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Unstable Homotopy Groups of Classical Groups (odd primary components).

by

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Introduction.

The purpose of this paper is to compute the p-primary components of unstable homotopy groups of classical groups (in this paper p always denotes an odd prime). In [2] B. Harris has shown the following direct sum decompositions and isomorphism, so we are enough to compute only for unitary and symplectic groups.

$$(0.1)\quad \begin{aligned} &{}^{p}\pi_i(SU(2n))\cong{}^{p}\pi_i(Sp(n))+{}^{p}\pi_i(SU(2n)/Sp(n)),\\ &{}^{p}\pi_i(SU(2n+1))\cong{}^{p}\pi_i(SO(2n+1))\\ &\qquad\qquad +{}^{p}\pi_i(SU(2n+1)/SO(2n+1)),\\ &{}^{p}\pi_i(SO(2n))\cong{}^{p}\pi_i(SO(2n-1))+{}^{p}\pi_i(S^{2n-1}),\\ &{}^{p}\pi_i(SO(2n+1))\cong{}^{p}\pi_i(Sp(n)),\end{aligned}$$

where ${}^{p}\pi_i$ stands for a subgroup of the i-th homotopy group π_i with an index prime to p and having no q-primary part for $q\neq p$.

Before stating our results we define functions $N(n, k)$ and $N'(n, k)$ of integers n, k, $0\leqq k<p^2-1$ for each odd prime p. Let $t=\left[\dfrac{k}{p-1}\right]$ and $n+k=q-i$, where $q\equiv 0 \pmod p$ and $1\leqq i\leqq p$, then we define $N(n, k)$ and $N'(n, k)$ as follows

$$N(n,k)=\begin{cases} 0 & i>t,\\ \min(\nu_p(q), t-i+1) & i\leqq t,\quad t<p,\\ \min(\nu_p(q)-1, p) & i=1,\quad t=p,\\ \min(\nu_p(q), p-i+2) & i\neq 1,p,\quad t=p,\end{cases}$$

$$N'(n,k)=\begin{cases}\min(\nu_p(q-p^2),2) & i=t=p,\\ \nu_p((n+k)!)-t+N(n,k) & t<p \text{ or } t=p, i=1,\\ \nu_p((n+k)!)-p-1+N(n,k) & t=p, i\neq 1,\end{cases}$$

where $\nu_p(x)$ is defined for any non zero rational number x as the exponent of p in the factorization of x into prime powers and we define $\nu_p(0)=0$.

Theorem 1. *For* $0<k<p^2-1$, $k<(n+1)(p-1)$

$$(0.2)\quad {}^p\pi_{2n+2k-1}(SU(n))\cong\begin{cases}Z_{p^{N(n,k)}} & k<p^2-2 \text{ or } n\not\equiv 0 \pmod p\\ Z_{p^{N(n,k)}}+Z_p & k=p^2-2,\ n\equiv 0 \pmod p).\end{cases}$$

For $0\leqq k<p^2-1$, $k<(n+1)(p-1)-1$

$$(0.3)\quad {}^p\pi_{2n+2k}(SU(n))\cong\begin{cases}Z_{p^{N'(n,k)}} & k<p(p-1)-1 \text{ or } k=p^2-2\\ & \text{or } n+k\equiv -2 \pmod p)\\ Z_{p^{N'(n,k)}}+Z_p & p(p-1)-1\leqq k<p^2-2,\\ & n+k\not\equiv -2 \pmod p).\end{cases}$$

Theorem 2. *Let* $2k<p^2-1$, *then*

for $k\leqq(n+1)(p-1)$: ${}^p\pi_{4n+4k-1}(Sp(n))=0$,

for $k<(n+1)(p-1)$: ${}^p\pi_{4n+4k}(Sp(n))$

$$\cong\begin{cases}0 & 2k<p(p-1) \text{ or } n+k\equiv -1 \pmod p)\\ Z_p & p(p-1)\leqq 2k,\ n+k\not\equiv -1 \pmod p),\end{cases}$$

for $k<(n+1)(p-1)$: ${}^p\pi_{4n+4k+1}(Sp(n))\cong Z_{p^{N(2n+1,2k)}}$,

for $k<(n+1)(p-1)$: ${}^p\pi_{4n+4k+2}(Sp(n))\cong Z_{p^{N'(2n+1,2k)}}$.

The most part of this paper is devoted to prove the formula (0.2). In case of $i=1$, $t<p$, *H.* Matsunaga [4] proved (0. 2) and the idea of our proof is essentially due to [4]. In §1 we shall reduce our problem to the computation of homotopy groups of a stunted complex projective space and a simple complex $X_n^{0,t}$, using Bott periodicity theorem and I. Yokota's cellular decomposition of special unitary groups. Attaching maps of cells of $X_n^{0,t}$ are considered in §2, there we shall use the theory of functional Chern character (Toda [9]), or Adams' invariant e_C ([1]), in the form of Lemma 2. 1 and

in order to calculate the Chern characters in a complex projective space we use Proposition 2. 2. (This proposition is proved in the last section.) In §3 we compute the homotopy groups of some elementary complexes and prove (0. 2). Then (0. 3) and theorem 2 are proved easily in §4 and in §5 respectively.

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§1. Reduction to a simple complex.

At first we consider the following exact sequence;

$$\pi_{2n+2k}(SU(n+k+1))\to\pi_{2n+2k}(SU(n+k+1)/SU(n))$$
$$\to\pi_{2n+2k-1}(SU(n))\xrightarrow{i_*}\pi_{2n+2k-1}(SU(n+k+1))$$

where $\pi_{2n+2k-1}(SU(n+k+1))\cong Z$, $\pi_{2n+2k}(SU(n+k+1))=0$ by Bott periodicity and $\pi_{2n+2k-1}(SU(n))$ is a finite group so i_* is zero map. Therefore:

(1. 1) $$\pi_{2n+2k-1}(SU(n))\cong\pi_{2n+2k}(SU(n+k+1)/SU(n)).$$

By the cellular decomposition of special unitary groups ([10]), the (4n+3)-skeleton of $SU(n+k+1)/SU(n)$ has the cell structure of $S(CP(n+k)/CP(n-1))$ where S is the suspension and $CP(i)$ i-dimensional complex projective space. So if n is sufficiently large with respect to k, $\pi_{2n+2k-1}(SU(n))\cong\pi_{2n+2k-1}(CP(n+k)/CP(n-1))$. But by I. M. James' following generalized Freudenthal-Serre suspension theorem ([3] Th. 3. 2):

(1. 2) $${}^p\pi_i(SU(n+k+1)/SU(n))$$
$$\cong{}^p\pi_{i+2Nb_{k+1}}(SU(n+k+1+Nb_{k+1})/SU(n+Nb_{k+1}))$$

for $i<2p(n+1)-3$, where b_{k+1} is the James number and N is an arbitrary natural number, so we have

(1. 3). *Under the assumption of* (0. 2) *it is sufficient to prove* (0. 2) *for sufficiently large n.*

For, taking $N\cdot b_{k+1}$ a multiple of sufficiently large power of p, the value $N(n, k)$ does not change. In the future, we always assume n sufficiently large then:

$$^{p}\pi_{2n+2k-1}(SU(n))\cong{}^{p}\pi_{2n+2k-1}(CP(n+k)/CP(n-1)).$$

Now we quote the results on stable homotopy groups of spheres ([7] Th. 4.15)

$$(1.4)\quad \begin{aligned} &{}^{p}\pi_{N+2t(p-1)-1}(S^{N})\cong Z_{p}=\{\alpha_{t}\} && 1\leqq t<p^{2},\ t\not\equiv 0 \pmod p,\\ &{}^{p}\pi_{N+2rp(p-1)-1}(S^{N})\cong Z_{p}^{2}=\{\alpha'_{rp}\} && 1\leqq r<p-1,\ p\cdot\alpha'_{rp}=\alpha_{rp},\\ &{}^{p}\pi_{N+2p(p-1)-2}(S^{N})\cong Z_{p}=\{\beta\} && \\ &{}^{p}\pi_{N+2(p+1)(p-1)-3}(S^{N})\cong Z_{p}=\{\alpha_{1}\cdot\beta'\} && \beta'=S^{2p-3}\beta,\\ &{}^{p}\pi_{N+i}(S^{N})=0 \quad i<2(p+1)(p-1) && \text{and except above cases,} \end{aligned}$$

where $\alpha_{i}, \alpha'_{ip}, \beta$ denotes generators and α_{i} are defined inductively, using the secondary composition, by $\alpha_{i}=\{\alpha_{1}, p, \alpha_{i-1}\}$.

