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Special functions connected with representations of the infinite dimensional motion group

By

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Summary. We consider Hermite polynomials from the stand point of representation theory of groups. We show that the correspondence between the space of multiple Wiener integrals [2] and the decomposition of their Fourier transform introduced by M. G. Krein [5] is obtained by the relations which follow from the representations of the infinite dimensional motion group introduced by A. Orihara [6]. We are lead to various formulae for Hermite polynomials, for instance, orthogonality, differential equation, addition formula, integral representations, Wiener transform and so on, systematically from the relations, and in addition we calculate the matrix elements of the representations of the infinite dimensional motion group and point out that they are the limits of matrix elements of the representations of the finite dimensional Euclidian motion groups.

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§1. Introduction. The Gaussian measure and Hermite polynomials connected with this measure play important roles in probability theory. But they are in an exceptional situation from the stand point that special functions are considered in a unified way by means of representation theory of groups [12]; that is, Hermite polynomials can be investigated by the representation theory of the infinite dimensional motion group (def. 2) [6]. The multiple Wiener integrals well known in probability theory are in this sense the space of irreducible representation of the infinite dimensional rotation (def. 1) [13]. The infinite dimensional Gaussian measure is understood as the limit of the uniform measures on finite dimensional spheres and the Hermite polynomials are obtained as the limit of the Gegenbauer polynomials which are the matrix elements of the representations of the finit dimentional Euclidian motion group [10].

In §2, we define the infinite dimensional rotation group and and the infinite dimensional motion group. We consider the representations of the infinite dimensional motion group on two Hilbert spaces $L^2(L', d\mu_c)$ and $\mathcal{F}$, and we discuss the relations between them. In §3, we have an orthogonal decomposition of $\mathcal{F}$ and define Hermite polynomials as members of a complete orthogonal base of $L^2(L', d\mu_c)$, making use of the relations obtained in §2. In §4, we have the various formulae for Hermite polynomials from the results of §2 and §3. In §5, we calculate matrix elements of the representations of the infinite dimensional motion group and show that they are the limits of matrix elements of the representations of the finite dimensional Euclidian motiin groups investigated by N. Ya. Vilenkin [11].

Remark. We can consider Charlier polynomials in relation with the Poisson white noise. Also in this case, we are lead to orthogonarity and addition formula for Charlier polynomials in an analogous way as in the case of Hermite polynomials.

§2. Representations of the infinite-dimensional motion group. In this section we define the infinite dimensional rotion group and the infinite dimensional motion group. We consider the representations of the infinite dimensional motion group in two way, and we obtain the relation between them.

Let L be an infinite dimensional real nuclear space, and H be its completion by a fixed continuous Hilbertian norm $\| \ \|$. Then we have the relation

$$L \subset H \subset L' .$$

(L' is the dual space of L.)

Definition 1. [7] *The infinite dimensional rotation group* $0(\infty)$ is defined by the following two conditions:

i) $u \in 0(\infty)$ is a linear homeomorphic map from L to L.

ii) This homeomorphic map induces orthogonal transformation of H.

Identifing the adjoint operator u^* of u with u^{-1}, we can regard $0(\infty)$ the group of transformation acting on L'.

Definition 2. [6] *The infinite dimensional motion group* on L' is the set of pairs of elements of $0(\infty)$ and of L with the group operator defined as follows:

$$(u_1, \varphi_1)(u_2, \varphi_2) = (u_1 u_2, u_1 \varphi_2 + \varphi_1), \quad \text{for} \quad u_i \in 0(\infty), \quad \varphi_i \in L.$$

We denote the infinite dimensional motion group by G_∞.

The Gaussian measure μ_c on L' with the variance c^2 ($c > 0$) is defined by the caracteristic functional $\chi(\xi)$ on L as follows:

$$\chi(\xi) = e^{-\frac{c^2\|\xi\|^2}{2}} = \int_{L'} e^{i\langle X, \xi\rangle} d\mu_c(X), \tag{1}$$

where $\xi \in L$, $X \in L'$, and $\langle X, \xi\rangle$ means the canonical bilinear form. The Gaussian measure μ_c is characterised by the following properties [8]:

(G–1) μ_c is $0(\infty)$–invariant.

(G–2) μ_c is L–quasi-invariant, that is, for all $\varphi \in L$, we have

$$\frac{d\mu_{c,\varphi}}{d\mu_c} = e^{-\frac{\langle X, \varphi\rangle}{c^2} - \frac{\|\varphi\|^2}{2c^2}},$$

where $\mu_{c,\varphi}$ is defined by $\mu_{c,\varphi}(X) = \mu_c(X + \varphi)$.

(G–3) μ_c is $0(\infty)$–ergodic, that is, μ_c is one of the smallest of $0(\infty)$–invariant measures.

We denote $L^2(L', d\mu_c)$ the Hilbert space of all complex-valued functions on L' square integrable with respect to the Gaussian measure μ_c.

Definition 3. Let $g(x_1, \cdots, x_n)$ be an n variable polynomial.

We call function L' expressed by

$$F(X) = g(\langle X, \xi_1\rangle, \cdots, \langle X, \xi_n\rangle)$$

a polynomial on L', where $\xi_1, \cdots, \xi_n$ are an orthonormal system of H belonging to L.

Lemma 1. *The set of all polynomials on L' is dense in $L^2(L', d\mu_c)$.*

Lemma 2. *Let us define*

$$F(X) = g(\langle X, \xi_1\rangle, \cdots, \langle X, \xi_n\rangle),$$
$$F'(X) = g'(\langle X, \xi_1\rangle, \cdots, \langle X, \xi_n\rangle).$$

When $F(X)$ and $F'(X)$, two polynomials on L', coincide with each other as elements of $L^2(L', d\mu_c)$, so do $g(x)$ and $g'(x)$ as polynomials.

These lemmas are trivial and so we omit the proofs.

Now we consider the representations of the group G_∞ on $L^2(L', d\mu_c)$.

Put

$$T_g F(X) = e^{-\frac{i\langle X,\varphi\rangle}{2c^2}} F(u^{-1}X), \tag{2}$$

$$S_g F(X) = e^{-\frac{\|\varphi\|^2+2\langle Y,\varphi\rangle}{4c^2}} F(u^{-1}(X+\varphi)), \tag{3}$$

for $g=(u, \varphi)$, $F(X)\in L^2(L', d\mu_c)$.

Then, $(T_g, L^2(L', d\mu_c))$ and $(S_g, L^2(L', d\mu_c))$ are unitary representations of G_∞ in virtue of the properties (G$-$1) and (G$-$2) respectively.

We define Wiener transform W from $L^2(L', d\mu_c)$ to itself as follows:

$$WF(X) = \int_{L'} g(\sqrt{2}\langle X, \xi_1\rangle + i\langle Y, \xi_1\rangle, \cdots, \sqrt{2}\langle X, \xi_n\rangle + i\langle Y, \xi_n\rangle) \times d\mu_c(X) = \tilde{g}(\langle Y, \xi_1\rangle, \cdots, \langle Y, \xi_n\rangle) = \tilde{F}(Y), \tag{4}$$

where $F(X)=g(\langle X, \xi_1\rangle, \cdots, \langle X, \xi_n\rangle)$ and $g(x_1\cdots x_n)$, $\tilde{g}(x_1\cdots x_n)$ are two n variable usual polynomials.

By simple calculations we have

$$\|F(X)\|_{L^2(L', d\mu_c)} = \|\tilde{F}(Y)\|_{L^2(L', d\mu_c)}.$$

Since polynomials on L' are dense in $L^2(L', d\mu_c)$, W can be extended to an isometric map defined on $L^2(L', d\mu_c)$.

