} \qquad \text{homotopy commute}$$

except for the cases in which X_i is S^{N+2k-4}, i.e., the case $i=3$, $j \leq k-2$. The problem here is that the maps of $S^{N+2k-4+j}$ into $C_{j+3,j}$ are not defined, since we do not yet have a C-presentation for the k-fold product $\{\hat{\alpha}_1(2k+1), \alpha_1, \cdots, \alpha_1, \alpha_{p-(k-1)}\}_C$. However, the required maps may be constructed from the F-presentation, as in the proof of 3.3. Thus, we now have a C-presentation for $p\delta'_k$ as an element of the product $\{\hat{\alpha}_1(2k+1), \alpha_1, \cdots, \alpha_1, \alpha_{p-(k-1)}\}_C$. However, our construction of the maps $P_{4,k-3} \to \Omega^{k+1}S^{2k+1}$ and $P_{3,k-2} \to \Omega^{k+1}S^{2k+1}$ as factoring through $\Omega^{k-3}S^{2k-3}$ and $\Omega^{k-1}S^{2k-1}$ respectively (as in the proof of 6.4) allow us to conclude that $p\delta'_k = \pm m_{p-k} S^2 \delta_{k-1}$ where δ_{k-1} is in the k-fold product $\{\tilde{\alpha}_1(2k-1), \alpha_1, \cdots, \alpha_1, \alpha_{p-(k-1)}\}_F$, as in 6.3, when everything is looked at in the homotopy of S^{2k+1}. Note that $S^2\delta_{k-1}$ is well-defined for the given presentation; any two choices of δ_k differ by an element in the double suspension so that any two $S^2\delta_{k-1}$ differ by an element in the quadruple suspension, which is zero by our induction hypothesis. Our induction hypothesis also implies that, since δ_{k-1} generates $G(N-1, k-1)$, $S^2\delta_{k-1}$ is non-zero. Hence, δ'_k is of order p^2, which completes the induction. This proves the following.

Theorem 6.6. $\Pi_{2p(p-1)-2+2k+1}(S^{2k+1}, p) \simeq Z_{p^2}$ for $2 \leq k \leq p-1$. *A generator is given by the* δ_k *of* 6.4. *The image of this group* *under double suspension is cyclic of order p; its image under quadruple suspension is zero if $k<p-1$.*

Corollary 6.7. *The lowest dimensional sphere in whose homotopy groups there is an element which suspends to β_1 is S^{2p-1}, in whose homotopy there is an element δ_{p-1} of order p^2 which double suspends to β_1 in the homotopy of S^{2p+1}.*

References

[1] J. F. Adams, *Cohomology Operations.* Mimeographed notes, Seattle Topology Conference, 1963.

[2] A. Borel and J. P. Serre, *Groupes de Lie et puissances réduites de Steenrod*, Amer. J. Math., **75** (1953) 409-448.

[3] B. Eckmann and P. J. Hilton, *Groupes d'homotopie et dualité.* C. R. Acad. Sci. Paris, **246** (1958) 2555-2558, 2991-2993.

[4] K. A. Hardie, *On the Hopf-Toda invariant*, Trans. Amer. Math. Soc. **112** (1964) 43-54.

[5] P. J. Hilton, *Remark on loop spaces*, Proc. Amer. Math. Soc., **15** (1964) 596-600.

[6] S. T. Hu, *Homotopy Theory;* New York: Academic Press, 1959.

[7] W. S. Massey, *Some higher order cohomology operations*, pp. 145-154, Symposium internacional de topología algebraica; Mexico City: Universidad Nacional Autonoma de Mexico and UNESCO, 1958.

[8] F. P. Peterson, *Generalized cohomotopy groups*, Amer. J. Math. **78** (1956) 259-281.

[9] ——————, *Functional cohomology operations*, Trans. Amer. Math. Soc. **86** (1957) 197-210.

[10] E. H. Spanier, *Secondary operations on mappings and cohomology*, Ann. Math. **75** (1962) 260-282.

[11] H. Toda, *Generalized Whitehead products and the homotopy groups of spheres*, J. Inst. Poly. Osaka City Univ., **3** (1952) 43-82.

[12] ———, *On double suspension E^2*, J. Inst. Poly. Osaka City Univ., **7** (1956) 103-145.

[13] ———, *p-primary components of homotopy groups III*, Mem. Coll. Sci. Univ. of Kyoto, **31** (1958) 191-210.

[14] ———, *p-primary components of homotopy groups IV*,, Mem. Coll. Sci. Univ. of Kyoto, **32** (1959) 297-332.

[15] ———, *On unstable homotopy of spheres and classical groups*, Proc. Nat. Acad. Sci. **46** (1960) 1102-1105.

[16] ———, *Composition Methods in Homotopy Groups of Spheres*, Princeton: Princeton University Press, Annals of Math. Studies, No. 49, 1962.

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