Let $K=S\cup e_{1}\cup\cdots\cup e_{m}$ be a CW-complex such that $S=S^{N}$ (N: large) and e_{i} are $N+2i(p-1)$-cells. By use of stable homotopy exact sequences it follows easily from (1.4) the following (1.5).

$$(1.5)\quad {}^{p}\pi_{N+j}(K)\begin{cases} =0 & \text{if } 0<j<2(p+1)(p-1)-3,\ j\not\equiv -1, 0 \pmod{2(p-1)} \text{ and } j\neq 2p(p-1)-2,\\ \cong Z & \text{if } j=2k(p-1) \text{ and } 0\leqq k\leqq m\leqq p. \end{cases}$$

Lemma 1.1. *Let K be a simply connected finite CW-complex and the order of the attaching map of each cell of K (in this paper we identify an attaching map and its homotopy class) be finite then there existe a finite CW-conplex K' and a cellular map f of K' into K satisfying the following conditions*:

(i) *f induces the C_{p}-isomarphism f_{*}: $\pi_{i}(K')\to\pi_{i}(K)$ (C_{p} is a class of finite abelian groups whose orders are prime to p.)*

(ii) *The order of the attaching map of each cell of K' is a power of p. Especially if the dimension of each cell of K is even that is $K=S^{2n}\cup\cdots\cup e^{2k}$ and $n<k, k-n<p^{2}-2$, then by (1.4) K' is a one point union of complexes K_{i} and the dimension of each cell of K_{i} equals to $2i$ modulo $2(p-1)$, $0\leqq i\leqq p-2$.*

Proof. The case $K=S^{m}$ is trivial and we construct the complex K' and the map f inductively. Suppose $K=K_0\cup_\gamma e^n$ and the complex K_0', the map f_0 of K_0' into K_0 satisfying (i), (ii) are already constructed. If the order of γ is $p^r q$, $q\not\equiv 0 \pmod p$ then $q\cdot\gamma \in {}^p\pi_{n-1}(K_0)$, so there exists $\gamma'\in {}^p\pi_{n-1}(K_0')$ such that $f_{0*}(\gamma')=q\cdot\gamma$. We define $K'=K_0\cup_{\gamma'} e^n$, $f\,|\,K_0=f_0$, $f\,|\,e^n: e^n\to e^n$ a map of degree q. Here we may assume that K has no 1-cell, then by our construction, K_0' is simply connected, $f_{0*}:\pi_2(K')\to\pi_2(K)$ is onto and f induces an isomorphism of the homology mod p. In virtue of Serre's $\mathcal{C}$-theory [6], K' and f clearly satisfy (i), (ii). q.e.d.

$CP(n+k)/CP(n-1)=S^{2n}\cup e^{2n+2}\cup\cdots\cup e^{2n+2k}$ satisfies the condition of Lemma 1.1 so if $k<p^2-2$, it follows from (1.5) that $(CP(n+k)/CP(n-1))'$ has the following cell structure up to homotopy type

$$(CP(n+k)/CP(n-1))'=\left[\bigvee_{i=0}^{l} X^{0,t}_{n+i}\right]\vee\left[\bigvee_{j=l+1}^{p-2} X^{0,t-1}_{n+j}\right],$$

where $\vee$ denotes one point union of complexes, $n+k=n+l+t(p-1)$, $0\leqq l\leqq p-2$ and $X^{0,t}_{n+i}=S^{2n+2i}\cup e^{2n+2i+2(p-1)}\cup\cdots\cup e^{2n+2i+2t(p-1)}$. So by (1.5),

$$f_*:{}^p\pi_{2n+2l+2t(p-1)-1}(X^{0,t}_{n+l})\to{}^p\pi_{2n+2k-1}(CP(n+k)/CP(n-1))$$

is an isomorphism. In the sequel we get the following isomorphisms.

Proposition 1.2. *Under the condition of* (0.2)

$${}^p\pi_{2n+2k-1}(SU(n))\cong{}^p\pi_{2m+2l+2t(p-1)-1}(X^{0,t}_{m+l})$$
$${}^p\pi_{2n+2p^2-5}(SU(n))\cong{}^p\pi_{2m+2p^2-5}((X^{0,p}_{m}\vee X^{0,p-1}_{m+p-2})\cup e^{2m+2p^2-4})$$

for some large m with $N(n,k)=N(m,k)$.

In computing these groups we may assume $l=0$ and n sufficiently large.

§2. Attaching maps of $X^{0,t}_n$.

We shall recall the definition and some properties of the functional Chern character CH ([9] §6), or Adams' invariant e_C ([1]), CH is a homomorphism of $\pi_{2a+2b-1}(S^{2a})$ into the rational numbers modulo 1: Q/Z, defined as follows. Let γ be any element of $\pi_{2a+2b-1}(S^{2a})$, consider its mapping cone $C_\gamma = S^{2a} \cup_\gamma e^{2a+2b}$ and if ξ is an element of $\widetilde{K}(C_\gamma)$ such that $ch_a\xi = S^{2a}$, $ch_{a+b}\xi = \lambda \cdot e^{2a+2b}$ then we define $CH(\gamma) = \{\lambda\} \in Q/Z$. This does not depend on the choice of ξ. By definition CH is evidently an invariant of double suspension and the following properties are known.

(i) *There exists an element* α_1 *of* ${}^p\pi_{2N+2p-3}(S^{2N})$ *such that* $CH(\alpha_1) = \frac{1}{p}$. ([1] Cor. 8. 4)

(ii) *If* $\alpha \in \pi_{2a-2}(S^{2b-1})$, $\beta \in \pi_{2b-1}(S^{2c})$ *and* $(q\iota)\cdot\alpha = 0$, $\beta \cdot (q\iota) = 0$, $q \in Z$ *(*ι *is the homotopy class of the identity map of* S^{2b-1}*) then*

$$CH\{\beta, q\iota, \alpha\} = \pm qCH(S\alpha)CH(\beta) \qquad ([1]\,\text{Th. } 11.1)$$

Therefore by (1.4) $CH(\alpha_i) = \pm \frac{1}{p}$, $CH(\alpha'_{ip}) = \pm \frac{1}{p^2}$ (replacing the generator α'_{ip} if it is necessary) and CH: ${}^p\pi_{2N+2k(p-1)-1}(S^{2N}) \to Q/Z$ is injective if $k < p(p-1)$; in other words, in the complex $S^{2N} \cup_\gamma e^{2N+2i(p-1)}$, $\gamma \in {}^p\pi_{2N+2i(p-1)-1}(S^{2N})$ $i < p(p-1)$, γ is trivial if and only if $CH(\gamma) = 0$.

