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On the general Riemann's period relation for square integrable harmonic differentials on open Riemann surfaces

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Introduction

In 1947 Ahlfors [3] proved for a parabolic Riemann surface F the existence of an exhaustion and canonical homology basis of F such that for $\omega_1, \omega_2 \in \Gamma_h$

$$(\omega_1, \omega_2^*) = \lim_{n\to\infty} \sum_{i=1}^{p(n)} \left(\int_{A_i} \hat{\omega}_1 \int_{B_i} \bar{\hat{\omega}}_2 - \int_{A_i} \bar{\hat{\omega}}_2 \int_{B_i} \hat{\omega}_1 \right),$$

where $\hat{\omega}_1$ and $\hat{\omega}_2$ are the piecewise harmonic differentials adequately modified from ω_1 and ω_2 which depend on the exhaustion. From the above relation it follows that the bilinear relation for square integrable harmonic differentials having only a finite number of non-vanishing periods holds. In 1956 for a parabolic surface with some conditions Kusunoki [6] proved the validity of the bilinear relation for differentials in the class Γ_h such that the number of their non-vanishing periods is not necessarily finite. Later on, some conditions which insure the validity of this relation are found by several authors (Pfluger [13], Accola [1], Kobori and Sainouchi [5], Marden [7], Matsui [8] [9], M. Mori [10]). On the other hand, for the case of hyperelliptic surface of infinite genus this relation was investigated by P. J. Myrberg [11] and Pfluger [12]. In this paper we shall give some metric conditions for the validity of the bilinear relation on open Riemann surfaces, which include a part of our earlier results [5].

In § 1 we establish a general Riemann's relation for square integrable harmonic differentials which are not necessarily semi-exact on the surface. In § 2 for a special choice of canonical homology basis the general Riemann's relation is discussed by means of the same way as in § 1, but there we do not assume that the exhausting regions are canonical. In § 3 we consider an another way to establish the general Riemann's relation and compare the result obtained there with that in § 1. In the following we shall use the same notations and terminologies for the classes of differentials as in Ahlfors and Sario [4].

§ 1. General Riemann's relation

1. F being an arbitrary open Riemann surface, we consider an exhaustion $\{F_n\}$ $(n=1, 2, \cdots)$ of F by regular regions and corresponding canonical homology basis $\{A_i, B_i\}$ $(i=1, 2, \cdots)$ such that $A_1, B_1, \cdots, A_{p(n)}, B_{p(n)}$ form a canonical homology basis of F_n (mod ∂F_n) and $A_i \times B_j = \delta_{ij}$, $A_i \times A_j = B_i \times B_j = 0$. Let $F_n^{(i)}$ $(i= 1, 2, \cdots, m(n))$ be components of $F_{n+1} - \bar{F}_n$ and $u_n(p)$ be the harmonic function in $F_{n+1} - \bar{F}_n$ such that

$$u_n(p) = \begin{cases} 0 & \text{on } \partial F_n \\ \mu_n & \text{on } \partial F_{n+1}, \end{cases}$$

and its conjugate harmonic function $v_n(p)$ has the variation 2π on ∂F_{n+1}, that is, $\int_{\partial F_{n+1}} dv_n = 2\pi$. The quantity μ_n is the harmonic modulus of the open set $F_{n+1} - \bar{F}_n$. Similarly, the harmonic modulus $\mu_n^{(i)}$ of $F_n^{(i)}$ may be defined. If we choose adequately an additive constant of $v_n(p)$, the function $u_n(p) + iv_n(p)$ maps conformally $F_n^{(i)}$ with a finite number of slits onto a slit rectangle $0 < u_n < \mu_n$, $b_i < v_n < a_i + b_i$, where a_i and b_i are constants satisfying the following conditions

$$a_i = 2\pi \mu_n / \mu_n^{(i)}, \quad \sum_{i=1}^{m} a_i = 2\pi$$

and

$$b_1 = 0, \quad b_i = \sum_{k=1}^{i-1} a_k \qquad (1 < i \leqq m).$$

The function $u_n(p)+iv_n(p)$ maps conformally $F_{n+1}-\bar{F}_n$ with a finite number of slits onto a slit rectangle $0<u_n<\mu_n$, $0<v_n<2\pi$. The function $u(p)+iv(p)$ defined by $u_n(p)+iv_n(p)+\sum_{j=1}^{n-1}\mu_j$ on each $F_{n+1}-\bar{F}_n$ $(n=1, 2, \cdots)$ maps one to one and conformally $F-\bar{F}_1$ with at most an enumerable number of suitable slits onto a strip domain $0<u<R=\sum_{j=1}^{\infty}\mu_j$, $0<v<2\pi$ with at most an enumerable number of slits. This strip domain thus obtained is the graph of F associated with the exhaustion in Noshiro's sense.

2. Next we suppose that the exhaustion $\{F_n\}$ is canonical, that is, each contour of $\partial F_n=\bigcup_{i=1}^{m}\gamma_n^{(i)}$ is a dividing cycle. Let $D_n^{(i)}$ $(i=1, \cdots, m)$ be annuli each of which includes a contour and are disjoint each other. We put $D_n=\bigcup_{i=1}^{m}D_n^{(i)}$ and assume that D_n $(n=1, 2, \cdots)$ are disjoint each other. We construct the graph of $D=\bigcup_n D_n$ associated with the sequence $\{D_n\}$ and denote the harmonic modulus of $D_n^{(i)}$ and D_n by $\nu_n^{(i)}$ and ν_n, respectively. Also we denote by $u_0(p)+iv_0(p)$ the function which maps $\bigcup_n D_n$ onto the strip domain $0<u_0<R_0=\sum_{n=1}^{\infty}\nu_n$, $0<v_0<2\pi$. For any r $(0<r<R_0)$ the locus γ_r of the points of F satisfying $u_0(p)=r$ consists of a finite number of closed analytic curves $\gamma_r^{(i)}$ $(i=1, \cdots, m(r))$.