We have the following relation between T_g and S_g:

$$T_g W = W S_g \,. \tag{5}$$

In fact, if $F(X)$ is a polynomial on L', we have

$$\begin{aligned} WS_gF &= We^{-\frac{\|\varphi\|^2+2\langle X,\varphi\rangle}{4c^2}}F(u^{-1}(X+\varphi)) \\ &= \int_{L'} e^{-\frac{\|\varphi\|^2+2\sqrt{2}\langle X,\varphi\rangle+2i\langle Y,\varphi\rangle}{4c^2}}F(\sqrt{2}u^{-1}X+iu^{-1}Y+u^{-1}\varphi)d\mu_c(X)\,. \end{aligned}$$

If we put $\sqrt{2}X=\sqrt{2}uX'-\varphi$, then in virtue of (G-2), this is equal to

$$\begin{aligned} &= \int_{L'} e^{-\frac{i\langle Y,\varphi\rangle}{2c^2}}F(\sqrt{2}X'+iu^{-1}Y)d\mu_c(X') \\ &= T_g\, WF\,. \end{aligned}$$

Because of the property (G-3) of the Gaussian measure μ_c, the representations $(T_g, L^2(L', d\mu_c))$ and $(S_g, L^2(L', d\mu_c))$ are irreducible [6].

Now we denote by $\mathcal{F}$ the smallest (complex) Hilbert space with the reproducing kernel $\chi(\xi-\varphi)$ [1] [3]. $\mathcal{F}$ has the following properties:

(F-1) For any fixed $\varphi\in L$, $\chi(\cdot-\varphi)\in\mathcal{F}$.

(F-2) For any $f\in\mathcal{F}$, $(f(\cdot), \chi(\cdot-\varphi))_{\mathcal{F}}=f(\varphi)$.

(F-3) $\mathcal{F}$ is spanned by $\{\chi(\cdot-\varphi)\,;\,\varphi\in L\}$.

(F-4) $\|\sum_{j=1}^{n} a_j\chi(\xi-\varphi_j)\|_{\mathcal{F}}^2 = \sum_{j,k}^{n} a_i\bar{a}_k\chi(\varphi_j-\varphi_k)$.

Lemma 3 [3]. *Put*

$$UF(X) = \int_{L'} e^{i\langle X,\xi\rangle}F(X)d\mu_c(X)\,, \tag{6}$$

for $F(X)\in L^2(L', d\mu_c)$. *Then* U *is a one to one linear isometric map from* $L^2(L', d\mu_c)$ *onto* $\mathcal{F}$.

Proof. For any complex numbers a_j and for any $\{\varphi_j\}\subset L$,

$$\begin{aligned} U\sum_{j=1}^{n} a_j e^{i\langle X,\varphi_j\rangle} &= \sum_{j=1}^{n} a_j\int_{L'} e^{i\langle X,\xi\rangle+i\langle X,\varphi_j\rangle}d\mu_c(X) \\ &= \sum_{j=1}^{n} a_j\chi(\xi+\varphi_j)\,. \end{aligned} \tag{7}$$

Since $\{\sum_{j=1}^{n} a_j e^{i\langle X,\varphi_j\rangle};\ a_j$ complex number, $\varphi_j \in L\}$ is dense in $L^2(L', d\mu_c)$, we have the conclusion of this lemma from the relation (6) and the property (F-4). Q.E.D.

We denote by $(t_g, \mathcal{F})$ and $(s_g, \mathcal{F})$ the representations of G_∞ on $\mathcal{F}$ obtained from $(T_g, L^2(L', d\mu_c))$ and $(S_g, L^2(L', d\mu_c))$ by the transformation U respectively. Then we have

$$UT_gU^{-1}f(\xi) = t_gf(\xi) = f\left(u^{-1}\left(\xi - \frac{\varphi}{2c^2}\right)\right), \tag{8}$$

$$US_gU^{-1}f(\xi) = s_gf(\xi) = e^{-\frac{\|\varphi\|^2}{4c^2} - i(\xi,\varphi)} f\left(u^{-1}\left(\xi - \frac{i\varphi}{2c^2}\right)\right), \tag{9}$$

for $f(\xi) \in \mathcal{F}$, $UF(X) = f(\xi)$, $g = (u, \varphi)$. In fact

$$\begin{aligned} t_gf(\xi) &= UT_gU^{-1}f(\xi) = Ue^{-\frac{i\langle X,\varphi\rangle}{2c^2}}F(u^{-1}X) \\ &= \int_{L'} e^{i\langle X,\xi\rangle} e^{-\frac{i\langle X,\varphi\rangle}{2c^2}} F(u^{-1}X)d\mu_c(X) \\ &= \int_{L'} e^{i\left\langle uY,\xi-\frac{\varphi}{2c^2}\right\rangle} F(Y)d\mu_c(Y) = f\left(u^{-1}\left(\xi - \frac{\varphi}{2c^2}\right)\right). \end{aligned}$$

$$\begin{aligned} s_gf(\xi) &= US_gU^{-1}f(\xi) = Ue^{-\frac{\|\varphi\|^2+2\langle X,\varphi\rangle}{4c^2}}F(u^{-1}(X+\varphi)) \\ &= \int_{L'} e^{i\langle X,\xi\rangle - \frac{\|\varphi\|^2+2\langle X,\varphi\rangle}{4c^2}} F(u^{-1}(X+\varphi))d\mu_c(X). \end{aligned}$$

If we put $X = uY - \varphi$, then owing to (G-2), this is equal to

$$\begin{aligned} &= e^{-\frac{\|\varphi\|^2}{4c^2} - i(\xi,\varphi)} \int_{L'} e^{i\langle Y,u^{-1}\xi\rangle + \left\langle Y,\frac{u^{-1}\varphi}{2c^2}\right\rangle} F(Y)d\mu_c(Y) \\ &= e^{-\frac{\|\varphi\|^2}{4c^2} - i(\xi,\varphi)} f\left(u^{-1}\left(\xi - \frac{i\varphi}{2c^2}\right)\right). \end{aligned}$$

Now we consider the case of $g = (I, r\varphi)$, where r runs the real numbers and I means the identity of $0(\infty)$.

Put

$$T_g = T_r^\varphi, \quad S_g = S_r^\varphi, \quad t_g = t_r^\varphi, \quad s_g = s_r^\varphi;$$

then they satisfy the following relations:

i) $T_q \cdot T_r = T_{q+r}$.

ii) $T_0 = 1$ (the identity operator).

iii) $\lim_{r \to 0} \| T_r - I \| = 0\,.$ (10)

It is sufficient to show that T_r^φ satisfies the condition (10) owing to the relations (5), (8), and (9).

$$
\begin{aligned}
T_q^\varphi \cdot T_r^\varphi F(X) &= T_q^\varphi e^{-i\frac{\langle X, r\varphi\rangle}{2c^2}} F(X) \\
&= e^{-\frac{i\langle X, q\varphi\rangle}{2c^2} - \frac{i\langle X, r\varphi\rangle}{2c^2}} F(X) \\
&= e^{-\frac{i(q+r)\langle X, \varphi\rangle}{2c^2}} F(X) \\
&= T_{q+r}^\varphi F(X)\,.
\end{aligned}
$$

It is obvious that T_0 is equal to the identity operator, and finally

$$
\begin{aligned}
&\lim_{r \to 0} \| T_r^\varphi F(X) - F(X) \| \\
&\qquad = \lim_{r \to 0} \int_{L'} |(e^{i\langle X, r\varphi\rangle} - 1) F(X)|^2 d\mu_c(X) = 0\,.
\end{aligned}
$$

Therefore we have the following expressions according to Stone's theorem:

$$
\begin{aligned}
T_r^\varphi &= e^{irA_\varphi}\,, \\
S_r^\varphi &= e^{irB_\varphi}\,, \\
t_r^\varphi &= e^{ira_\varphi}\,, \\
s_r^\varphi &= e^{irb_\varphi}\,,
\end{aligned}
\tag{11}
$$

where A_φ and B_φ are self-adjoint operators defined on $L^2(L', d\mu_c)$ and a_φ and b_φ are self-adjoint operators defined on $\mathfrak{F}$.