Let Q_p denotes the ring of rational numbers whose denominators are prime to p, and we define a homomorphism ch_n of $\widetilde{K}(X) \otimes Q_p$ into $H^{2n}(X: Q)$ by an evident manner, that is for any $\eta = \Sigma a_i\xi_i \in \widetilde{K}(X) \otimes Q_p$, $a_i \in Q_p$ $\xi_i \in \widetilde{K}(X)$, $ch_n\eta = \Sigma a_i ch_n \xi_i$. Then the next lemma is a trivial restatement of the above fact.

Lemma 2.1. *Let* $X = S^{2N} \cup_\gamma e^{2N+2k(p-1)}$, $\gamma \in {}^p\pi_{2N+2k(p-1)-1}(S^{2N})$ $k < p(p-1)$, *and* ξ *is an element of* $K(X) \otimes Q_p$ *such that* $ch_N\xi = a \cdot S^{2N}$, $a \in Z$ $a \not\equiv 0 \pmod p$, *then* γ *is trivial if and only if* $\nu_p(ch_{N+k(p-1)}\xi) \geqq 0$. *(Here we identify* $\lambda \cdot e^{2N+2k(p-1)} \in H^{2N+2k(p-1)}(X;\ Q)$ *with* $\lambda \in Q$; *such an identification will be made frequently.)*

Let $\widetilde{\xi}$ be the dual bundle to the canonical line bundle over $CP(n)$ and $x \in H^2(CP(n);\ Z)$ be the Chern class of $\widetilde{\xi}$ then it is well known that $\widetilde{K}(CP(n))$ (respectively $\widetilde{H}^*(CP(n);\ Z)$) is a truncated polynomial ring with the single generator $\xi = \widetilde{\xi} - 1$ (x) and a single relation $\xi^{n+1} = 0$ $(x^{n+1} = 0)$. The next exact sequence shows that we can identify $\widetilde{K}(CP(n)/CP(m))$, $m<n$, $(\widetilde{H}^*(CP(n)/CP(m)))$ with an ideal in $\widetilde{K}(CP(n))(\widetilde{H}^*(CP(n)))$ generated by ξ^{m+1} (x^{m+1}).

$$0\to\widetilde{K}(CP(n)/CP(m))\to\widetilde{K}(CP(n))\to\widetilde{K}(CP(m))\to 0$$
$$(0\to\widetilde{H}^*(CP(n)/CP(m))\to\widetilde{H}^*(CP(n))\to\widetilde{H}^*(CP(m))\to 0)$$

Obviously $ch\,\xi=e^x-1$. The following proposition will be proved in §6.

Proposition 2.2. *There exists an element* $\eta\in\widetilde{K}(CP(\infty))\otimes Q_p$ *such that*

$$ch\,\eta^n\equiv\sum_{k=0}^{p}\left(-\frac{1}{p}\right)^k\frac{n(n+kp-1)!}{k!(n+kp-k)!}x^{n+k(p-1)}\ (\mathrm{mod}\ x^{n+p^2-1}),$$

that is, $ch_n\eta^n=x^n$ *and*

$$ch_{n+t(p-1)}\eta^{n+s(p-1)}$$
$$=\frac{(n+s(p-1))\prod_{i=1}^{t-s-1}(n+t(p-1)+i)}{(-1)^{t-s}(t-s)!\,p^{t-s}}x^{n+t(p-1)}$$

for $0\leqq t-s\leqq p$.

REMARK. (i) Among the factors of the numerators of the last formula, *at most* one factor is a multiple of p, since $n+s(p-1)=n+t(p-1)+(s-t)p+t-s$. (ii) In the future we consider η^n as an element of $K(CP(m)/CP(n-1))\otimes Q_p$, $m\geqq n$.

Now let us consider the attaching maps of cells of $X_n^{0,t}$, for the simplicity, we denote e_i the $(2n+2i(p-1))$-cell of $X_n^{0,t}$ i.e. $X_n^{0,t}=S_0\cup e_1\cup\cdots\cup e_t$ and $X_n^{i,j}$, $0\leqq i\leqq j\leqq t$, is the complex obtained from $X_n^{0,j}$ smashing the subcomplex $X_n^{0,i-1}$ to a point. Le $\gamma_i\in{}^p\pi_{2n+2i(p-1)}(X_n^{0,i-1})$ be the attaching map of the cell e_i of $X_n^{0,i}$ and $\gamma'_i\in{}^p\pi_{2n+2i(p-1)-1}(S_{i-l})$, $S_{i-1}=X_n^{i-1,i-1}$, the attaching map of e_i of $X_n^{i-1,i}$, we say γ_i is *essential* (*trivial*) to e_{i-1} if $\gamma'_i\neq 0$ $(\gamma'_i=0)$. Also we denote $P_n^{i,j}$ the stunted complex projective space $CP(n+j(p-1))/CP(n+i(p-1)-1)$ then the map f induces naturally a map f of $X_n^{i,j}$ into $P_n^{i,j}$, $0\leqq i\leqq j\leqq t$.

Proposition 2.3. *Let* $n+t(p-1)=q-i$, $q\equiv 0(\mathrm{mod}\,p)$, $1\leqq i\leqq p$, $t\leqq p$, *then in the complex* $X_n^{j-1,j}=S_{j-1}\cup_{\gamma'_j}e_j$, $\gamma'_j=0$ *if and only if*

$j=t-i+1$.

Proof. Let us consider the next commutative diagram and let $\xi=(f^!\otimes 1)\eta^{n+(j-1)(p-1)}\in\widetilde{K}(X_n^{j-1,j})\otimes Q_p$:

$$\begin{array}{ccc}\widetilde{K}(P_n^{j-1,j})\otimes Q_p & \xrightarrow{f^!\otimes 1} & \widetilde{K}(X_n^{j-1,j})\otimes Q_p \\ \downarrow ch & & \downarrow ch \\ \widetilde{H}^*(P_n^{j-1,j};\ Q) & \xrightarrow{f^*} & \widetilde{H}^*(X_n^{j-1,j};\ Q),\end{array}$$

then by construction of f and by Proposition 2.2

$$\begin{aligned} ch_{n+(j-1)(p-1)}\xi &= f^*ch_{n+(j-1)(p-1)}\eta^{n+(j-1)(p-1)} \\ &= a\cdot S_{j-1} \quad a\in Z,\ a\not\equiv 0 \pmod p, \\ \nu_p(ch_{n+j(p-1)}\xi) &= \nu_p(f^*ch_{n+j(p-1)}\eta^{n+(j-1)(p-1)}) \\ &= \nu_p(ch_{n+j(p-1)}\eta^{n+(j-1)(p-1)}) \\ &= \nu_p((n+(j-1)(p-1))/p) \\ &= \nu_p(n+(j-1)(p-1))-1, \end{aligned}$$

so by Lemma 2.1 $\gamma'_j=0$ if and only if $n+(j-1)(p-1)\equiv 0 \pmod p$, and $n+(j-1)(p-1)=q-(t-j+1)p+t+1-i-j$ therefore $\gamma'_j=0$ if and only if $j=t-i+1$. q. e. d.

(This proposition is also proved easily by using the reduced power operation.)