Now let us suppose ω_1 is in Γ_{hse} and ω_2 is in Γ_h. Then ω_2 has, in general, a non-vanishing period along a dividing cycle. We set $d_r^{(i)}=\int_{\gamma_r^{(i)}}\omega_2$ and $\theta_r^{(i)}=\int_{\gamma_r^{(i)}}dv_0$, then for each i $d_r^{(i)}$ and $\theta_r^{(i)}$ are equal to constants $d_n^{(i)}$ and $\theta_n^{(i)}$ respectively when r is contained in the interval $\sum_{j=1}^{n-1}\nu_j<r<\sum_{j=1}^{n}\nu_j$, and moreover $\sum_{i=1}^{m}d_n^{(i)}=0$ and $\sum_{i=1}^{m}\theta_n^{(i)}=2\pi$. Particularly,

$$\theta_r^{(i)}=\int_{\gamma_r^{(i)}}dv_0=2\pi\frac{\nu_n}{\nu_n^{(i)}}.$$

We shall set $\Lambda_0(r)=\max_i\int_{\gamma_r^{(i)}}dv_0$ and define a differential $S\omega_2$ in D as follows:

$$S\omega_2=\omega_2-\frac{d_r^{(i)}}{\theta_r^{(i)}}dv_0\quad\left(\sum_{j=1}^{n-1}\nu_j<r<\sum_{j=1}^{n}\nu_j\right)\qquad\text{in } D_n^{(i)}.$$

At first we shall prove the following

Proposition 1. *Suppose ω_1 is in Γ_{hse} and ω_2 is in Γ_h. If the integral $\int_0^{R_0}\frac{dr}{\Lambda_0(r)}$ is divergent, then there exist a canonical exhaustion $\{\Omega_{n'}\}$ with canonical homology basis $\{A_i, B_i\}$ $(i=1,2,\cdots)$ and numbers $m_{n'}^{(i)}$ such that*

$$(*)\qquad \begin{aligned}(\omega_1, \omega_2^*) &= \lim_{n'\to\infty}\Big\{\sum_{i=1}^{p(n')}\Big(\int_{A_i}\omega_1\int_{B_i}\bar\omega_2-\int_{A_i}\bar\omega_2\int_{B_i}\omega_1\Big)-R_{n'}\Big\}\,,\\ R_{n'} &= \sum_{i=1}^{m(n')}\frac{m_{n'}^{(i)}\bar d_{n'}^{(i)}}{\theta_{n'}^{(i)}}\,,\end{aligned}$$

where $d_{n'}^{(i)}=\int_{\beta_{n'}^{(i)}}\omega_2$ and $\theta_{n'}^{(i)}=\int_{\beta_{n'}^{(i)}}dv_0$ $(\partial\Omega_{n'}=\bigcup_i \beta_{n'}^{(i)})$.

Proof. For any r $(0<r<R_0)$ we put $R_r=F_n\cup\{p\,|\,u_0(p)\leqq r\}$ if r is contained in the interval $\sum_{j=1}^{n-1}\nu_j<r\leqq\sum_{j=1}^{n}\nu_j$, then R_r and F_n have the same homology basis $\{A_i, B_i\}$ $(i=1,2,\cdots,p(r))$. Set $a_i=\int_{A_i}\omega_1$ and $b_i=\int_{B_i}\omega_1$, and define $T_r\omega_1$ as follows:

$$T_r\omega_1=\sum_{i=1}^{p(r)}(b_i\sigma_{R_r}(A_i)-a_i\sigma_{R_r}(B_i))\,,$$

where $\sigma_{R_r}(A_i)$ and $\sigma_{R_r}(B_i)$ are the period reproducers in $\Gamma_{h0}(\bar R_r)$ for A_i and B_i, respectively. Then by means of the Green formula we have

$$(1)\qquad \begin{aligned}(\omega_1, \omega_2^*)_{R_r} &= (T_r\omega_1, \omega_2^*)_{R_r}+(\omega_1-T_r\omega_1, \omega_2^*)_{R_r}\\ &=\sum_{i=1}^{p(r)}\Big(\int_{A_i}\omega_1\int_{B_i}\bar\omega_2-\int_{A_i}\bar\omega_2\int_{B_i}\omega_1\Big)-\int_{\partial R_r}u_r\bar\omega_2\,,\end{aligned}$$

where $u_r(p)=\int_{p_0}^{p}(\omega_1-T_r\omega_1)$ with a fixed point $p_0\in F_1$. By the definition of $S\omega_2$

$$\int_{\gamma_r^{(i)}}u_r\bar\omega_2=\int_{\gamma_r^{(i)}}u_r\overline{S\omega_2}+\frac{\bar d_r^{(i)}}{\theta_r^{(i)}}\int_{\gamma_r^{(i)}}u_r\,dv_0\,.$$

We put $m_r^{(i)}=\int_{\gamma_r^{(i)}}u_r\,dv_0$, where the integral is considered in the graph of D with variable u_0+iv_0. Then (1) becomes to

$$(\omega_1, \omega_2^*) = \sum_{i=1}^{p(r)} \left(\int_{A_i} \omega_1 \int_{B_i} \overline{\omega}_2 - \int_{A_i} \overline{\omega}_2 \int_{B_i} \omega_1\right) - \sum_{i=1}^{m} \frac{m_r^{(i)} \overline{d}_r^{(i)}}{\theta_r^{(i)}} - \int_{\partial R_r} u_r \overline{S\omega_2}\,.$$

Since $\int_{\gamma_r^{(i)}} S\omega_2 = 0$ and $\sigma_{R_r}(A_i)$ and $\sigma_{R_r}(B_i)$ belong to $\Gamma_{h0}(\bar{R}_r)$, we have

$$\begin{aligned} \left|\int_{\gamma_r^{(i)}} u_r \overline{S\omega_2}\right| &\leqq \int_{\gamma_r^{(i)}} |du_r| \int_{\gamma_r^{(i)}} |S\omega_2| = \int_{\gamma_r^{(i)}} |\omega_1 - T_r\omega_1| \int_{\gamma_r^{(i)}} |S\omega_2| \\ &= \int_{\gamma_r^{(i)}} |\omega_1| \int_{\gamma_r^{(i)}} |S\omega_2|\,. \end{aligned}$$

We set $\omega_1 = a_1 du_0 + b_1 dv_0$ and $S\omega_2 = a_2 du_0 + b_2 dv_0$, then by the successive applications of the Schwarz inequality we obtain

$$\begin{aligned} L(r) = \left|\int_{\partial R_r} u_r \overline{S\omega_2}\right| &\leqq \sum_{i=1}^{m} \int_{\gamma_r^{(i)}} |\omega_1| \int_{\gamma_r^{(i)}} |S\omega_2| \\ &\leqq \Lambda_0(r) \sum_{i=1}^{m} \left(\int_{\gamma_r^{(i)}} |b_1|^2 dv_0\right)^{\frac12} \left(\int_{\gamma_r^{(i)}} |b_2|^2 dv_0\right)^{\frac12} \\ &\leqq \Lambda_0(r) \left(\int_0^{2\pi} |b_1|^2 dv_0\right)^{\frac12} \left(\int_0^{2\pi} |b_2|^2 dv_0\right)^{\frac12}. \end{aligned}$$