We have the following four expressions easily:

$$
2c^2 A_\varphi F(X) = -\langle X, \varphi\rangle F(X)\,, \tag{12}
$$

if there exists a positive number δ such that for $|r| < \delta$, $e^{r\langle X, \varphi\rangle} F(X)$ belongs to $L^2(L', d\mu_c)$.

$$
2c^2 B_\varphi F(X) = i\langle X, \varphi\rangle F(X) + 2c^2 D_\varphi F(X)\,, \tag{15}
$$

if $F(X)$ belongs to the domain of A_φ and $D_\varphi F(X)$ exists, where

$$
D_\varphi F(X) = \lim_{r \to 0} \frac{F(X + r\varphi) - F(X)}{ir}
$$

by definition, if the limit exists in the sense of norm of $L^2(L', d\mu_c)$.

$$
2c^2 a_\varphi f(\xi) = -d_\varphi f(\xi)\,, \tag{14}
$$

where

$$d_\varphi = \lim_{r\to 0}\frac{f(\xi+r\varphi)-f(\xi)}{ir},$$

if the limit exists in the norm of $\mathcal{F}$.

$$2c^2 b_\varphi f(\xi) = -2c^2(\xi, \varphi)f(\xi)-id_\varphi f(\xi). \tag{15}$$

if $f(\xi)$ belongs to the domain of d_φ and if there exists a positive number δ such that for $|r|<\delta$, $e^{r(\varphi,\xi)}f(\xi)$ belongs to $\mathcal{F}$.

From the rerations (8) and (9) we have

$$UA_\varphi U^{-1} = a_\varphi, \tag{16}$$

$$UB_\varphi U^{-1} = b_\varphi; \tag{17}$$

that is,

$$U\langle X, \varphi\rangle U^{-1} = d_\varphi, \tag{18}$$

$$U(i\langle X, \varphi\rangle+2c^2D_\varphi)U^{-1} = -2c^2(\xi, \varphi)-id_\varphi. \tag{19}$$

From the rerations (18) and (19) we have

$$c^2UD_\varphi U^{-1}+iU\langle X, \varphi\rangle U^{-1} = -c^2(\xi, \varphi). \tag{20}$$

Further from the relation (5), we have

$$-W^{-1}\langle X, \varphi\rangle W = i\langle X, \varphi\rangle+2c^2D_\varphi. \tag{21}$$

§ 3. **Hermite polynomials.** In this section we have an orthogonal decomposition of $\mathcal{F}$ making use of the relations (18) and (20), and we have Hermite polynomials as a complete orthogonal base of $L^2(L', d\mu_c)$ in virtue of the orthogonal decomposition of $\mathcal{F}$.

Let $\{\xi_j\,; j=1, 2, \cdots, \xi_j\in L\}$ be a complete orthonormal base of H.

Lemma 4 [5]. *$\mathcal{F}$ can be decomposed as follows:*

$$\mathcal{F} = \sum_{n=0}^{\infty}\oplus\mathcal{F}_n \tag{22}$$

where $\mathcal{F}_n$ is spanned by $\{(\xi, \xi_{k_1})^{l_1}\cdots(\xi, \xi_{k_j})^{l_j}e^{-\frac{c^2\|\xi\|^2}{2}}\,;\ l_1+\cdots+l_j=n$, l_i: nonneganive integer, k_i: integer$\}$. Put

$$\varphi_{k_1\cdots k_j}^{l_1\cdots l_j}(\xi) = \frac{c^n e^{-\frac{c^2\|\xi\|^2}{2}}}{\sqrt{l_1!\cdots l_j!}}(\xi, \xi_{k_1})^{l_1}\cdots(\xi,\xi_{k_j})^{l_j}. \tag{23}$$

Then $\{\varphi_{k_1\cdots k_j}^{l_1\cdots l_j},\ l_1+\cdots+l_j=n\}$, *is a complete orthonormal base of* $\mathcal{F}_n$.

Proof. From the relations (12) and (18), the domain of d_{ξ_k} ($\mathfrak{D}(d_{\xi_k})$) contains $e^{-\frac{c^2\|\xi\|^2}{2}}$ and

$$\begin{aligned} d_{\xi_k} e^{-\frac{c^2\|\xi\|^2}{2}} &= \lim_{r\to 0} \frac{1}{ir}\left(e^{-\frac{c^2\|\xi+r\xi_k\|^2}{2}} - e^{-\frac{c^2\|\xi\|^2}{2}}\right) \\ &= \lim_{r\to 0} \frac{1}{ir} e^{-\frac{c^2\|\xi\|^2}{2}}\left(e^{-c^2(\xi,\xi_k)r-\frac{c^2r^2}{2}}-1\right) \\ &= ic^2(\xi,\xi_k)e^{-\frac{c^2\|\xi\|^2}{2}}. \end{aligned} \tag{24}$$

Generally, $\mathfrak{D}(d_{\xi_k})$ contains $(\xi,\xi_k)^n e^{-\frac{c^2\|\xi\|^2}{2}}$ and

$$i^n c^{2n}(\xi,\xi_k)^n e^{-\frac{c^2\|\xi\|^2}{2}} = P_n(d\xi_k)e^{-\frac{c^2\|\xi\|^2}{2}}, \tag{25}$$

where P_n is a polynomial of degree $\leqq n$ with 1 for the coefficient in the highest term.

At first we show that $(\xi,\xi_k)^{l_1}e^{-\frac{c^2\|\xi\|^2}{2}}$ and $(\xi,\xi_k)^{l_2}e^{-\frac{c^2\|\xi\|^2}{2}}$ are mutually orthogonal if $l_1 \neq l_2$.

We assume $l_1 \geqq l_2$. Then

$$\begin{aligned} &\left((\xi,\xi_k)^{l_1}e^{-\frac{c^2\|\xi\|^2}{2}}, (\xi,\xi_k)^{l_2}e^{-\frac{c^2\|\xi\|^2}{2}}\right)_{\mathcal{F}} \\ &= \left((\xi,\xi_k)^{l_1}e^{-\frac{c^2\|\xi\|^2}{2}}, i^{-l_2}c^{-2l_2}P_{l_2}(d_{\xi_k})e^{-\frac{c^2\|\xi\|^2}{2}}\right)_{\mathcal{F}}, \end{aligned}$$

d_{ξ_k} being self-adjoint operator,

$$= \left(\bar{P}_{l_2}(d_{\xi_k})\cdot(\xi,\xi_k)^{l_1}e^{-\frac{c^2\|\xi\|^2}{2}}, i^{-l_2}c^{-2l_2}e^{-\frac{c^2\|\xi\|^2}{2}}\right)_{\mathcal{F}}, \tag{$*$}$$

$\bar{P}(d_{\xi_k})(\xi,\xi_k)^{l_1}e^{-\frac{c^2\|\xi\|^2}{2}}$ forms a polynomial of (ξ,ξ_k) at least degree $(l_1-l_2)\times e^{-\frac{c^2\|\xi\|^2}{2}}$ and if $l_1-l_2>0$, then in virtue of the property (F-2), we have $(*)=0$.

If $l=l_1=l_2$, then

$$P_l(d_{\xi_k})(\xi,\xi_k)^l = l!\, i^{-l}+(\xi,\xi_k)Q_{l-1}((\xi,\xi_k)),$$

where Q_l is a polynomial at most degree l, and we have

$$(*) = (l!i^{-l}e^{-\frac{c^2\|\xi\|^2}{2}},\ i^{-l}c^{-2l}e^{-\frac{c^2\|\xi\|^2}{2}})_{\mathcal{F}}+$$
$$+((\xi,\ \xi_k)Q_{l-1}((\xi,\ \xi_k))e^{-\frac{c^2\|\xi\|^2}{2}},\ i^{-l}c^{-2l}e^{-\frac{c^2\|\xi\|^2}{2}})_{\mathcal{F}}$$
$$= \frac{l!}{c^{2l}}. \tag{26}$$

Next, considering the fact that

$$d_{\xi_{k_1}}(\xi,\ \xi_{k_2})^l e^{-\frac{c^2\|\xi\|^2}{2}} = (\xi,\ \xi_{k_2})^l d_{\xi_{k_1}} e^{-\frac{c^2\|\xi\|^2}{2}} \quad \text{if} \quad k_1 \neq k_2\,,$$

we get the last conclusion. Q.E.D.