Proposition 2.4. *Under the assumption of Proposition 2.3 let p^x be the order of $\gamma_t\in {}^p\pi_{2n+2t(p-1)-1}(X_n^{0,t-1})$ then*

$$x=\begin{cases}\max(t-\nu_p(q),\ i-1) & t<p,\ i\leqq t, \\ t & i>t, \\ \max(p+1-\nu_p(q),\ i-1) & t=p,\ i\neq p, \\ \max(p+1-\nu_p(q-p^2),\ p-1) & t=p,\ i=p.\end{cases}$$

Proof. Let $X_\varepsilon^0(\varepsilon=0,1)$ be the complex obtained from $X_n^{0,t-1}$ attaching a $(2n+2t(p-1))$-cell by the map $p^{x-\varepsilon}\cdot\gamma_t$ and we naturally define a map g_ε of X_ε^0 into $X_n^{0,t}$ that is $g_\varepsilon|X_n^{0,t-1}$ is the identity map and $g_\varepsilon|e^{2n+2t(p-1)}$: $e^{2n+2t(p-1)}\to e_t$ is a map of degree $p^{x-\varepsilon}$. Let $X_\varepsilon^j(j=0,1,2,\cdots,t-1)$ be the complex obtained from X_ε^0 by smashing the subcomplex $X_n^{0,j-1}$ to a point and the map g_ε of X_ε^j into

$X_n^{j,t}$ is also defined naturally. Then

(i) for $\varepsilon=1$, $p^{x-1}\cdot\gamma_t\neq 0$, so there exists $j(0\leqq j\leqq t-1)$ such that the attaching map of the $(2n+2t(p-1))$-cell of X_1^0 is reducible to $X_n^{0,j}$ but not reducible to $X_n^{0,j-1}$; that is X_1^j contains a subcomplex $X_1=S^{2n+2j(p-1)}\cup e^{2n-2t(p-1)}$ (for the simplicity we denote $X_1=S_j\cup e_t$) and the attaching map of e_t is not trivial:

(ii) for $\varepsilon=0$, $p^x\cdot\gamma_t=0$, so for any $j(0\leqq j\leqq t-1)$ the complex X_0^j contains a subcomplex $X_0=S_j\cup e_t$ and the attaching map of e_t is trivial.

We shall restate (i) and (ii) using the next commutative diagram and Lemma 2.1, where i_ε is the natural inclusion $X_\varepsilon\subset X_\varepsilon^0$.

$$\begin{array}{ccccccc}
\widetilde{K}(P_n^{j,t})\otimes Q_p & \xrightarrow{f^!\otimes 1} & \widetilde{K}(X_n^{j,t})\otimes Q_p & \xrightarrow{g_\varepsilon^!\otimes 1} & \widetilde{K}(X_\varepsilon^j)\otimes Q_p & \xrightarrow{i_\varepsilon^!\otimes 1} & \widetilde{K}(X_\varepsilon)\otimes Q_p \\
\downarrow ch & & \downarrow ch & & \downarrow ch & & \downarrow ch \\
\widetilde{H}^*(P_n^{j,t};Q) & \xrightarrow{f^*} & \widetilde{H}^*(X_n^{j,t};Q) & \xrightarrow{g_\varepsilon^*} & \widetilde{H}^*(X_\varepsilon^j:Q) & \xrightarrow{i_\varepsilon^*} & \widetilde{H}^*(X_\varepsilon:Q)
\end{array}$$

Let $\xi_\varepsilon=(i_\varepsilon^!\otimes 1)(g_\varepsilon^!\otimes 1)(f^!\otimes 1)\eta^{n+j(p-1)}$ then by definitions of f, g_ε and i_ε

$$\begin{aligned}
ch_{n+j(p-1)}\xi_\varepsilon &= i_\varepsilon^*\cdot g_\varepsilon^*\cdot f^*(ch_{n+j(p-1)}\eta^{n+j(p-1)}) \\
&= a\cdot S_j, \quad a\in Z,\ a\not\equiv 0 \pmod p), \\
\nu_p(ch_{n+t(p-1)}\xi_\varepsilon) &= \nu_p(i_\varepsilon^*\cdot g_\varepsilon^*\cdot f^*(ch_{n+t(p-1)}\eta^{n+j(p-1)})), \\
&= x-\varepsilon+\nu_p(ch_{n+t(p-1)}\eta^{n+j(p-1)}),
\end{aligned}$$

so by Lemma 2.1.

(i)′ $0\leqq{}^\exists j\leqq t-1,\ x-1+\nu_p(ch_{n+t(p-1)}\eta^{n+j(p-1)})<0$,

(ii)′ $0\leqq{}^\forall j\leqq t-1,\ x+\nu_p(ch_{n+t(p-1)}\eta^{n+j(p-1)})\geqq 0$.

Therefore $x=\underset{0\leqq j\leqq t-1}{\mathrm{Max}}(-\nu_p(ch_{n+t(p-1)}\eta^{n+j(p-1)}))$.

By Proposition 2.2

$$\nu_p(ch_{n+t(p-1)}\eta^{n+j(p-1)})=\begin{cases}
\nu_p(q)-p-1 & t=p,\ i\neq p,\ j=0, \\
\nu_p(q-p^2)-p-1 & t=i=p,\ j=0, \\
\nu_p(q)-t+j & 0\leqq j\leqq t-i-1 \text{ and } (t,j)\neq(p,0), \\
\nu_p(q-ip)-i & j=t-i, \\
j-t & t-i+1\leqq j\leqq t-1,
\end{cases}$$

then the proposition is a direct consequence. q. e. d.

§3. Proof of (0.2).

Proposition 3.1. *Consider a CW-complex* $X=S\cup e_1\cup e_2\cup\cdots\cup e_m$ *where S is an N-sphere (N large) and* e_i $(1\leq i\leq m)$ *are* $(N+2i(p-1))$*-cells. Let us assume that the attaching map of* e_i *has an order which is a power of p and is essential to* e_{i-1} *for any* i $(1\leq i\leq m)$, *then*

(i) ${}^{p}\pi_{N+2t(p-)-1}(X)\cong Z_{p^{m+1}}$ $\quad 0\leq m<t<p$,

(ii) ${}^{p}\pi_{N+2p(p-1)-1}(X)\cong Z_{p^{m+2}}$ $\quad 0\leq m\leq p-2$,

(iii) ${}^{p}\pi_{N+2p(p-1)-1}(X)\cong Z_{p^p}$ $\quad m=p-1$,

(iv) *in cases of* (i), (ii) *a map is a generator if and only if it is essential to the* $(N+2m(p-1))$*-cell of X.*

The proof will be given at the end of this section.

Lemma 3.2. *Consider the next exact sequence where G is an abelian group:*

$$0\rightarrow Z_{p^j}\rightarrow G\xrightarrow{\varphi} Z_{p^i}\rightarrow 0.$$

Let γ *be an element of G such that* $\varphi(\gamma)$ *is a generator of* Z_{p^i} *and the order of* γ *is* p^k *then* $G/\{\gamma\}\cong Z_{p^{i+j-k}}$ *where* $\{\gamma\}$ *is the subgroup G generated by* γ.

Proof. If G has only one generator the lemma is trivial. If G has generators α and β then $\gamma=x\alpha+y\beta$. But $\varphi(\gamma)=x\varphi(\alpha)+y\varphi(\beta)$ is a generator of $Zx\bar{\alpha}$ so we can assume $x\not\equiv 0(\bmod p)$, $x\alpha$ and β generate G, and $x\bar{\alpha}=-y\bar{\beta}$($\bar{\lambda}$ denotes the element of $G/\{\gamma\}$ correspodding to $\lambda\in G$). Therefore $\bar{\beta}$ generates $G/\{\gamma\}$ so considering orders of G and $G/\{\gamma\}$, we get the lemma. q. e. d.

Proof of (0.2). Clearly ${}^{p}\pi_{2n+2t(p-1)}(X_n^{0,t})$ is isomorphic with ${}^{p}\pi_{2n+2t(p-1)-1}(X_n^{0,t-1})/\{\gamma_t\}$.

(i) If $i>t$, by Proposition 2, 3, $X_n^{0,t-1}$ satisfies the condition of Proposition 3. 1 and γ_t is a generator of ${}^{p}\pi_{2n+2t(p-1)-1}(X_n^{0,t-1})$. Therefore ${}^{p}\pi_{2n+2t(p-1)-1}(X_n^{0,t})=0$.