Hence, again applying the Schwarz inequality, we get

$$\begin{aligned} \int_0^{R_0} \frac{L(r)}{\Lambda_0(r)} dr &\leqq \int_0^{R_0} \left\{\left(\int_0^{2\pi} |b_1|^2 dv_0\right)^{\frac12} \left(\int_0^{2\pi} |b_2|^2 dv_0\right)^{\frac12}\right\} dr \\ &\leqq \left(\int_0^{R_0}\int_0^{2\pi} (|a_1|^2 + |b_1|^2) dv_0 dr\right)^{\frac12} \left(\int_0^{R_0}\int_0^{2\pi} (|a_2|^2 + |b_2|^2) dv_0 dr\right)^{\frac12} \\ &= \|\omega_1\|_D \|S\omega_2\|_D\,. \end{aligned}$$

Consequently, under the condition $\|S\omega_2\|_D < \infty$ we have

$$\varliminf_{r \to R_0} L(r) = 0\,,$$

and so there exists a sequence $\{r_{n'}\}$ such that

$$L(r_{n'}) = \left|\int_{\partial R_{r_{n'}}} u_{r_{n'}} \overline{S\omega_2}\right| \to 0 \qquad (r_{n'} \to R_0)\,.$$

Thus if the finiteness of $\|S\omega_2\|_D$ is proved, the proof of proposition will be completed. Now let us prove that $\|S\omega_2\|_D$ is finite.

Since

$$||\omega_2 - S\omega_2||^2_{D_n^{(i)}} = \left(\frac{|d_n^{(i)}|}{\theta_n^{(i)}}\right)^2 ||dv_0||^2_{D_n^{(i)}} = \left(\frac{|d_n^{(i)}|}{\theta_n^{(i)}}\right)^2 \int_{\partial D_n^{(i)}} u_0 dv_0$$
$$= \nu_n \frac{|d_n^{(i)}|^2}{\theta_n^{(i)}} = \frac{1}{2\pi}\nu_n^{(i)}|d_n^{(i)}|^2,$$

we have

$$||S\omega_2||^2_{D_n^{(i)}} \leqq (||\omega_2||_{D_n^{(i)}} + ||\omega_2 - S\omega_2||_{D_n^{(i)}})^2$$
$$\leqq 2\left(||\omega_2||^2_{D_n^{(i)}} + \frac{1}{2\pi}\nu_n^{(i)}|d_n^{(i)}|^2\right).$$

Hence

$$||S\omega_2||^2_D = \sum_{n=1}^{\infty}\sum_{i=1}^{m} ||S\omega_2||^2_{D_n^{(i)}} \leqq 2||\omega_2||^2_D + \frac{1}{\pi}\sum_{n=1}^{\infty}\sum_{i=1}^{m}\nu_n^{(i)}|d_n^{(i)}|^2.$$

On the other hand, we have

$$|d_n^{(i)}| = \left|\int_{\gamma_n^{(i)}} \omega_2\right| = |(\omega_2, \sigma(\gamma_n^{(i)})*)_{D_n^{(i)}}| \leqq ||\omega_2||_{D_n^{(i)}} ||\sigma(\gamma_n^{(i)})||_{D_n^{(i)}},$$

where $\sigma(\gamma_n^{(i)})$ is the period reproducer in $\Gamma_{h0}(\bar{D}_n^{(i)})$ for $\gamma_n^{(i)}$. Since $||\sigma(\gamma_n^{(i)})||^2 = \lambda(\gamma_n^{(i)}) = 2\pi/\nu_n^{(i)}$, where $\lambda(\gamma_n^{(i)})$ is the extremal length of the family of cycles which are homologous to $\gamma_n^{(i)}$ (Accola [2]), we get

$$\sum_{n=1}^{\infty}\sum_{i=1}^{m}\nu_n^{(i)}|d_n^{(i)}|^2 \leqq 2\pi\sum_{n=1}^{\infty}\sum_{i=1}^{m}||\omega_2||^2_{D_n^{(i)}}.$$

Thus we have

$$||S\omega_2||^2_D \leqq 4||\omega_2||^2.$$

Therefore, we obtained the above mentioned result, q. e. d.

Next, let us consider the general case, that is, suppose ω_1 and ω_2 are both in Γ_h. The following theorem is obtained.

Theorem 1. *Suppose ω_i $(i=1,2)$ are in Γ_h and ω_i'' are the Γ^*_{hm}-component of ω_i in the orthogonal decomposition $\Gamma_h = \Gamma_{hse} \dotplus \Gamma^*_{hm}$. Then if the integral $\int_0^{R_0}\frac{dr}{\Lambda_0(r)}$ is divergent, there exist a canonical exhaustion $\{\Omega_{n'}\}$ with canonical homology basis $\{A_i, B_i\}$ $(i=1,2,\cdots,p(n'))$ and numbers $m_{n'}^{(i)}$ and $l_{n'}^{(i)}$ such that*

$$(\ast)\qquad \begin{aligned}(\omega_1, \omega_2^*) = \lim_{n'\to\infty} \{\sum_{i=1}^{p(n')} (\int_{Ai} \omega_1 \int_{Bi} \overline{\omega}_2 - \int_{Ai} \overline{\omega}_2 \int_{Bi} \omega_1) \\ - \sum_{i=1}^{p(n')} (\int_{Ai} \omega_1'' \int_{Bi} \overline{\omega}_2'' - \int_{Ai} \overline{\omega}_2'' \int_{Bi} \omega_1'') + R'_{n'}\} .\end{aligned}$$

$$R'_{n'} = \sum_{i=1}^{m(n')} \frac{\overline{l}_{n'}^{(i)} c_{n'}^{(i)} - m_{n'}^{(i)} \overline{d}_{n'}^{(i)}}{\theta_{n'}^{(i)}},$$

where $c_{n'}^{(i)} = \int_{\beta_{n'}^{(i)}} \omega_1$ *and* $d_{n'}^{(i)} = \int_{\beta_{n'}^{(i)}} \omega_2 \qquad (\partial\Omega_{n'} = \bigcup_{i=1}^{m} \beta_{n'}^{(i)})$.