Put

$$U^{-1}\mathcal{F}_n = L_n\,,$$
$$U^{-1}\varphi^{l_1 \cdots l_j}_{k_1 \cdots k_j}(\xi) = i^{-n}\Phi^{l_1 \cdots l_j}_{k_1 \cdots k_j}(X)\,. \quad (l_1+\cdots+l_j=n) \tag{27}$$

Owing to lemma 3, we have

$$L^2(L',\ d\mu_c) = \sum_{n=0}^{\infty} L_n$$

and $\{\Phi^{l_1 \cdots l_j}_{k_1 \cdots k_j};\ l_1+\cdots+l_j=n\}$ is a complete orthonormal system of L_n.

Theorem 1. *$\Phi^{l_1 \cdots l_j}_{k_1 \cdots k_j}(X)$ is expressed as a polynomial on L' as follows:*

$$\Phi^{l_1 \cdots l_j}_{k_1 \cdots k_j}(X) = (2^n l_1! \cdots l_j!)^{-1/2} \prod_{i=1}^{j} H_{l_i}\left(\frac{\langle X,\ \xi_{k_i}\rangle}{c\sqrt{2}}\right), \tag{28}$$

where $H_n(x)$ is a polynomial in one uariable of degree n satisfying the following recurrence formula:

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)\,,$$
$$H_0(x) = 1\,,$$
$$H_1(x) = 2x\,. \tag{29}$$

Remark. $H_n(x)$ satisfying (29) is the so-called Hermite polynomial of degree n. However we consider the recurrence formula (29) the definition of the Hermite polynomial $H_n(x)$.

Proof. Owing to lemma 3 and the relation (1), we have

$$\Phi^0_k(X) = 1\,.$$

From the relation (18),

$$\begin{aligned}\langle X, \xi_k\rangle &= U^{-1}d_{\xi_k}\varphi_k^0(\xi)\\ &= U^{-1}ic^2(\xi, \xi_k)e^{-\frac{c^2\|\xi\|^2}{2}}.\end{aligned}$$

Therefore we have

$$U^{-1}\varphi_k^1(\xi) = \frac{\langle X, \xi_k\rangle}{ic}.$$

We assume $l \geqq 1$, then we have from the relation (18),

$$\begin{aligned}\langle X, \xi_k\rangle i^{-l}\Phi_k^l(X) &= U^{-1}d_{\xi_k}\varphi_k^l(\xi)\\ &= U^{-1}d_{\xi_k}\frac{c^l}{\sqrt{l!}}(\xi, \xi_k)^l e^{-\frac{c^2\|\xi\|^2}{2}}\\ &= U^{-1}\Big(\frac{\sqrt{l}\,c^l}{i\sqrt{(l-1)!}}(\xi, \xi_k)^{l-1}e^{-\frac{c^2\|\xi\|^2}{2}}\\ &\quad -\frac{\sqrt{(l+1)}c^{l+2}}{i\sqrt{(l+1)!}}(\xi, \xi_k)^{l+1}e^{-\frac{c^2\|\xi\|^2}{2}}\Big);\end{aligned}$$

that is,

$$\langle X, \xi_k\rangle\Phi_k^l(X) = \sqrt{l}\,c\Phi_k^{l-1}(X) + \sqrt{l+1}\,c\Phi_k^{l+1}(X). \tag{30}$$

Put

$$\Phi_k^l(X) = \frac{1}{\sqrt{2^l l!}}H_k(X);$$

then we have

$$\begin{aligned}&H_0(X) = 1,\\ &H_1(X) = \frac{2}{c}\langle X, \xi_k\rangle,\\ &\frac{\sqrt{2}}{c}\langle X, \xi_k\rangle H_l(X) = 2lH_{l-1}(X) + H_{l+1}(X).\end{aligned} \tag{51}$$

Hence we have

$$H_l(X) = H_l\Big(\frac{\langle X, \xi_k\rangle}{c\sqrt{2}}\Big), \tag{52}$$

where $H_l(x)$ is a polynomial in one variable of degree l satisfing the recurrence formula (29).

Next we consider the case $i^{-(l_1+l_2)}\Phi_{k_1,k_2}^{l_1,l_2} = U^{-1}\varphi_{k_1,k_2}^{l_1,l_2}(\xi)$. Analogously to the last case, we have

$$\begin{aligned}
&\langle X, \xi_{k_1}\rangle i^{-(l_1+l_2)}\Phi^{l_1,l_2}_{k_1,k_2} \\
&\quad = U^{-1} d_{\xi_{k_1}} \varphi^{l_1,l_2}_{k_1,k_2} \\
&\quad = U^{-1}\left\{\frac{c^{l_1+l_2}\sqrt{l_1}}{i\sqrt{(l_1-1)!\,l_2!}}(\xi, \xi_{k_1})^{l_1-1}\cdot(\xi, \xi_{k_2})^{l_2} e^{-\frac{c^2\|\xi\|^2}{2}} - \frac{\sqrt{l_1+1}\,c^{l_1+l_2+2}}{i\sqrt{(l_1+1)!\,l_2!}}\right. \\
&\qquad\qquad \left. \times (\xi, \xi_{k_1})^{l_1+1}\cdot(\xi, \xi_{k_2})^{l_2} e^{-\frac{c^2\|\xi\|^2}{2}}\right\},
\end{aligned}$$

and

$$\begin{aligned}
i^{-l_2}\langle X, \xi_{k_1}\rangle \Phi^{0,l_2}_{k_1,k_2} &= U^{-1}\left(-\frac{c^{2+l_2}}{i\sqrt{l_2!}}(\xi, \xi_{k_1})\cdot(\xi, \xi_{k_2})^{l_2} e^{-\frac{c^2\|\xi\|^2}{2}}\right) \\
&= \left(-\frac{c}{i^{2+l_2}}\right)\Phi^{1,l_2}_{k_1,k_2};
\end{aligned}$$

that is,

$$\Phi^{1,l_2}_{k_1,k_2} = \frac{\langle X, \xi_{k_1}\rangle}{c}\Phi^{0,l_2}_{k_1,k_2}.$$

Hence we have

$$\begin{aligned}
\Phi^{l_1,l_2}_{k_1,k_2}(X) &= \frac{1}{\sqrt{2^{l_1} l_1!}} H_{l_1}\left(\frac{\langle X, \xi_{k_1}\rangle}{c\sqrt{2}}\right)\Phi^{0,l_2}_{k_1,k_2} \\
&= \frac{1}{\sqrt{2^{l_1+l_2} l_1!\, l_2!}} H_{l_1}\left(\frac{\langle X, \xi_{k_1}\rangle}{c\sqrt{2}}\right) H_{l_2}\left(\frac{\langle X, \xi_{k_2}\rangle}{c\sqrt{2}}\right).
\end{aligned}$$

Consequently we have the conclusion of the theorem by the same way. Q.E.D.

§ 4. **The various formulae for Hermite polynomials.** In this section, we prove the various formulae for Hermite polynomials making use of the relations (18) and (20), map U and theorem 1.

Orthogonality.

From $(\Phi^n_1(X), \Phi^m_1(X))_{L^2(L', d\mu_c)} = \delta_{n,m}$ and theorem 1 we have orthogonality of Hermite polynomials:

$$\int_{-\infty}^{\infty} H_n(x) H_m(x) \frac{1}{\sqrt{\pi}} e^{-x^2} dx = \delta_{n,m} 2^n n!. \tag{33}$$

Differential equation.