(ii) If $i=1$, $X_n^{0,t-1}$ satisfies the condition of Proposition 3. 1 and ${}^p\pi_{2n+2t(p-1)-1}(X_n^{0,t-1})=Z_{p^t}$ hence ${}^p\pi_{2n+2t(p-1)-1}(X_n^{0,t})=Z_{p^{t-x}}$ for $t\leqq p$.

(iii) If $1<i\leqq t$, $X_n^{0,t-i}$ and $X_n^{t-i+1,t-1}$ satisfy the condition of Proposition 3. 1. Consider the stable homotopy exact sequence of the pair $(X_n^{0,t-1}, X_n^{0,t-i})$:

$$
\begin{aligned}
&{}^p\pi_{2n+2t(p-1)}(X_n^{t-i+1,t-1})\rightarrow{}^p\pi_{2n+2t(p-1)-1}(X_n^{0,t-i})\\
&\rightarrow{}^p\pi_{2n+2t(p-1)-1}(X_n^{0,t-1})\xrightarrow{j_*}{}^p\pi_{2n+2t(p-1)-1}(X_n^{t-i+1,t-1}).
\end{aligned}
$$

By (1. 5) the first group of this sequence is 0 and, by Proposition 2. 3 and Proposition 3. 1, (iv), $j_*(\gamma_t)$ is a generator of ${}^p\pi_{2n+2t(p-1)-1}(X_n^{t-i+1,t-1})$. Further ${}^p\pi_{2n+2t(p-1)-1}(X_n^{0,t-i})\cong Z_{p^{t-i+1}}$ for $t<p$, ${}^p\pi_{2n+2p(p-1)-1}(X_n^{0,p-i})\cong Z_{p^{p-i+2}}$, and ${}^p\pi_{2n+2t(p-1)-1}(X_n^{t-i+1,t-1})\cong Z_{p^{i-1}}$ for $t\leqq p$. Therefore by Lemma 3. 2 ${}^p\pi_{2n+2t(p-1)-1}(X_n^{0,t})\cong Z_{p^{t-x}}$ for $t<p$ and ${}^p\pi_{2n+2p(p-1)-1}(X_n^{0,p})\cong Z_{p^{p+1-x}}$.

Let $k<p^2-2$. By Proposition 1. 2,

$$
{}^p\pi_{2n+2k-1}(SU(n))\cong{}^p\pi_{2(n-1)+2t(p-1)-1}(X_{n+l}^{0,t})\cong\begin{cases}Z_{p^{t-x}} & \text{for } t<p,\\ Z_{p^{p+1-x}} & \text{for } t=p,\end{cases}
$$

where $n+k=n+l+t(p-1)$, $0\leqq l\leqq p-2$ and x is given by Proposition 2. 4 for $q-i=(n+l)+t(p-1)=n+k$. By definition of $N(n,k)$, we have $N(n,k)=t-x$ or $=p+1-x$ for $t<p$ or $t=p$ respectively. Thus (0. 2) is proved for $k<p^2-2$.

When $k=p^2-2$ let us consider the complex $(X_n^{0,p}\vee X_{n+p-2}^{0,p-1})\cup_\gamma e^{2n+2p^2-4}$. As remarked in §1 we assume n sufficiently large, so ${}^p\pi_{2n+2p^2-5}(X_n^{0,p}\vee X_{n+p-2}^{0,p-1})\cong{}^p\pi_{2n+2p^2-5}(X_n^{0,p})+{}^p\pi_{2n+2p^2-5}(X_{n+p-2}^{0,p-1})$ therefore we can consider that γ is a sum of the attaching map γ_p of $(2n+2p^2-4)$-cell of $X_{n+p-2}^{0,p}$ and an element of ${}^p\pi_{2n+2p^2-5}(X_n^{0,p})$.

Lemma 3.3.

$$
{}^p\pi_{2n+2p^2-5}(X_n^{0,p})\cong\begin{cases}0 & n\not\equiv 0(\operatorname{mod} p)\\ Z_p & n\equiv 0(\operatorname{mod} p).\end{cases}
$$

Proof. Consider the exact sequence:

$$
{}^p\pi_{2n+2p^2-4}(X_n^{2,p})\rightarrow{}^p\pi_{2n+2p^2-5}(X_n^{0,1})\rightarrow{}^p\pi_{2n+2p^2-5}(X_n^{0,p})\rightarrow{}^p\pi_{2n+2p^2-5}(X_n^{2,p})
$$

where ${}^p\pi_{2n+2p^2-4}(X_n^{2,p})\cong{}^p\pi_{2n+2p^2-5}(X_n^{2,p})=0$ by (1. 5), therefore

$${}^p\pi_{2n+2p^2-5}(X_n^{0,p}) \cong {}^p\pi_{2n+2p^2-5}(X_n^{0,1}) \cong {}^p\pi_{2n+2p^2-5}(S^{2n} \underset{\gamma}{\cup} e^{2n+2(p-1)}),$$

As the proof of Proposition 2.3 shows, $\gamma_1=0$ if and only if $n\equiv 0$ (mod p) and in this case

$${}^p\pi_{2n+2p^2-5}(X^{0,1}) \cong {}^p\pi_{2n+2p^2-5}(S^{2n} \vee S^{2n+2(p-1)}) \cong Z_p.$$

When $n\not\equiv 0 (\mathrm{mod}\, p)$ we can assume $\gamma_1=\alpha_1$ and we denote $X_n^{0,1}=S\underset{\alpha_1}{\cup}e$. Let E be a $(2n+2p-2)$-cell, ∂E its boundary and $x:(E, \partial E)\to(S\underset{\alpha_1}{\cup}e, S)$ the characteristic map of $(2n+2p-2)$-cell of $X_n^{0,1}$. Let us consider the commutative diagramm:

$$\begin{array}{ccccc}
{}^p\pi_{2n+2p^2-4}(S\underset{\alpha_1}{\cup}e, s) & \overset{\partial}{\longrightarrow} & {}^p\pi_{2n+2p^2-5}(S) \longrightarrow {}^p\pi_{2n+2p^2-5}(S\underset{\alpha_1}{\cup}e)\longrightarrow 0 \\
\uparrow x_* & & \uparrow (\chi|\partial E)_* \\
{}^p\pi_{2n+2p^2-4}(E, \partial E) & \longrightarrow & {}^p\pi_{2n+2p^2-5}(\partial E)
\end{array}$$

where the low is exact, $(\chi|\partial E)_*=\alpha_{1*}$ and ${}^p\pi_{2n+2p^2-4}(S\underset{\alpha_1}{\cup}e, S) \cong {}^p\pi_{2n+2p^2-4}(S^{2n+2p-2}) \cong Z_p=\{\beta\}$, therefore $\partial(\beta)=\alpha_1\beta$ and ∂ is an isomorphism. This proves the lemma. q.e.d.

Now let us turn to the proof of (0.2).

(i) If $n\not\equiv 0(\mathrm{mod}\, p)$, by lemma 3.3 ${}^p\pi_{2n+2p^2-5}((X_n^{0,p}\vee X_{n+p-2}^{0,p-1})\underset{\gamma}{\cup} e^{2n+2p^2-4}) \cong {}^p\pi_{2n+2p^2-5}(X_{n+p-2}^{0,p})$ so the problem is reduced to the case $k< p^2-2$.