Proof. By means of the orthogonal decomposition $\Gamma_h = \Gamma_{hse} \dot{+} \Gamma_{hm}^*$ we put

$$\omega_i = \omega_i' + \omega_i'', \quad \omega_i' \in \Gamma_{hse}, \quad \omega_i'' \in \Gamma_{hm}^* \qquad (i = 1, 2)$$

and also set

$$c_r^{(i)} = \int_{\gamma_r^{(i)}} \omega_1 = \int_{\gamma_r^{(i)}} \omega_1'' \qquad (\sum_{i=1}^{m} c_r^{(i)} = 0)$$

$$d_r^{(i)} = \int_{\gamma_r^{(i)}} \omega_2 = \int_{\gamma_r^{(i)}} \omega_2'' \qquad (\sum_{i=1}^{m} d_r^{(i)} = 0).$$

Then, by the orthogonality of Γ_{hse} and Γ_{hm}^*, we have

$$\begin{aligned}(\omega_1, \omega_2^*) &= (\omega_1', \omega_2^*) + (\omega_1'', \omega_2'^*) + (\omega_1'', \omega_2''^*) \\ &= (\omega_1', \omega_2^*) - \overline{(\omega_2', \omega_1''^*)}.\end{aligned}$$

By an application of the Green formula to the canonical region R_r, we obtain

$$(2)\qquad \begin{aligned}&(\omega_1', \omega_2^*)_{R_r} - \overline{(\omega_2', \omega_1''^*)_{R_r}} \\ &= \sum_{i=1}^{p(r)} (\int_{Ai} \omega_1' \int_{Bi} \overline{\omega}_2 - \int_{Ai} \overline{\omega}_2 \int_{Bi} \omega_1') - \overline{\sum_{i=1}^{p(r)} (\int_{Ai} \omega_2' \int_{Bi} \overline{\omega}_1'' - \int_{Ai} \overline{\omega}_1'' \int_{Bi} \omega_2')} \\ &\qquad - \int_{\partial R_r} u_r^{(1)} \overline{\omega}_2 + \int_{\partial R_r} \overline{u_r^{(2)} \overline{\omega}_1''} \\ &= \sum_{i=1}^{p(r)} (\int_{Ai} \omega_1' \int_{Bi} \overline{\omega}_2 - \int_{Ai} \overline{\omega}_2 \int_{Bi} \omega_1') - \int_{\partial R_r} u_r^{(1)} \overline{\omega}_2 + \int_{\partial R_r} \overline{u}_r^{(2)} \omega_1'' \\ &\qquad + \sum_{i=1}^{p(r)} (\int_{Ai} \omega_1'' \int_{Bi} (\overline{\omega}_2 - \overline{\omega}_2'') - \int_{Ai} (\overline{\omega}_2 - \overline{\omega}_2'') \int_{Bi} \omega_1'') \\ &= \sum_{i=1}^{p(r)} (\int_{Ai} \omega_1 \int_{Bi} \overline{\omega}_2 - \int_{Ai} \overline{\omega}_2 \int_{Bi} \omega_1) - \sum_{i=1}^{p(r)} (\int_{Ai} \omega_1'' \int_{Bi} \overline{\omega}_2'' - \int_{Ai} \overline{\omega}_2'' \int_{Bi} \omega_1'') \\ &\qquad - \int_{\partial R_r} u_r^{(1)} \overline{\omega}_2 + \int_{\partial R_r} \overline{u}_r^{(2)} \omega_1'',\end{aligned}$$

where $u_r^{(k)}(p)=\int_{p_0}^{p}(\omega_k'-T_r\omega_k')$ with $T_r\omega_k'=\sum_{i=1}^{p(r)}(b_i^{(k)}\sigma_{R_r}(A_i)-a_i^{(k)}\sigma_{R_r}(B_i))$, $a_i^{(k)}=\int_{Ai}\omega_k'$, $b_i^{(k)}=\int_{Bi}\omega_k'$ $\quad (k=1,2)$.

We define a differential $Q\omega_1''$ in D and numbers $l_r^{(i)}$ by the same way as before, that is,

$$Q\omega_1''=\omega_1''-\frac{c_r^{(i)}}{\theta_r^{(i)}}dv_0 \qquad (\sum_{j=1}^{n-1}\nu_j<r<\sum_{j=1}^{n}\nu_j) \quad \text{in } D_n^{(i)}$$

and

$$l_r^{(i)}=\int_{\gamma_r^{(i)}}u_r^{(2)}dv_0\,.$$

Then

$$\begin{aligned}\int_{\partial R_r}u_r^{(1)}\bar{\omega}_2 &= \sum_{i=1}^{m}\Big(\int_{\gamma_r^{(i)}}u_r^{(1)}\overline{S\omega_2}+\frac{\bar{d}_r^{(i)}}{\theta_r^{(i)}}\int_{\gamma_r^{(i)}}u_r^{(1)}dv_0\Big)=\sum_{i=1}^{m}\int_{\gamma_r^{(i)}}u_r^{(1)}\overline{S\omega_2}\\ &\quad+\sum_{i=1}^{m}\frac{m_r^{(i)}\bar{d}_r^{(i)}}{\theta_r^{(i)}}\end{aligned}\tag{3}$$

$$\begin{aligned}\int_{\partial R_r}\bar{u}_r^{(2)}\omega_1'' &= \sum_{i=1}^{m}\Big(\int_{\gamma_r^{(i)}}\bar{u}_r^{(2)}Q\omega_1''+\frac{c_r^{(i)}}{\theta_r^{(i)}}\int_{\gamma_r^{(i)}}\bar{u}_r^{(2)}dv_0\Big)=\sum_{i=1}^{m}\int_{\gamma_r^{(i)}}\bar{u}_r^{(2)}Q\omega_1''\\ &\quad+\sum_{i=1}^{m}\frac{\bar{l}_r^{(i)}c_r^{(i)}}{\theta_r^{(i)}}\,,\end{aligned}$$

where $S\omega_2$ is a differential defined as before and $m_r^{(i)}=\int_{\gamma_r^{(i)}}u_r^{(1)}dv_0$. We put

$$L'(r)=|\sum_{i=1}^{m}\int_{\gamma_r^{(i)}}u_r^{(1)}\overline{S\omega_2}|,\quad L''(r)=|\sum_{i=1}^{m}\int_{\gamma_r^{(i)}}\bar{u}_r^{(2)}Q\omega_1''|\,.$$

Then, by the same way as in the proof of proposition 1 we have

$$\int_0^{R_0}\frac{L'(r)+L''(r)}{\Lambda_0(r)}dr\leqq\|\omega_1'\|_D\|S\omega_2\|_D+\|\omega_2'\|_D\|Q\omega_1''\|_D\,.$$

The finiteness of $\|S\omega_2\|_D$ is already proved in proposition 1 and also that of $\|Q\omega_1''\|_D$ can be proved analogously. Hence there exists a sequence $\{r_{n'}\}$ such that

$$\lim_{n'\to\infty}(L'(r_{n'})+L''(r_{n'}))=0\,.\tag{4}$$

Finally,

$$\lim_{r\to R_0}\{(\omega_1', \omega_2^*)_{R_r} - \overline{(\omega_2', \omega_1''^*)_{R_r}}\} = (\omega_1, \omega_2^*).$$

Thus, by (2), (3) and (4) we obtain the desired result, q. e. d.