Making use of the relation (20), we have

$$D_{\xi_1} U^{-1}(\xi, \xi_1)^n e^{-\frac{c^2\|\xi\|^2}{2}} = -U^{-1}(\xi, \xi_1)^{n+1} e^{-\frac{c^2\|\xi\|^2}{2}} - \frac{i}{c^2}\langle X, \xi_1\rangle U^{-1}(\xi, \xi_1)^n e^{-\frac{c^2\|\xi\|^2}{2}}.$$

Therefore,

$$H_n'\left(\frac{\langle X, \xi_1\rangle}{c\sqrt{2}}\right) = -H_{n+1}\left(\frac{\langle X, \xi_1\rangle}{c\sqrt{2}}\right) + \frac{\sqrt{2}\langle X, \xi_1\rangle}{c} H_n\left(\frac{\langle X, \xi_1\rangle}{c\sqrt{2}}\right).$$

In virtue of lemma 2, we have

$$H_n'(x) = -H_{n+1}(x) + 2xH_n(x). \tag{34}$$

From (29) and (34), we have

$$H_n'(x) = 2nH_{n-1}(x). \tag{35}$$

Therefore we have the following differential equation from (34) and (35) which the Hermite polynomial satisfies.

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0. \tag{36}$$

Addition formula.

Put $\quad \varphi = a_1\xi_1 + a_2\xi_2 + \cdots + a_n\xi_n, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = 1.$
Then

$$\begin{aligned} UH_l\left(\frac{\langle X, \varphi\rangle}{c\sqrt{2}}\right) &= 2^{l/2} i^l c^l (\xi, \varphi)^l e^{-\frac{c^2\|\xi\|^2}{2}} \\ &= 2^{l/2} i^l c^l (\xi, \sum_{i=1}^{n} a_i\xi_i)^l e^{-\frac{c^2\|\xi\|^2}{2}} \\ &= 2^{l/2} i^l c^l \sum_{l_1+\cdots+l_n=l} \frac{l!\, a_1^{l_1} a_2^{l_2} \cdots a_n^{l_n}}{l_1! \cdots !_n!} (\xi, \xi_1)^{l_1} \cdots (\xi, \xi_n)^{l_n} e^{-\frac{c^2\|\xi\|^2}{2}}. \end{aligned}$$

Owing to theorem 1,

$$= U \sum_{l_1+\cdots+l_n=l} \frac{l!}{l_1! \cdots l_n!} \prod_{i=1}^{n} a_i^{l_i} H_{l_i}\left(\frac{\langle X, \xi_i\rangle}{c\sqrt{2}}\right).$$

Therefore owing to lemma 3, we have

$$H_l\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{l_1+\cdots+l_n=l} \frac{l!\, a_1^{l_1} \cdots a_n^{l_n}}{l_1! \cdots l_n!} H_{l_1}(x_1) \cdots H_{l_n}(x_n), \tag{37}$$

where $a_1^2 + \cdots + a_n^2 = 1$.

Generating function.

$$U \sum_{n=0}^{\infty} \frac{t^n H_n\left(\frac{\langle X, \xi_1\rangle}{c\sqrt{2}}\right)}{n!} = \sum_{n=0}^{\infty} \frac{2^{n/2} i^n t^n c^n (\xi, \xi_1)^n}{n!} e^{-\frac{c^2\|\xi\|^2}{2}}$$

$$= e^{\sqrt{2}\,ic(\xi,\xi_1)t-\frac{c^2\|\xi\|^2}{2}}$$

$$= e^{-t^2}e^{-\frac{c^2\left\|\xi-\frac{\sqrt{2}}{c}it\xi_1\right\|^2}{2}}$$

$$= e^{-t^2}Ue^{\frac{2t\langle X,\,\xi_1\rangle}{\sqrt{2}c}}.$$

Thus, we have

$$\sum_{n=0}^{\infty}\frac{t^n H_n(x)}{n!} = e^{2tx-t^2}.$$

Integral representations.

We have the following integral formula putting $\xi=\lambda\xi_1$ from the relation $U\Phi_1^n=i^n\varphi_1^n$

$$\int_{-\infty}^{\infty} e^{i\lambda x}H_n\Big(\frac{x}{c\sqrt{2}}\Big)\frac{e^{-\frac{x^2}{2c^2}}}{\sqrt{2\pi c^2}}\,dx = i^n 2^{n/2}c^n\lambda^n e^{-\frac{c^2\lambda^2}{2}}. \tag{38}$$

Put $U\langle X,\xi_1\rangle^n = h_n(\xi)$.

If $n=0$ obviously we have

$$h_0(\xi) = e^{-\frac{c^2\|\xi\|^2}{2}}.$$

From the relation (20) we have

$$U\langle X,\xi_1\rangle = ic^2(\xi,\xi_1)U + ic^2UD_{\xi_1}.$$

therefore

$$h_1(\xi) = ic^2(\xi,\xi_1)e^{-\frac{c^2\|\xi\|^2}{2}}, \qquad \text{and}$$

$$\begin{aligned} U\langle X,\xi_1\rangle^{n+1} &= U\langle X,\xi_1\rangle\langle X,\xi_1\rangle^n \\ &= c^2iUD_{\xi_1}\langle X,\xi_1\rangle^n + c^2i(\xi,\xi_1)U\langle X,\xi_1\rangle^n \\ &= c^2Un\langle X,\xi_1\rangle^{n-1} + c^2i(\xi,\xi_1)U\langle X,\xi_1\rangle^n. \end{aligned}$$

Hence we have

$$\begin{aligned} h_{n+1}(\xi) &= nc^2h_{n-1}(\xi) + c^2i(\xi,\xi_1)h_n(\xi), \\ h_1(\xi) &= ic^2(\xi,\xi_1)e^{-\frac{c^2\|\xi\|^2}{2}}, \\ h_0(\xi) &= e^{-\frac{c^2\|\xi\|^2}{2}}. \end{aligned} \tag{39}$$

Comparing with the formula (29) and (39), we have

$$U\langle X,\xi_1\rangle^n = h_n(\xi) = i^n\Big(\frac{c}{\sqrt{2}}\Big)^n H_n\Big(\frac{c(\xi,\xi_1)}{\sqrt{2}}\Big)e^{-\frac{c^2\|\xi\|^2}{2}}. \tag{40}$$

Putting $\xi_1 = -\xi_1$, we have

$$H_n(-x) = (-1)^n H_n(x). \tag{41}$$

Further, putting $\xi = \lambda\xi_1$, we have

$$\int_{-\infty}^{\infty} e^{i\lambda x} x^n \frac{e^{-\frac{x^2}{2c^2}}}{\sqrt{2\pi c^2}} dx = i^n \left(\frac{c}{\sqrt{2}}\right)^n H_n\left(\frac{c\lambda}{\sqrt{2}}\right) e^{-\frac{c^2\lambda^2}{2}}. \tag{42}$$

We have so called Gauss transform putting $\frac{x}{c} = t + i\lambda$ in (42):

$$\int_{-\infty}^{\infty} (t+i\lambda)^n \frac{e^{-t^2/2}}{\sqrt{2\pi}} dt = i^n 2^{-n/2} H_n\left(\frac{\lambda}{\sqrt{2}}\right). \tag{43}$$