(ii) If $n\equiv 0(\mathrm{mod}\, p)$, $n+p^2-2\equiv -2(\mathrm{mod}\, p)$, so by Proposition 2.3 $X_{n+p-2}^{0,p-2}$ satisfies the condition of Proposition 3.1, and the next exact sequences show that ${}^p\pi_{2n+2p^2-5}(X_{n+p-1}^{0,p-1}) \cong Z_{p^p}+Z_p$ (we denote $n'=n+p-2$ and e_i the $(2n'+2i(p-1))$-cell of $X_{n'}^{0,p-1}$.).

$$\begin{aligned}
&{}^p\pi_{2n'+2p(p-1)}(S_{p-1})\to{}^p\pi_{2n'+2p(p-1)-1}(X_{n'}^{0,p-2})\to{}^p\pi_{2n'+2p(p-1)-1}(X_{n'}^{0,p-1}) \\
&\qquad\qquad\qquad\overset{j_*}{\longrightarrow}{}^p\pi_{2n'+2p(p-1)-1}(S_{p-1}), \\
&{}^p\pi_{2n+'2p(p-1)}(S_{p-2}\vee S_{p-1})\to{}^p\pi_{2n'+2p(p-1)-1}(X_{n'}^{0,p-3})\to{}^p\pi_{2n'+2p(p-1)-1}(X_{n'}^{0,p-1}), \\
&\qquad\qquad\qquad\to{}^p\pi_{2n'+2p(p-1)-1}(S_{p-2}\vee S_{p-1}).
\end{aligned}$$

In fact left hand sides of these sequences are zero, ${}^p\pi_{2n'+2p(p-1)-1}(X_{n'}^{0,p-2}) \cong Z_{p^p}$, $j_*(\gamma_p)=\gamma'_p\neq 0$ so j_* is surjective, and ${}^p\pi_{2n'+2p(p-1)}(S_{p-2}\vee S_{p-1})\cong Z_p+Z_p$.

Therefore ${}^p\pi_{2n+2p^2-5}(X_n^{0,p}\vee X_{n+p-2}^{0,p-1})=Z_p+Z_{p^p}+Z_p$. Let λ, μ, ν be generators, then $\gamma=\lambda+p^{p-x}\mu+\nu$ or $\gamma=\lambda+p^{p-x}\mu$ and in any case:

$$
\begin{aligned}
{}^{p}\pi_{2n+2p^2-5}((X_n^{0,p}\vee X_{n+p-2}^{0,p-1})\underset{\gamma}{\cup}e^{2n+2p^2-4}) &\cong {}^{p}\pi_{2n+2p^2-5}(X_n^{0,p}\vee X_{n+p-2}^{0,p-1})/\{\gamma\} \\
&\cong Z_{p^{p-x+1}}+Z_p.
\end{aligned}
$$

Thus by Proposition 2.4 and the definition of $N(n,k)$, (0.2) is proved for $k=p^2-2$.

Proof of Proposition 3.1.

Proof of (iv). Let us consider the exact sequence:

$$
\begin{aligned}
{}^{p}\pi_{N+2t(p-1)}(S_m)\to{}^{p}\pi_{N+2t(p-1)-1}(S\cup e_1\cup\cdots\cup e_{m-1})&\xrightarrow{i_*} \\
{}^{p}\pi_{N+2t(p-1)-1}(S\cup e_1\cup\cdots\cup e_m)&\xrightarrow{j_*}{}^{p}\pi_{N+2t(p-1)-1}(S_m)
\end{aligned}
$$

where ${}^{p}\pi_{N+2t(p-1)}(S_m)=0$, so if (i) and (ii) hold and if γ and γ' are generators of ${}^{p}\pi_{N+2t(p-1)-1}(S\cup e_1\cup\cdots\cup e_{m-1})$ and ${}^{p}\pi_{N+2t(p-1)-1}(S\cup e_1\cup\cdots\cup e_m)$ respectively, then $i_*(\gamma)=ap\cdot\gamma'$, $a\not\equiv 0 \pmod p$, and $j_*(\gamma')\neq 0$ that is γ' is essential to the $(N+2m(p-1))$-cell of $S\cup e_1\cup\cdots\cup e_m$. And the converse is trivial.

Proof of (i) (*special case*). Let N be a suitably large integer such that $N+t(p-1)=q-p$, $t<p$, $q\equiv 0 \pmod p$, and consider the complex $X_N^{0,t}$. By Proposition 2.3, $X_N^{0,i}(0\leqq i\leqq t)$ satisfy the condition of Proposition 3.1 and an easy exact sequence argument shows inductively that the order of ${}^{p}\pi_{2N+2t(p-1)-1}(X_N^{0,i})$ equals to p^{i+1} for $0\leqq i<t$. But by Proposition 2.4 the order of γ_t equals to p^t, hence ${}^{p}\pi_{2n+2t(p-1)-1}(X_N^{0,t-1})\cong Z_{p^t}$.

When $m<t-1$ let us consider the following exact sequence:

$$
\begin{aligned}
&{}^{p}\pi_{2N+2t(p-1)}(X_N^{m+1,t-1})\to{}^{p}\pi_{2N+2t(p-1)-1}(X_N^{0,m}) \\
&\qquad\to{}^{p}\pi_{2N+2t(p-1)-1}(X_N^{0,t-1})\xrightarrow{j_*}{}^{p}\pi_{2N+2t(p-1)-1}(X_N^{m+1,t-1})
\end{aligned}
$$

where ${}^{p}\pi_{2n+2t(p-1)}(X_N^{0,t-1})=0$, ${}^{p}\pi_{2N+2t(p-1)-1}(X_N^{0,t-1})\cong Z_{p^t}$, ${}^{p}\pi_{2N+2t(p-1)-1}(X_N^{m+1,t-1})\cong Z_{p^{t-m-1}}$, and by (iv) $j_*(\gamma_t)$ is a generator, so j_* is surjective. Therefore ${}^{p}\pi_{2M+2t(p-1)-1}(X_N^{0,m})\cong Z_{p^{m+1}}$.

Proof of (ii) (*special case.*) Let N be a suitably large integer such that $N+p(p-1)=q-1$, $\nu_p(q)=1$ and consider the comlpex $X_N^{0,p}$. By Proposition 2.3, $\gamma'=0$ so we can consider γ_p as an element of ${}^p\pi_{2N+2p(p-1)-1}(X_N^{0,p-2})$. By the same argument as (i) the order of ${}^p\pi_{2N+2p(p-1)-1}(X_N^{0,p-2})$ is at most p^p. But by Proposition 2.4, the order of γ_p is p^p, so ${}^p\pi_{2n+2p(p-1)-1}(X_N^{0,p-2})\cong Z_{p^p}$. When $m<p-2$, (ii) holds by the same argument as the case (i).

Proof of (i), (ii) (*general case*). Let $X=S\cup e_1\cdots\cup e_m$ be a complex satisfying the condition of the proposition. By iterating suspensions we can assume the dimension of S is equal to $2N$ where N is an integer considered above (note that we can take such an integer N arbitrarily large). The attaching map of e_m generates ${}^p\pi_{2N+2m(p-1)-1}(X^{m-1})$, $X^{m-1}=S\cup e_1\cup\cdots\cup e_{m-1}$, provided ${}^p\pi_i(X^{m-1})\cong{}^p\pi_i(X_N^{0,m-1})$. Then it is easy to construct inductively a map f of $X_N^{0,m}$ into X which induces a C_p-isomorphism $f_*:\pi_i(X_N^{0,m})\to\pi_i(X)$, $i\geqq 0$. This shows that the general case is reduced to the special one.

Proof of (iii). Let $X=S\cup e_1\cdots\cup e_{p-1}$ be a complex satisfying the condition of the proposition. To prove (iii) it is sufficient to prove that j_* is trivial in the exact sequence:

$$0\to{}^p\pi_{N+2p(p-1)-1}(S\cup e_1\cup\cdots\cup e_{p-2})\to{}^p\pi_{N+2p(p-1)-1}(X)\xrightarrow{j_*}{}^p\pi_{N+2p(p-1)-1}(S_{p-1}).$$

If there exists an element $\gamma\in{}^p\pi_{N+2p(p-1)}(X)$ such that $j_*(\gamma)\neq 0$. Then, in the cohomology group $H^*(X\cup_\gamma e^{N+2p(p-1)}; Z_p)$, $\mathcal{P}^1(e_{p-1})\neq 0$ but by the condition of Proposition 3.1 $\mathcal{P}^{p-1}(S)\neq 0$, therefore $\mathcal{P}^1\mathcal{P}^{p-1}(S)\neq 0$ which contradicts to Adem's relation.