The bilinear relation in theorem 1 contains the quantities $l_{n'}^{(i)}$ and $m_{n'}^{(i)}$ which depend upon ω_1' resp. ω_2' and moduli $\theta_{n'}^{(i)}$ other than periods of ω_i and ω_i'' $(i=1,2)$. Thus the above relation may not be called the bilinear relation in a proper sense. But, the above relation may be regarded as a generalization of theorem 1 given in Kobori and Sainouchi [5]. Indeed, let both ω_1 and ω_2 be in Γ_{hse}, then $\omega_1''=\omega_2''\equiv 0$ and $c_{n'}^{(i)}=d_{n'}^{(i)}=0$, hence our theorem is reduced to the previous one.

In the theorem 1, suppose ω_1 is in $\Gamma_{he}\cap\Gamma_{hse}^*$ and set $\omega_2=-\omega_1^*$, then $\|\omega_1\|^2=0$. Hence we have

Corollary. *If* $\displaystyle\int_0^{R_0}\frac{dr}{\Lambda_0(r)}$ *is divergent for a canonical exhaustion, then F belongs to the class* O_{KD}, *and so,* $\Gamma_{hse}=\Gamma_{h0}$ *and* $\Gamma_{hm}=\Gamma_{he}$.

3. Next we shall give a sufficient condition analogous to that obtained by Accola [1].

Proposition 2. *If* $\min\limits_i \nu_n^{(i)}\geqq M>0$ (*M: constant*) *for any n, then there exists a canonical exhaustion* $\{\Omega_n\}$ *such that* Ω_n *has the same canonical homology basis* $\{A_i, B_i\}$ $(i=1,2,\cdots,p(n))$ *as* F_n *and the general Riemann's period relation* (⁑) *holds for* $\omega_1, \omega_2\in\Gamma_h$.

Proof. We use the same notations as in the proof of theorem 1, then by the same way as in theorem 1 we have

$$(5)\qquad \int_{\sum_{j=1}^{n-1}\nu_j}^{\sum_{j=1}^{n}\nu_j}\frac{L'(r)+L''(r)}{\Lambda_0(r)}dr \leqq \|\omega_1'\|_{D_n}\|S\omega_2\|_{D_n}+\|\omega_2'\|_{D_n}\|Q\omega_1''\|_{D_n}.$$

By the mean value theorem we can find a r_n $(\sum_{j=1}^{n-1}\nu_j<r_n<\sum_{j=1}^{n}\nu_j)$ such that the left hand side of (5) is equal to

$$\nu_n\frac{L'(r_n)+L''(r_n)}{\Lambda_0(r_n)} = (L'(r_n)+L''(r_n))\frac{\min\limits_i \nu_n^{(i)}}{2\pi}.$$

On the other hand, the right hand side of (5) is smaller than $2(\|\omega_1'\|_{D_n}\|\omega_2\|_{D_n}+\|\omega_2'\|_{D_n}\|\omega_1''\|_{D_n})$, and so we get

$$L'(r_n)+L''(r_n) \leqq \frac{4\pi}{\min\limits_i \nu_n^{(i)}}(\|\omega_1'\|_{D_n}\|\omega_2\|_{D_n}+\|\omega_2'\|_{D_n}\|\omega_1''\|_{D_n})$$
$$\leqq \frac{4\pi}{M}(\|\omega_1'\|_{D_n}\|\omega_2\|_{D_n}+\|\omega_2'\|_{D_n}\|\omega_1''\|_{D_n}).$$

Since the right hand side of this inequality tends to zero ($n\to\infty$), we obtain a sequence $\{r_n\}$ such that

$$\lim_{n\to\infty}(L'(r_n)+L''(r_n)) = 0.$$

Thus the proof of proposition 2 is completed.

Corollary. *If* $\inf\limits_n(\min\limits_i \nu_n^{(i)})>0$ *for a canonical exhaustion, then*

$$(\omega_1, \omega_2^*) = \lim_{n\to\infty}\sum_{i=1}^{p(n)}\left(\int_{A_i}\omega_1\int_{B_i}\overline{\omega}_2-\int_{A_i}\overline{\omega}_2\int_{B_i}\omega_1\right)$$

holds for any two $\omega_1, \omega_2\in\Gamma_{hse}$.

We remark that this corollary is included in a sufficient condition obtained by Accola [1].

§ 2. General Riemann's relation for the special choice of homology basis

1. In the proof of theorem 1 that the exhaustion $\{F_n\}$ is cononical is essential. In the case that the exhaustion is not necessarily canonical, the restriction of a semiexact differential to region F_n is not in general semiexact on F_n. But if we choose a special canonical homology basis with respect to an exhaustion $\{F_n\}$ of F such that the cycles on ∂F_n are weakly homologous to a linear combination of A-cycles only and if the index n of ∂F_n is large, each of index of corresponding A-cycles is large (Ahlfors [3]), then a semiexact differential with only a finite number of non-vanishing A-periods becomes also semiexact on F_n for a sufficiently large n. In following propositions we shall use such a homology basis.

2. Now let $\{F_n\}$ be an exhaustion of F and for each n $\gamma(r_{nk})$ be a set of finite number of level curves:

$$u(p) = r_{nk} \quad (\sum_{j=1}^{n-1} \mu_j = r_{n1} < r_{n2} < \cdots < r_{nk} < \cdots < r_{n\nu} = \sum_{j=1}^{n} \mu_j)$$

such that at least one critical point of $u(p)$ is contained in $\gamma(r_{nk})$ $(k \neq 1, \nu)$, where $u(p)$ is the function defined in § 1. We shall consider the relatively compact regions F_{nk} bounded by $\gamma(r_{nk})$ $(n=1, 2, \cdots, k=1, 2, \cdots, \nu(n))$, then we may suppose that those regions constitute an exhaustion $\{F_{nk}\}$ $(F_{n1}=F_n,\ F_{n\nu}=F_{n+1})$ such that each component $F_{nk}^{(i)}$ of $F_{nk}-\bar{F}_{nk-1}$ is a ring domain (cf. Ahlfors [3]). Now let us introduce the above mentioned homology basis with respect to the exhaustion $\{F_{nk}\}$, then the region bounded by $\gamma(r)$ $(r_{nk} \leqq r < r_{n\,k+1})$ has the same homology basis as that of F_{nk}. Let R_r be the region $\bar{F}_1 \cup \{p \mid u(p) < r\}$ and we set

$$\Lambda(r) = \max_i \int_{\gamma_r^{(i)}} dv = \max \theta_r^{(i)} \quad \text{and} \quad d_r^{(i)} = \int_{\gamma_r^{(i)}} \omega_2\,,$$

where $\gamma_r^{(i)}$ $(i=1, 2, \cdots, m(r))$ are components of ∂R_r and $\omega_2 \in \Gamma_h$.