Next we consider

$$U(\langle X, \xi_1\rangle + i\langle X, \xi_2\rangle)^n = k_n(\xi).$$

$$\begin{aligned} k_n(\xi) &= U\{\langle X, \xi_1\rangle(\langle X, \xi_1\rangle + i\langle X, \xi_2\rangle)^{n-1} \\ &\qquad + i\langle X, \xi_2\rangle(\langle X, \xi_1\rangle + i\langle X, \xi_2\rangle)^{n-1}\} \\ &= (d_{\xi_1} + i_{\xi_2}) k_{n-1}(\xi) \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &= (d_{\xi_1} + id_{\xi_2})^n e^{-\frac{c^2\|\xi\|^2}{2}}. \end{aligned}$$

$$\begin{aligned} (d_{\xi_1} + id_{\xi_2}) e^{-\frac{c^2\|\xi\|^2}{2}} &= \lim_{r\to 0} \frac{1}{ir} \{e^{-\frac{c^2\|\xi + r\xi_1 + ir\xi_2\|^2}{2}} - e^{-\frac{c^2\|\xi\|^2}{2}}\} \\ &= ic^2\{(\xi, \xi_1) + i(\xi, \xi_2)\} e^{-\frac{c^2\|\xi\|^2}{2}}. \end{aligned}$$

$$\begin{aligned} &(d_{\xi_1} + id_{\xi_2}) \cdot ((\xi, \xi_1) + i(\xi, \xi_2))^l e^{-\frac{c^2\|\xi\|^2}{2}} \\ &\quad = [l\{(\xi, \xi_1) + i(\xi, \xi_2)\}^{l-1} + i^2 l\{(\xi, \xi_1) + i(\xi, \xi_2)\}^{l-1}] \\ &\qquad \times e^{-\frac{c^2\|\xi\|^2}{2}} + ((\xi, \xi_1) + i(\xi, \xi_2))^l (d_{\xi_1} + id_{\xi_2}) e^{-\frac{c^2\|\xi\|^2}{2}}. \\ &\quad = \{(\xi, \xi_1) + i(\xi, \xi_2)\}^l (d_{\xi_1} + id_{\xi_2}) e^{-\frac{c^2\|\xi\|^2}{2}}. \end{aligned}$$

Therefore, we have

$$(d_{\xi_1} + id_{\xi_2})^n e^{-\frac{c^2\|\xi\|^2}{2}} = i^n c^{2n} \{(\xi, \xi_1) + i(\xi, \xi_2)\}^n e^{-\frac{c^2\|\xi\|^2}{2}}.$$

Hence we have from theorem 1,

$$\begin{aligned} k_n(\xi) &= i^n c^{2n} \sum_{k=0}^{n} \binom{n}{k} i^k (\xi, \xi_1)^{n-k} (\xi, \xi_2)^k e^{-\frac{c^2\|\xi\|^2}{2}} \\ &= U 2^{-n/2} c^n \sum_{k=0}^{n} i^k \binom{n}{k} H_{n-k}\left(\frac{\langle X, \xi_1\rangle}{c\sqrt{2}}\right) H_k\left(\frac{\langle X, \xi_2\rangle}{c\sqrt{2}}\right). \end{aligned} \tag{44}$$

Thus we have from lemma 2,

$$2^n(x+iy)^n = \sum_{k=0}^{n} i^k \binom{n}{k} H_{n-k}(x)H_k(y). \tag{45}$$

From the relations (33) and (44) we have

$$\int_{-\infty}^{\infty}(x+iy)^n H_m(x)\frac{e^{-x^2}}{\sqrt{\pi}}dx = \left(\frac{i}{2}\right)^{n-m}\frac{n!}{(n-m)!}H_{n-m}(y). \tag{46}$$

Rodrigues' formula.

Making use of the relation (18),

$$\begin{aligned}(d_{\xi_1})^n e^{-\frac{c^2\|\xi\|^2}{2}} &= U\langle X, \xi_1\rangle^n U^{-1} e^{-\frac{c^2\|\xi\|^2}{2}} \\ &= U\langle X, \xi_1\rangle^n,\end{aligned}$$

owing to the relation (40)

$$= i^n\left(\frac{c}{\sqrt{2}}\right)^n H_n\left(\frac{c(\xi, \xi_1)}{\sqrt{2}}\right)e^{-\frac{c^2\|\xi\|^2}{2}}. \tag{47}$$

If we put $\xi=\lambda\xi_1$, we have

$$\begin{aligned}d_\xi\, e^{-\frac{c^2\|\xi\|^2}{2}}|_{\xi=\lambda\xi_1} &= \lim_{r\to 0}\frac{1}{ir}\{e^{-\frac{c^2\|\xi+r\xi_1\|^2}{2}} - e^{-\frac{c^2\|\xi\|^2}{2}}\}\,|_{\xi=\lambda\xi_1} \qquad (48)\\ &= \lim_{r\to 0}\frac{1}{ir}e^{-\frac{c^2\|\xi\|^2}{2}}\{e^{-c^2(\xi,\xi_1)r-\frac{r^2}{2}c^2}-1\}\,|_{\xi=\lambda\xi_1} \\ &= \frac{1}{i}\frac{d}{d\lambda}e^{-\frac{c^2\lambda^2}{2}}.\end{aligned}$$

Therefore we have from (47),

$$\left(\frac{d}{d\lambda}\right)^n e^{-\frac{c^2\lambda^2}{2}} = (-1)^n\left(\frac{c}{\sqrt{2}}\right)^n H_n\left(\frac{c\lambda}{\sqrt{2}}\right)e^{-\frac{c^2\lambda^2}{2}}.$$

Put $\frac{1}{c^2}=\sigma$, then we have

$$(-1)^n(\sqrt{2\sigma})^n e^{\frac{\lambda^2}{2\sigma}}\left(\frac{d}{d\lambda}\right)^n e^{-\frac{\lambda^2}{2\sigma}} = H_n\left(\frac{\lambda}{\sqrt{2\sigma}}\right) \equiv H_n(\lambda, \sigma). \tag{49}$$

This is Kakutani's formulation [4].

A relation between Hermite polynomial and Legendre poly-

nomial.

Put $UH_n\left(\frac{\langle X, \varphi\rangle}{c\sqrt{2}}\right)=m_n(\xi)$.

Making use of the relation, (20), we have

$$-(\xi, \varphi)m_n(\xi) = U\left(\frac{||\varphi||^2}{i\sqrt{2}c}H_n'\left(\frac{\langle X, \varphi\rangle}{c\sqrt{2}}\right)+\frac{i}{c^2}\langle X, \varphi\rangle H_n\left(\frac{\langle X, \varphi\rangle}{c\sqrt{2}}\right)\right).$$

Where we made use of the relation

$$D_\varphi H_n\left(\frac{\langle X, \varphi\rangle}{c\sqrt{2}}\right) = \lim_{r\to 0}\frac{1}{ir}\left\{H_n\left(\frac{\langle X, \varphi\rangle+r||\varphi||^2}{c\sqrt{2}}\right)-H_n\left(\frac{\langle X, \varphi\rangle}{c\sqrt{2}}\right)\right\}$$
$$= \frac{||\varphi||^2}{i\sqrt{2}c}H_n'\left(\frac{\langle X, \varphi\rangle}{c\sqrt{2}}\right).$$

From the relations (29) and (35), we have

$$-(\xi, \varphi)m_n(\xi)=U\left\{\frac{||\varphi||^2-1}{ic}n\sqrt{2}H_{n-1}\left(\frac{\langle X, \varphi\rangle}{c\sqrt{2}}\right)-\frac{1}{i\sqrt{2}c}H_{n+1}\left(\frac{\langle X, \varphi\rangle}{c\sqrt{2}}\right)\right\}.$$

Putting $\xi=0$, we have

$$m_{n+1}(0) = 2n(||\varphi||^2-1)m_{n-1}(0).$$

Obviously we have $\quad m_0(0) = 1$,

therefore we have

$$m_{2n}(0) = \frac{(2n)!}{n!}(||\varphi||^2-1)^n. \tag{50}$$

Putting $||\varphi||=y$, we have

$$\int_{-\infty}^{\infty} H_{2n}(yx)\frac{1}{\sqrt{\pi}}e^{-x^2}dx = \frac{(2n)!}{n!}(y^2-1)^n. \tag{51}$$

Since $\frac{d}{dy}H_{2n}(yx)=2\cdot 2nH_{2n-1}(yx)x$, we have

$$\int_{-\infty}^{\infty} x^nH_n(yx)\frac{1}{\sqrt{\pi}}e^{-x^2}dx = \frac{1}{2^n}\left(\frac{d}{dy}\right)^n(y^2-1)^n$$
$$= n!\,P_n(y), \tag{52}$$

where $P_n(y)$ is the Legendre polynomial of degree n.