Proposition 3. *Suppose ω_1 is in Γ_{hse} and has only a finite number of non-vanishing A-periods, on the other hand ω_2 is in Γ_h. Then if the integral $\int_0^R \frac{dr}{\Lambda(r)}$ is divergent, there exists an exhaustion such that the Riemann's period relation* (*) *holds for ω_1, ω_2 and in the right side of* (*) *only a finite number of terms $\int_{A_i} \omega_1 \int_{B_i} \bar{\omega}_2$ are nonzero.*

Proof. For a sufficiently large r $\omega_1 \in \Gamma_{hse}(\bar{R}_r)$, and so by an application of the Green formula to such a region R_r we have

$$(\omega_1, \omega_2^*)_{R_r} = \sum_{i=1}^{p(r)} (\int_{A_i} \omega_1 \int_{B_i} \bar{\omega}_2 - \int_{A_i} \bar{\omega}_2 \int_{B_i} \omega_1) - \int_{\partial R_r} u_r \bar{\omega}_2\,,$$

where only a finite number of terms $\int_{A_i} \omega_1 \int_{B_i} \bar{\omega}_2$ are nonzero and u_r is the same as that in proposition 1. We define a differential $S_0\omega_2$ in $F-\bar{F}_1$ as follows;

$$S_0\omega_2 = \omega_2 - \frac{d_r^{(i)}}{\theta_r^{(i)}}dv \qquad (r_{nk}<r<r_{n\,k+1}) \quad \text{in } F_{nk}^{(i)}.$$

Then

$$\|S_0\omega_2-\omega_2\|^2_{F^{(i)}_{nk}} = (|d_r^{(i)}|/\theta_r^{(i)})^2\|dv\|^2_{F^{(i)}_{nk}} = (|d_r^{(i)}|/\theta_r^{(i)})^2\int_{\partial F^{(i)}_{nk}} udv$$

$$= (r_{nk}-r_{n\,k-1})(|d_r^{(i)}|^2/\theta_r^{(i)}) \leqq \frac{(r_{nk}-r_{n\,k-1})}{\theta_r^{(i)}}\cdot\frac{2\pi\|\omega_2\|^2_{F^{(i)}_{nk}}}{\mu^{(i)}_{nk}},$$

where $\mu^{(i)}_{nk}$ is the harmonic modulus of $F^{(i)}_{nk}$. On the other hand, $\frac{2\pi}{\theta_r^{(i)}}(u(p)-r_{n\,k-1})$ is the harmonic function in $F^{(i)}_{nk}$ such that it vanishes on the inner boundary of $F^{(i)}_{nk}$ and its conjugate harmonic function has the variation 2π on the outer boundary of $F^{(i)}_{nk}$, and so the harmonic modulus $\mu^{(i)}_{nk}$ of $F^{(i)}_{nk}$ is equal to $2\pi\frac{r_{nk}-r_{n\,k-1}}{\theta_r^{(i)}}$. Thus we have

$$\|S_0\omega_2-\omega_2\|^2_{F^{(i)}_{kn}} \leqq \|\omega_2\|^2_{F^{(i)}_{nk}}.$$

Hence

$$\|S_0\omega_2-\omega_2\|^2_{F-F_1} \leqq \|\omega_2\|^2.$$

Consequently,

$$\|S_0\omega_2\|^2_{F-F_1} \leqq 4\|\omega_2\|^2.$$

Therefore, by the same way as in the proof of proposition 1 we get the desired result, q. e. d.

In the general case where ω_1 and ω_2 are both in Γ_h, assume that both ω_1' and ω_2' have only a finite number of non-vanishing A-periods, then by the same way as in the proof of theorem 1 we can prove the following

Theorem 2. *Suppose ω_i $(i=1,2)$ are in Γ_h and ω_i' are the Γ_{hse}-components of ω_i in the orthogonal decomposition $\Gamma_h=\Gamma_{hse}\dotplus\Gamma^*_{hm}$. Then if ω_i' $(i=1,2)$ have a finite number of non-vanishing A-periods and $\int_0^R \frac{dr}{\Lambda(r)}$ is divergent, then there exists an exhaustion such that the Riemann's period relation (‡) holds for ω_1 and ω_2.*

Remark ; (i) In this case the period relation also may be written as

$$(\omega_1, \omega_2^*) = \sum_{i=1}^{p}\left(\int_{A_i}\omega_1'\int_{B_i}\bar{\omega}_2 - \int_{A_i}\bar{\omega}_2'\int_{B_i}\omega_1\right) +$$
$$\lim_{n'\to\infty}\left\{\sum_{i=1}^{p(n')}\left(\int_{A_i}\omega_1''\int_{B_i}\bar{\omega}_2' - \int_{A_i}\bar{\omega}_2''\int_{B_i}\omega_1'\right) + R'_{n'}\right\},$$

where the first sum at the right hand is a finite sum.

(ii) Since $\int_{\beta_{n'}^{(i)}}\omega_1' = \int_{\beta_{n'}^{(i)}}\omega_2' = 0$ for a sufficiently large n', $c_{n'}^{(i)} = \int_{\beta_{n'}^{(i)}}\omega_1''$ and $d_{n'}^{(i)} = \int_{\beta_{n'}^{(i)}}\omega_2''$. So if ω_1 and ω_2 are in Γ_{hse} and have only a finite number of non-vanishing A-periods, then $c_{n'}^{(i)} = d_{n'}^{(i)} = 0$ for large n' and so we get the theorem II in [5].

§ 3. An another form for general Riemann's period relation

1. In the following two sections we always suppose that the exhaustion $\{F_n\}$ is canonical. For $\omega_1 \in \Gamma_{h0}$ and $\omega_2 \in \Gamma_h$, we put $\omega_1 = \omega_1' + \omega_1''$ $(\omega_1' \in \Gamma_{h0} \cap \Gamma_{hse}^*,\ \omega_1'' \in \Gamma_{hm})$ and $\omega_2 = \omega_2' + \omega_2''$ $(\omega_2' \in \Gamma_{hse},\ \omega_2'' \in \Gamma_{hm}^*)$. Let us denote by $\sigma(A_i)$ and $\sigma(B_i)$ the period reproducer in Γ_{h0} for A_i and B_i, respectively, and we set

$$T_n\omega_2' = \sum_{i=1}^{p(n)}(b_i\sigma(A_i) - a_i\sigma(B_i)),$$

where $a_i = \int_{A_i}\omega_2'$ and $b_i = \int_{B_i}\omega_2'$. Now let $O_n\omega_2'$ be the $\Gamma_{h0} \cap \Gamma_{h0}^*$-component of $(T_n\omega_2')^*$ in the orthogonal decomposition $\Gamma_{h0}^* = \Gamma_{h0} \cap \Gamma_{h0}^* \dotplus \Gamma_{he}^*$.

Proposition 4. *Let $\omega_2 = \omega_2' + \omega_2''$ be in Γ_h $(\omega_2' \in \Gamma_{hse},\ \omega_2'' \in \Gamma_{hm}^*)$ and $\|O_n\omega_2'\|$ be bounded as $n \to \infty$, then for any $\omega_1 \in \Gamma_{h0}$*

$$(\omega_1, \omega_2^*) = \lim_{n\to\infty}\sum_{i=1}^{p(n)}\left(\int_{A_i}\omega_1\int_{B_i}\bar{\omega}_2' - \int_{A_i}\bar{\omega}_2'\int_{B_i}\omega_1\right) - \lim_{n\to\infty}\sum_{i=1}^{m(n)}x_n^{(i)}\int_{\gamma_n^{(i)}}\bar{\omega}_2,$$

where $x_n^{(i)}$ are complex numbers such that if the harmonic measure of $\gamma_n^{(i)}$ in F_n is denoted by $\omega_{F_n}(\gamma_n^{(i)})$, $\omega_{F_n}'' = \sum_{i=1}^{m(n)}x_n^{(i)}\omega_{F_n}(\gamma_r^{(i)}) \to \omega_1''$.

Proof. By the orthogonality of decomposition we have

$$(\omega_1, \omega_2^*) = (\omega_1, \omega_2'^*) - \overline{(\omega_2'', \omega_1''^*)}.$$

According to the Accola's theorem, the bilinear relation

$$(\omega_1, \omega_2'^*) = \lim_{n\to\infty}(\omega_1, (T_n(\omega_2')*) = \lim_{n\to\infty}\sum_{i=1}^{p(n)}(\int_{A_i}\omega_1\int_{B_i}\overline{\omega}_2' - \int_{A_i}\overline{\omega}_2'\int_{B_i}\omega_1)$$

holds if and only if $||O_n\omega_2'||$ is bounded. On the other hand, by the definition of Γ_{hm} there exists $\omega''_{F_n}\in\Gamma_{hm}(\bar{F}_n)$ such that

$$||\omega_1'' - \omega''_{F_n}||_{F_n} \to 0 \qquad (n\to\infty).$$

The harmonic measure ω''_{F_n} is expressed as a linear combination of $\omega_{F_n}(\gamma_n^{(i)})$

$$\omega''_{F_n} = \sum_{i=1}^{m(n)} x_n^{(i)}\omega_{F_n}(\gamma_n^{(i)}),$$

where $x_n^{(i)}$ are complex numbers (Ahlfors and Sario [4]). Thus we obtain

$$(\omega_2'', \omega_1''^*) = \lim_{n\to\infty}(\omega_2'', \omega''^*_{F_n})_{F_n} = \lim_{n\to\infty}\sum_{i=1}^{m(n)}\bar{x}_n^{(i)}(\omega_2'', \omega_{F_n}(\gamma_n^{(i)})*)_{F_n}$$

$$= \lim_{n\to\infty}\sum_{i=1}^{m(n)}\bar{x}_n^{(i)}\int_{\gamma_n^{(i)}}\omega_2'' = \lim_{n\to\infty}\sum_{i=1}^{m(n)}\bar{x}_n^{(i)}\int_{\gamma_n^{(i)}}\omega_2.$$

Therefore we obtain the above mentioned proposition, q. e. d.

Remark. Suppose that the surface belongs to the class O_{KD}. Let ω_i be in Γ_h and $\omega_i=\omega_i'+\omega_i''$, where ω_1', $\omega_2'^*\in\Gamma_{h0}=\Gamma_{hse}$ and $\omega_1''^*$, $\omega_2''\in\Gamma_{hm}=\Gamma_{he}$. Then

$$(\omega_1, \omega_2^*) = (\omega_1', \omega_2^*)+(\omega_1'', \omega_2''^*).$$

Hence if $||O_n\omega_2'||$ is bounded, we have

$$(\omega_1, \omega_2^*) = \lim_{n\to\infty}\sum_{i=1}^{p(n)}(\int_{A_i}\omega_1'\int_{B_i}\overline{\omega}_2' - \int_{A_i}\overline{\omega}_2'\int_{B_i}\omega_1') - \lim_{n\to\infty}\sum_{i=1}^{m(n)}(x_n^{(i)}\int_{\gamma_n^{(i)}}\overline{\omega}_2 -$$

$$\bar{y}_n^{(i)}\int_{\gamma_n^{(i)}}\omega_1) = \lim_{n\to\infty}\sum_{i=1}^{p(n)}(\int_{A_i}\omega_1'\int_{B_i}\overline{\omega}_2 - \int_{A_i}\overline{\omega}_2\int_{B_i}\omega_1') -$$

$$\lim_{n\to\infty}\sum_{i=1}^{m(n)}(x_n^{(i)}\int_{\gamma_n^{(i)}}\overline{\omega}_2 - \bar{y}_n^{(i)}\int_{\gamma_n^{(i)}}\omega_1),$$

where $x_n^{(i)}$ and $y_n^{(i)}$ are complex numbers defined for the Γ_{hm}-components of ω_1' and ω_2 by the same way as in the proposition 4.

Suppose that $\int_0^{R_0}\frac{dr}{\Lambda_0(r)}$ is divergent as in §1. Then there exists an exhaustion $\{\Omega_{n'}\}$ such that for $\omega_1, \omega_2 \in \Gamma_{hse}$ the bilinear relation

$$(\omega_1, \omega_2^*) = \lim_{n'\to\infty} \sum_{i=1}^{p(n')} \left(\int_{A_i} \omega_1 \int_{B_i} \bar{\omega}_2 - \int_{A_i} \bar{\omega}_2 \int_{B_i} \omega_1 \right)$$

holds. Thus we can prove the following proposition.

Proposition 5. *If* $\int_0^{R_0} \frac{dr}{\Lambda_0(r)}$ *is divergent, then there exist an exhaustion* $\{\Omega_{n'}\}$ *and numbers* $x_{n'}^{(i)}$ *such that for* $\omega_1 \in \Gamma_{hse}$, $\omega_2 \in \Gamma_h$ *the relation*

$$(\omega_1, \omega_2^*) = \lim_{n'\to\infty} \sum_{i=1}^{p(n')} \left(\int_{A_i} \omega_1 \int_{B_i} \bar{\omega}_2' - \int_{A_i} \bar{\omega}_2' \int_{B_i} \omega_1 \right) - \lim_{n'\to\infty} \sum_{i=1}^{m(n')} x_{n'}^{(i)} \int_{\gamma_{n'}^{(i)}} \bar{\omega}_2$$

holds, where ω_2' *is the* Γ_{hse}*-component of* ω_2 *and* $x_{n'}^{(i)}$ *are complex numbers defined for the* Γ_{hm}*-component of* ω_1 *as before.*