Wiener transform.

Putting $\varphi=\xi_1$ in the relation (31), we have

$$-\frac{\langle Y, \xi_1\rangle}{2c^2} WH_n\Big(\frac{\langle X, \xi_1\rangle}{c\sqrt{2}}\Big) = W\frac{i\langle X, \xi_1\rangle}{2c^2} H_n\Big(\frac{\langle X, \xi_1\rangle}{c\sqrt{2}}\Big)$$
$$+ W\frac{1}{ic\sqrt{2}} H_n'\Big(\frac{\langle X, \xi_1\rangle}{c\sqrt{2}}\Big),$$

owing to the relations (29) and (35)

$$= -\frac{ni}{\sqrt{2}c} WH_{n-1}\Big(\frac{\langle X, \xi_1\rangle}{c\sqrt{2}}\Big)+\frac{i}{c2\sqrt{2}} WH_{n+1}\Big(\frac{\langle X, \xi_1\rangle}{c\sqrt{2}}\Big).$$

Therefore we have

$$\frac{\sqrt{2}\langle Y, \xi_1\rangle}{c} WH_n\Big(\frac{\langle X, \xi_1\rangle}{c\sqrt{2}}\Big) = -iWH_{n+1}\Big(\frac{\langle X, \xi_1\rangle}{c\sqrt{2}}\Big)+2niWH_{n+1}\Big(\frac{\langle X, \xi_1\rangle}{c\sqrt{2}}\Big).$$

Obviously we have $W1=1$, therefore we have from the recurrence formula (29),

$$WH_n\Big(\frac{\langle X, \xi_1\rangle}{c\sqrt{2}}\Big) = i^n H_n\Big(\frac{\langle X, \xi_1\rangle}{c\sqrt{2}}\Big). \tag{53}$$

The formula (53) means by definition

$$\int_{-\infty}^{\infty} H_n\Big(\frac{\sqrt{2}x+iy}{c\sqrt{2}}\Big)\frac{e^{-\frac{x^2}{2c^2}}}{\sqrt{2\pi}c}dx = i^n H_n\Big(\frac{y}{c\sqrt{2}}\Big). \tag{54}$$

Since W transforms Hermite polynomials to same polynomials, it is a unitary operator on $L^2(L', d\mu_c)$, hence the representations $(T_g, L^2(L', d\mu_c))$ and $(S_g, L^2(L', d\mu_c))$ are equivalent each other.

§5. **Matrix elements.** In this section, we calculate matrix elements of the representations of the infinite dimensional motion group, and have the various formulae for them in virtue of the properties representations. Further we show that they are the limits of matrix elements of the representations of the finite dimensional Euclidian motion groups [10].

Put $g=(I, \xi)$ and

$$\left(T_g \frac{H_k\Big(\frac{\langle X, \xi_1\rangle}{c\sqrt{2}}\Big)}{\sqrt{2^k k!}}, \frac{H_m\Big(\frac{\langle X, \xi_1\rangle}{c\sqrt{2}}\Big)}{\sqrt{2^m m!}}\right)_{L^2(L', d\mu_c)} = H_{k,m}(\xi). \tag{55}$$

Obviously $H_{k,m}(\xi)$ is expressed as follows.

$$H_{k,m}(\xi) = \frac{1}{\sqrt{2^{m+k}m!\,k!}}\int_{L'} e^{-\frac{i\langle X,\xi\rangle}{2c^2}} H_k\left(\frac{\langle X,\xi_1\rangle}{c\sqrt{2}}\right) H_m\left(\frac{\langle X,\xi_1\rangle}{c\sqrt{2}}\right) d\mu_c(X). \tag{56}$$

Putting $\xi = 2cx\xi_1$, we defined the special function $H_{k,m}(x)$ as follows.

$$H_{k,m}(x) = \frac{1}{\sqrt{2^{m+k}m!\,k!}}\int_{-\infty}^{\infty} e^{-\sqrt{2}xyi} H_k(y) H_m(y) \frac{e^{-y^2}}{\sqrt{\pi}} dy. \tag{57}$$

We have the following limit theorem between the matrix elements of n-dimensional motion groups and $H_{k,m}(x)$ from the property of the limit theorem between Gegenbauer and Hermite polynomial [10].

Theorem 2.

$$\lim_{n\to\infty} J^n_{k,m,0}(\sqrt{n}\,x) = H_{k,m}(x). \tag{58}$$

where $J^n_{k,m,0}(x)$ is the matrix element of the n-dimentional motion group (see [11]).

Properties of $H_{k,m}(x)$.

$$\sqrt{k+1}\, H_{k+1,m+1}(x) = \sqrt{m+1}\, H_{k,m}(x) - ixH_{k,m+1}(x). \tag{59}$$

$$i\frac{d}{dx} H_{k,m}(x) = \sqrt{m+1}\, H_{k,m+1}(x) + \sqrt{m}\, H_{k,m-1}(x). \tag{60}$$

$$H_{k,m}(x_1 + x_2) = \sum_{s=0}^{\infty} H_{k,s}(x_1) H_{s,m}(x_2). \tag{61}$$

$$\sum_{s=0}^{\infty} H_{k,s}(x)\overline{H_{s,m}(x)} = \delta_{k,m}. \tag{62}$$

$$e^{-\sqrt{2}xyi} H_m(y) = \sum_{k=0}^{\infty} \sqrt{\frac{m!}{k!}}\, 2^{\frac{m-k}{2}} H_{k,m}(x) H_k(y). \tag{63}$$

Proof. The formulae (59) and (60) are obtained by the formulae (29), (35) and (37).

The formulae (61) and (62) follow from the property of the representation: $T_{g_1g_2} = T_{g_1g_2}$.

Putting $g_1 = (I, 2cx\xi_1)$, $g_2 = (I, 2cy\xi_1)$, we have the formula (61) and putting $y = -x$, we have formula (62).

The formula (63) is obtained by the relation (57) and complete orthonormality of the system $\frac{1}{\sqrt{2^k k!}} H_k(x)$. Q.E.D.

Put $g=(u, 0)$, and

$$H^l_{k,m}(u) = \left(T_g \frac{H_k\left(\frac{\langle X, \xi_1\rangle}{c\sqrt{2}}\right) H_{l-k}\left(\frac{\langle X, \xi_2\rangle}{c\sqrt{2}}\right)}{\sqrt{2^l k!\,(l-k)!}}, \right.$$

$$\left. \frac{H_m\left(\frac{\langle X, \xi_1\rangle}{c\sqrt{2}}\right) H_{l-m}\left(\frac{\langle X, \xi_2\rangle}{c\sqrt{2}}\right)}{\sqrt{2^l m!\,(l-m)!}} \right)_{L^2(L', d\mu_c)}. \tag{64}$$

$$(l \geqq m, k),$$

where we put,

$$\begin{aligned} u\xi_1 &= \cos\theta\xi_1 + \sin\theta\xi_2 \\ u\xi_2 &= -\sin\theta\xi_1 + \cos\theta\xi_2 \\ u\xi_i &= \xi_i, \quad i=3, 4, \cdots. \end{aligned} \tag{65}$$

Then, we can write $H^l_{k,m}(u) = H^l_{k,m}(\cos\theta, \sin\theta)$ and it can be expressed as follows.

$$\begin{aligned} H^l_{k,m}(\cos\theta, \sin\theta) &= \frac{1}{2^l\sqrt{k!\,m!\,(l-k)!\,(l-m)!}} \\ &\times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_k(x\cos\theta + y\sin\theta) H_{l-k}(-x\sin\theta + y\cos\theta) \\ &\times H_m(x) H_{l-m}(y) \frac{e^{-(x^2+y^2)}}{\pi}\, dy dx. \end{aligned} \tag{66}$$

Analogously to theorem 2, we have the following limit theorem between the matrix element of n dimentional rotation group and $H^l_{k,m}(\cos\theta, \sin\theta)$.