2. We choose annuli $R_n^{(i)}$ $(i=1, 2, \cdots, m(n))$ in canonical region F_n so that $\gamma_n^{(i)} \subset \bar{R}_n^{(i)}$, $R_n^{(i)} \cap R_n^{(j)} = \phi$ $(i \neq j)$. Let $\mu(R_n^{(i)})$ be the harmonic modulus of $R_n^{(i)}$ and define μ_{F_n} to be the supremum of $\min_i \mu(R_n^{(i)})$ for all possible choices of $\bigcup_i R_n^{(i)}$. If $\mu_{F_n} \geqq M > 0$ (M: constant) for $n \to \infty$, then for $\sigma \in \Gamma_{hse}$ $\|O_n(\sigma)\|$ is bounded (cf. Accola [1]). Accordingly the proposition 4 is valid on the surface with the property $\mu_{F_n} \geqq M > 0$. We consider a surface with the condition

$$(\sharp) \qquad \int_0^{R_0} \frac{dr}{\Lambda_0(r)} = \infty \quad \text{and } \mu_{F_n} \geqq M > 0 \qquad \text{for any n.}$$

It is easy to see that surface with the property $\inf_n (\min_i \nu_n^{(i)}) > 0$ satisfies the condition ($\sharp$). There exists a surface with the condition ($\sharp$) which does not belong to O_{HD}. For example, we can find such a surface in the class of Schottkyan covering surfaces of a closed Riemann surface (cf. Tsuji [15]). On such a surface let us compare the result obtained in §1 with that in §3. For simplicity let ω_1 and ω_2 in Γ_{hse} and Γ_h, respectively. Then we get the following theorem.

Theorem 3. *On a surface with the condition* ($\sharp$) *there exists* *an exhaustion* $\{\Omega_{n'}\}$ *such that the relation*

$$\lim_{n'\to\infty}\{\sum_{i=1}^{p(n')}(\int_{A_i}\omega_1\int_{B_i}\overline{\omega}_2''-\int_{A_i}\overline{\omega}_2''\int_{B_i}\omega_1)-\sum_{i=1}^{m(n')}(\frac{m_{n'}^{(i)}}{\theta_{n'}^{(i)}}-x_{n'}^{(i)})\int_{\beta_{n'}^{(i)}}\overline{\omega}_2''\}=0$$

holds for $\omega_1\in\Gamma_{hse}$ *and* $\omega_2\in\Gamma_h$, *where* $\omega_2=\omega_2'+\omega_2''$ $(\omega_2'\in\Gamma_{hse},\ \omega_2''\in\Gamma_{hm}^*)$ *and* $x_{n'}^{(i)}$ *and* $m_{n'}^{(i)}$ *are numbers defined in the propositions* 4 *and* 1, *respectively.*

Proof. According to the first condition of (♯), from the proposition 1 it follows the existence of exhaustion $\{\Omega_{n'}\}$ such that

$$(\omega_1,\omega_2^*)=\lim_{n'\to\infty}\{\sum_{i=1}^{p(n')}(\int_{A_i}\omega_1\int_{B_i}\overline{\omega}_2-\int_{A_i}\overline{\omega}_2\int_{B_i}\omega_1)-\sum_{i=1}^{m(n')}\frac{m_{n'}^{(i)}}{\theta_{n'}^{(i)}}\int_{\beta_{n'}^{(i)}}\overline{\omega}_2''\}.$$

Similarly, by the proposition 4 we have

$$(\omega_1,\omega_2^*)=\lim_{n'\to\infty}\sum_{i=1}^{p(n')}(\int_{A_i}\omega_1\int_{B_i}\overline{\omega}_2'-\int_{A_i}\overline{\omega}_2'\int_{B_i}\omega_1)-\lim_{n'\to\infty}\sum_{i=1}^{m(n')}x_{n'}^{(i)}\int_{\beta_{n'}^{(i)}}\overline{\omega}_2''.$$

Thus we get the desired result, q. e. d.

Particularly, putting $\omega_1=\sigma(A_j)$ we have

Collorary. *On a surface with the condition* (♯) *there exists an exhaustion* $\{\Omega_{n'}\}$ *such that for* $\omega_2''\in\Gamma_{hm}^*$

$$\int_{A_j}\omega_2''=\lim_{n'\to\infty}\sum_{i=1}^{m(n')}\left(\overline{x_{n'}^{(i)}(\sigma(A_j))}-\frac{\overline{m_{n'}^{(i)}(\sigma(A_j))}}{\theta_{n'}^{(i)}}\right)\int_{\beta_{n'}^{(i)}}\omega_2'',$$

where $x_{n'}^{(i)}(\sigma(A_j))$ *and* $m_{n'}^{(i)}(\sigma(A_j))$ *are numbers defined for* $\sigma(A_j)$ *as before.*

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REFERENCES

[1] Accola, R.: The bilinear relation on open Riemann surfaces. Trans. Amer. Math. Soc. 96 (1960) 143-161.

[2] ―――――: Differentials and extremal length on Riemann surfaces. Proc. N.A.S. 46 (1960) 540-543.

[3] Ahlfors, L. V.: Normalintegrale auf offenen Riemannschen Flächen. Ann. Acad. Scient. Fenn. A.I. 35 (1947).

[4] Ahlfors, L. V. and Sario, L.: Riemann surfaces. Princeton (1960).

[5] Kobori, A. and Sainouchi, Y.: On the Riemann's relation on open Riemann surfaces. J. Math. Kyoto Univ. 2 (1962) 211–223.

[6] Kusunoki, Y.: On Riemann's period relations on open Riemann surfaces. Mem. Coll. Sci. Univ. Kyoto Ser. A. Math. 30 (1956) 1–22.

[7] Marden, A.: The bilinar relation on open Riemann surface. Trans. Amer. Math. Soc. 100 (1964) 225–239.

[8] Matsui, K.: A note on Riemann's period relation. Proc. Japan Acad. 40 (1964) 697–702.

[9] ————: A note on Riemann's period relations II. Ibid. 42 (1966) 41–45.

[10] Mori, M.: Contribution to the theory of differentials on open Riemann surfaces. J. Math. Kyoto Univ. 4 (1964) 77–97.

[11] Myrberg, P. J.: Über transzendente hyperelliptische Integrale erster Gattung. Ann. Acad. Scient. Fenn. 14 (1943).

[12] Pfluger, A.: Über die Riemannsche Periodenrelation auf transzendenten hyperelliptischen Flächen. Comm. Math. Helv. 30 (1956) 98-106.

[13] ————: Theorie der Riemannschen Flächen. Berlin (1957).

[14] Sainouchi, Y.: On the analytic semiexact differentials on open Riemann surface. J. Math. Kyoto Univ. 2 (1963) 277–293.

[15] Tsuji, M.: Potential theory. Tokyo (1959).