Theorem 3.

$$\lim_{n\to\infty} P^{n,l}_{k,m,0}(\cos\theta) = H^l_{k,m}(\cos\theta, \sin\theta), \tag{67}$$

where $P^{n,l}_{k,m,0}$ is matrix element of the n-dimentional rotation group (see [11]).

Properties of $H^l_{k,m}(\cos\theta, \sin\theta)$.

$$H^l_{k,m}(\cos\theta, \sin\theta) = (-1)^{k+m} H^l_{m,k}(\cos\theta, \sin\theta). \tag{68}$$

$$\sqrt{k}\,H^l_{k,m} = \sqrt{(l-m)}\sin\theta H^{l-1}_{k-1,m} + \sqrt{m}\cos\theta H^{l-1}_{k-1,m-1}\,. \tag{69}$$

$$\frac{d}{d\theta}H^l_{k,m} = \sqrt{k(l-k+1)}H^l_{k-1,m} - \sqrt{(k+1)(l-k)}H^l_{k-1,m}\,. \tag{70}$$

$$H^l_{k,m}(\cos(\alpha+\beta), \sin(\alpha+\beta)) = \sum_{s=0}^{l} H^l_{k,s}(\cos\alpha, \sin\alpha)H^l_{s,m}(\cos\beta, \sin\beta)\,. \tag{71}$$

$$\sum_{s=0}^{l}(-1)^{s+m}H^l_{k,s}(\cos\alpha, \sin\alpha)H^l_{s,m}(\cos\alpha, \sin\alpha) = \delta_{k,m}\,. \tag{72}$$

$$H_k(x\cos\theta + y\sin\theta)H_{l-k}(-x\sin\theta + y\cos\theta) = \sum_{m=0}^{l}\sqrt{\frac{k!\,(l-k)!}{m!\,(l-m)!}} \times H^l_{k,m}(\cos\theta, \sin\theta)H_m(x)H_{l-m}(y)\,. \tag{73}$$

$$\sum_{n=0}^{l} H^l_{k,n}(\cos\alpha, \sin\alpha)H_{n,m}(x_1\cos\alpha - x_2\sin\alpha)H_{l-n,l-m}(x_1\sin\alpha + x_2\cos\alpha) = \sum_{n=0}^{l} H^l_{n,m}(\cos\alpha, \sin\alpha)H_{kn,}(x_1)H_{l-k,l-n}(x_2)\,. \tag{74}$$

The formulae (68)-(73) are obtained same way as the formulae (59)-(63). We shall prove the formula (74).

Put

$$g_1 = (u, 0)\,, \quad g_2 = (I, \xi)\,, \qquad \text{where} \quad \xi = 2cx_1\xi_1 + 2cx_2\xi_2\,,$$

and u satisfies (65). From the property of representation $T_{g_1g_2} = T_{g_1}T_{g_2}$, we have

$$\left(T_{g_1g_2}H_k\left(\frac{\langle X, \xi_1\rangle}{c\sqrt{2}}\right)H_{l-k}\left(\frac{\langle X, \xi_2\rangle}{c\sqrt{2}}\right),\ H_m\left(\frac{\langle X, \xi_1\rangle}{c\sqrt{2}}\right)H_{l-m}\left(\frac{\langle X, \xi_2\rangle}{c\sqrt{2}}\right)\right)_{L^2(L', d\mu_c)} \times \frac{1}{2^l\sqrt{k!\,m!\,(l-k)!\,(l-m)!}}$$

$$= \left(e^{-\frac{i}{c}\{(x_1\cos\alpha - x_2\sin\alpha)\langle X,\xi_1\rangle + (x_1\sin\alpha + x_2\cos\alpha)\langle Y,\xi_2\rangle}H_k\left(\frac{\langle X, u\xi_1\rangle}{c\sqrt{2}}\right)\right.$$
$$\left.\times H_{l-k}\left(\frac{\langle X, u\xi_2\rangle}{c\sqrt{2}}\right),\ H_m\left(\frac{\langle X, \xi_1\rangle}{c\sqrt{2}}\right)H_{l-m}\left(\frac{\langle X, \xi_2\rangle}{c\sqrt{2}}\right)\right)_{L^2(L', d\mu_c)}$$
$$\times 2^{-l}(k!\,m!\,(l-k)!\,(l-m)!)^{-1/2}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\sqrt{2}i\{(x_1\cos\alpha - x_2\sin\alpha)y_1 + (x_1\sin\alpha + x_2\cos\alpha)y_2\}}$$
$$\times \sum_{n=0}^{l}\frac{H^l_{k,n}(\cos\alpha, \sin\alpha)H_n(y_1)H_{l-n}(y_2)H_m(y_1)H_{l-m}(y_2)}{\sqrt{2^l n!\,(l-n)!\,m!\,(l-m)!}\,\pi}e^{-(y_1^2+y_2^2)}dy_1dy_2$$

$$= \sum_{n=0}^{l} H^l_{k,n}(\cos\alpha, \sin\alpha)\,H_{n,m}(x_1\cos\alpha - x_2\sin\alpha)H_{l-n,l-m}(x_1\sin\alpha + x_2\cos\alpha)\,.$$

The other hand,

$$\Big(T_{g_2}H_k\Big(\frac{\langle X,\xi_1\rangle}{c\sqrt{2}}\Big)H_{l-k}\Big(\frac{\langle X,\xi_2\rangle}{c\sqrt{2}}\Big),$$

$$T_{g_1^{-1}}H_m\Big(\frac{\langle X,\xi_1\rangle}{c\sqrt{2}}\Big)H_{l-m}\Big(\frac{\langle X,\xi_2\rangle}{c\sqrt{2}}\Big)\Big)_{L^2(L',d\mu_c)}2^{-l}(k!m!(l-k)!(l-m)!)^{-1/2}$$

$$=\sum_{n=0}^{l}\Big(e^{-\frac{i}{c}(x_1\langle X,\xi_1\rangle+x_2\langle X,\xi_2\rangle)}H_k\Big(\frac{\langle X,\xi_1\rangle}{c\sqrt{2}}\Big)H_{l-k}\Big(\frac{\langle X,\xi_2\rangle}{c\sqrt{2}}\Big),$$

$$H_n\Big(\frac{\langle X,\xi_1\rangle}{c\sqrt{2}}\Big)H_{l-n}\Big(\frac{\langle X,\xi_2\rangle}{c\sqrt{2}}\Big)\Big)_{L^2(L',d\mu_c)}$$

$$\times\Big(H_n\Big(\frac{\langle X,\xi_1\rangle\cos\alpha+\langle X,\xi_2\rangle\sin\alpha}{c\sqrt{2}}\Big)$$

$$\times H_{l-n}\Big(\frac{-\langle X,\xi_1\rangle\sin\alpha+\langle X,\xi_2\rangle\cos\alpha}{c\sqrt{2}}\Big),$$

$$H_m\Big(\frac{\langle X,\xi_1\rangle}{c\sqrt{2}}\Big)H_{l-m}\Big(\frac{\langle X,\xi_2\rangle}{c\sqrt{2}}\Big)\Big)_{L^2(L',d\mu_c)}$$

$$\times 2^{-2l}(n!(l-n)!)^{-1}(k!m!(l-k)!(l-m!)^{-1/2}$$

$$=\sum_{n=0}^{l}H_{k,n}(x_1)H_{l-k,l-n}(x_2)H^l_{n,m}(\cos\alpha,\sin\alpha).$$

Hence we have the formula (74).

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