§4. Proof of (0.3).

Lemma 4.1. *For* $k<(n+1)(p-1)-1$

$${}^p\pi_{2n+2k+1}(SU(n+k+1)/SU(n))\cong\begin{cases} Z+Z_p & p(p-1)\leqq k<p^2-2 \text{ and } n+k\not\equiv -2 \pmod p,\\ Z & k\leqq p^2-2 \text{ and except above cases.}\end{cases}$$

The proof will be given later in this section.

Let us consider the following commutative diagram:

$$
\begin{array}{ccc}
\pi_{2n+2k+1}(SU(n+k+1)) & \xrightarrow{p_{1*}} & \pi_{2n+2k+1}(SU(n+k+1)/SU(n)) \\
\| & & \downarrow p_* \\
\pi_{2n+2k+1}(SU(n+k+1)) & \xrightarrow{p_{2*}} & \pi_{2n+2k+1}(S^{2n+2k+1})
\end{array}
$$

$$
\begin{array}{ccc}
\to \pi_{2n+2k}(SU(n)) & \to & \pi_{2n+2k}(SU(n+k+1)) \\
\downarrow & & \| \\
\to \pi_{2n+2k}(SU(n+k)) & \to & \pi_{2n+2k}(SU(n+k+1))
\end{array}
$$

where lows are exact and by Bott periodicity theorem $\pi_{2n+2k+1}(SU(n+k+1))\cong Z$, $\pi_{2n+2k}(SU(n+k+1))=0$, so, when ${}^p\pi_{2n+2k+1}(SU(n+k+1)/SU(n))\cong Z$, to prove (0. 3) it is sufficient to know the degree of p_{1*}. But by the theorem of Borel-Hirzebruch $\pi_{2n+2k}(SU(n+k))\cong Z_{(n+k)!}$ (e. g. [8]), the degree of p_{2*} is $(n+k)!$ so it is sufficient to know the degree of p_*. Now consider the following commutative diagram:

$$
\begin{array}{ccccc}
\pi_{2n+2k+1}(SU(n+k+1)/SU(n)) & \xrightarrow{p_*} & \pi_{2n+2k+1}(S^{2n+2k+1}) & \to & \pi_{2n+2k}(SU(n+k)/SU(n)) \\
\uparrow & & \uparrow & & \uparrow \\
\pi_{2n+2k}(CP(n+k)/CP(n-1)) & \to & \pi_{2n+2k}(S^{2n+2k}) & \to & \pi_{2n+2k-1}(CP(n+k-1)/CP(n-1)) \\
 & & \uparrow f_* & & \uparrow f_* \\
 & & \pi_{2n+2k}(S^{2n+2k}) & \xrightarrow{\partial} & \pi_{2n+2k-1}(X^{0,t-1}_{n+})
\end{array}
$$

where $k<p^2-2$, lows are exact and vertical arrows are C_p-isomorphisms. Clearly $\partial(\iota)=\gamma_t$ so the degree of p_* is equal to a product of the order of γ_t and an integerj prime to p. (The case $k=p^2-2$ follows similarly.) Therefore

$$
{}^p\pi_{2n+2k}(SU(n))\cong Z_{p^{N'}} \quad N'=\nu_p((n+k)!)-x.
$$

When ${}^p\pi_{2n+2k-1}(SU(n+k+1)/SU(n))\cong Z+Z_p$, i. e. when $p(p-1)\leqq k<p^2-2$ and $n+k\not\equiv -2(\operatorname{mod} p)$, ${}^p\pi_{2n+2k}(SU(n))\cong Z_{p^{N'}}+Z_p$ or $Z_{p^{N'+1}}$ corresponding to whether the image of p_{1*} is contained or not contained in a complement of Z_p in ${}^p\pi_{2n+2k+1}(SU(n+k+1)/SU(n))$. But as will be shown in the last of the next section, ${}^p\pi_{2n+2k}(SU(n))$ has a direct summand Z_p or $Z_{p^{N'}}$ hence ${}^p\pi_{2n+2k}(SU(n))\cong Z_{p^{N'}}+Z_p$. By Proposition 2. 4 and the definition of $N'(n,k)$ this proves the formula (0. 3).

Proof of Lemma 4.1. As in §1, by the theorem of James ([3]), if $k<(n+1)(p-1)-1$ we can assume n sufficiently large and by the cellular decomposition of special unitary groups, ${}^p\pi_{2n+2k+1}(SU(n+k+1)/SU(n))\cong{}^p\pi_{2n+2k}(CP(n+k)/CP(n-1))$ so the case $k<p(p-1)-1$ the lemma follows easily from (1.4). When $p(p-1)-1\leqq k<p^2-2$ we get, by use of the reduction in §1,

$$
\begin{aligned}
{}^p\pi_{2n+2k}(CP(n+k)/CP(n-1)) &\cong {}^p\pi_{2n+2k}(S^{2n}\cup e^{2n+2}\cup\cdots\cup e^{2n+2k})\\
&\cong Z+{}^p\pi_{2n'+2p(p-1)-2}(X^{0,p-1}_{n'}),
\end{aligned}
$$

where $n+k=n'+p(p-1)-1$. To compute ${}^p\pi_{2n'+2p(p-1)-2}(X^{0,p-1}_{n'})$ we consider the next exact sequence:

$$
\begin{aligned}
{}^p\pi_{2n'+2p(p-1)-1}(X^{0,p-1}_{n'}) &\xrightarrow{j_*} {}^p\pi_{2n'+2p(p-1)-1}(X^{1,p-1}_{n'})\to{}^p\pi_{2n'+2p(p-1)-2}(S^{2n'})\\
&\to{}^p\pi_{2n'+2p(p-1)-2}(X^{0,p-1}_{n'})\to{}^p\pi_{2n'+2p(p-1)-2}(X^{1,p-1}_{n'}),
\end{aligned}
$$

where by (1.4) and (1.5) ${}^p\pi_{2n'+2p(p-1)-2}(X^{1,p-1}_{n'})=0$ and ${}^p\pi_{2n'+2p(p-1)-2}(S^{2n'})\cong Z_p$ so ${}^p\pi_{2n'+2p(p-1)-2}(X^{0,p-1}_{n'})\cong Z_p$ or 0, if j_* is surjective or not surjective respevtively. If $\gamma_1=0$ then by Proposition 3.1 and 2.3, for $\gamma_p\in{}^p\pi_{2n'+2p(p-1)-1}(X^{0,p-1}_{n})$, $j_*(\gamma_p)$ is a generator of ${}^p\pi_{2n'+2p(p-1)-1}(X^{1,p-1}_{n'})$ hence j_* is surjective. If $\gamma'_p=0$ then the proof of Proposition 3.1 (iii) shows that a generator of ${}^p\pi_{2n'+2p(p-1)-1}(X^{0,p-1}_{n'})$ is a map which is essential to the $(2n+2(p-2)(p-1))$-cell of $X^{0,p-1}_{n'}$ therefore j_* is not surjective. When $\gamma'_j=0, 1<j<p$, let us consider the following commutative diagram:

$$
\begin{array}{ccccc}
{}^p\pi_{2n'+2p(p-1)-1}(X^{0,j-1}_{n'}) & \to & {}^p\pi_{2n+2p(p-1)-1}(X^{0,p-1}_{n'}) & \xrightarrow{j_{2*}} & {}^p\pi_{2n'+2p(p-1)-1}(X^{j,p-1}_{n'})\\
\downarrow j_{1*} & & \downarrow j_* & & \|\\
{}^p\pi_{2n'+2p(2-1)-1}(X^{1,j-1}_{n'}) & \to & {}^p\pi_{2n'+2p(p-1)-1}(X^{1,p-1}_{n'}) & \to & {}^p\pi_{2n'+2p(p-1)-1}(X^{j,p-1}_{n'})
\end{array}
$$

where lows are exact. By Proposition 2.3 and 3.1, (iv) j_{1*} is surjective and j_{2*} is also surjective for $j_{2*}(\gamma_p)$ is a generator of ${}^p\pi_{2n'+2p(p-1)-2}(X^{j,p-1}_{n'})$, therefore j_* is surjective. In the sequel ${}^p\pi_{2n'+2p(p-1)-2}(X^{0,p-1}_{n'})\cong 0$ or Z_p corresponding to whether γ'_p is zero or not zero. But by Proposition 2.3 $\gamma'_p=0$ if and only if $n'+p(p-1)\equiv-1 \pmod p$ that is if and only if $n+k\equiv-2 \pmod p$. This proves the lemma for $k<p^2-2$.

In case of $k=p^2-2$, it is easily seen that ${}^p\pi_{2n+2p^2-4}(CP(n+p^2-2)/CP(n-1))\cong Z+{}^p\pi_{2n+2p^2-4}(X_n^{0,p})$ but the second term vanishes. In fact the proof of Lemma 3.3 shows that in the exact sequence:

$$0\to{}^p\pi_{2n+2p^2-4}(X_n^{0,p})\to{}^p\pi_{2n+2p^2-4}(X_n^{1,p})\xrightarrow{\partial}{}^p\pi_{2n+2p^2-5}(S^{2n})$$

∂ is an isomorphism if $n\not\equiv 0 (\bmod p)$ and in this case ${}^p\pi_{2n+2p^2-4}(X_n^{0,p})=0$, if $n\equiv 0 \ (\bmod p)$ then $n+p^2-2\equiv -2(\bmod p)$ so ${}^p\pi_{2n+2p^2-4}(X_n^{1,p})=0$ and ${}^p\pi_{2n+2p^2-4}(X_n^{0,p})=0$. q. e. d.

§5. Proof of Theorem 2.

As in §1 we shall reduce the problem to the computation of homotopy groups of Stiefel manifolds. By the theorem of James ([3] Th. 3. 2):

$${}^p\pi_i(Sp(n+k)/Sp(n))\cong{}^p\pi_{i+4Nc_k}(Sp(n+k+Nc_k)/Sp(n+Nc_k)) \quad \text{for } i<4p(n+1)-3,$$

we always assume, under the assumption of Theorem 2, n sufficiently large.

By the cellular decomposition of symplectic groups ([10]), the (8n+9)-skeleton of $Sp(n+k+1)/Sp(n)$ has the cell structure $S^{4n+3}\cup e^{4n+7}\cup\cdots\cup e^{4(n+k)+3}$ and the map i^*: $H^{4n+4i+3}(SU(2n+2k+2)/SU(2n+1))\to H^{4n+4i+3}(Sp(n+k+1)/Sp(n))$ induced from the natural injection i: $Sp(n+k+1)/Sp(n)\to SU(2n+2k+2)/SU(2n+1)$ has the degree ± 1. Let $X=S^{4n+2}\cup e^{4n+6}\cup\cdots\cup e^{4(n+k)+2}$ be a complex such that $SX=(Sp(n+k+1)/Sp(n))^{8n+9}$, where S denotes the suspension and $(K)^i$ denotes the i-skeleton of K, and (X', f') be a complex and a map of X' into X constructed by Lemma 1.1. Then if $2k<(p+1)(p-1)$, X' has the following cell structure.

$$X'=\left[\bigvee_{i=0}^{t} X'^{0,t}_{2n+2i+1}\right]\vee\left[\bigvee_{j=t+1}^{[\frac{p-2}{2}]} X'^{0,t-1}_{2n+2j+1}\right],$$
$$X'^{0,t}_{2n+2i+1}=S^{4n+4i+2}\cup e^{4n+4i+2+2(p-1)}\cup\cdots\cup e^{4n+4i+2+2t(p-1)}.$$

Let i': $X\to CP(2n+2k+1)/CP(2n)$ be a map such that $Si'=i\,|\,SX$ then clearly $\nu_p(x)=\nu_p(f'^*\cdot i'^*(x))$ for $x\in H^{4n+4i+2}(CP(2n+2k+$

$1)/CP(2n):Q)$ $0\leqq i\leqq k$, thus if we replace, in the arguments of §2, $CP(n+k)/CP(n-1)$, $X_n^{0,t}$ and η by X, $X'^{0,t}_{2n+1}$ and $i'(\eta)$ respectively then we can easily see that the attaching maps of $X'^{0,t}_{2n+1}$ are quite similar to that of $X^{0,t}_{2n+1}$. So Theorem 2 is virtually a corollary of Theorem 1.

Consider the following exact sequence:

$$ {}^{p}\pi_{4n+i}(Sp(\infty))\rightarrow{}^{p}\pi_{4n+i}(Sp(\infty)/Sp(n))\rightarrow{}^{p}\pi_{4n+i-1}(Sp(n)) \xrightarrow{i_*}{}^{p}\pi_{4n+i-1}(Sp(\infty)),$$

where by Bott periodicity theorem ${}^{p}\pi_{4n+i}(Sp(\infty))=0$ $i\neq 4k+3$ and i_* is trivial for $i\geqq 1$. Therefore:

$$ {}^{p}\pi_{4n+i-1}(Sp(n))\cong{}^{p}\pi_{4n+i}(Sp(\infty)/Sp(n)) \quad i\neq 4k+3,\ i\geqq 1.$$

Case 1. $i=4k$. By (1. 5)

$$ {}^{p}\pi_{4n+4k}(Sp(\infty)/Sp(n))\cong{}^{p}\pi_{4n+4k}(S^{4n+3}\cup e^{4n+7}\cup\cdots\cup e^{4(n+k)+3})=0.$$

Case 2. $i=4k+2$.

$$\begin{aligned} {}^{p}\pi_{4n+4k+2}(Sp(\infty)/Sp(n)) &\cong{}^{p}\pi_{4n+4k+2}(S^{4n+3}\cup e^{4n+7}\cup\cdots\cup e^{4(n+k)+3}) \\ &\cong{}^{p}\pi_{2(2n+1+2l)}(X'^{0,t}_{2n+1+2l}) \quad 2k=2l+t(p-1),\ t=\left[\frac{2k}{p-1}\right] \\ &\cong Z_{p^{N(2n+1,2k)}}. \end{aligned}$$

Case 3. $i=4k+1$

$$\begin{aligned} {}^{p}\pi_{4n+4k+1}(Sp(\infty)/Sp(n)) &\cong{}^{p}\pi_{4n+4k+1}(S^{4n+3}\cup e^{4n+7}\cup\cdots\cup e^{4(n+k)+3}) \\ &\cong\begin{cases} 0 & 2k<p(p-1), \\ {}^{p}\pi_{2(2n+1+2l)+2p(p-1)-2}(X'^{0,p}_{2n+2l+1}) & 2k=2l+p(p-1), \end{cases} \end{aligned}$$

where ${}^{p}\pi_{4n+4k}(X'^{0,p}_{2n+2l+1})=0$ if $n+k\equiv -1 (\mathrm{mod}\ p)$ and ${}^{p}\pi_{4n+4k}(X'^{0,p}_{2n+2l+1})\cong Z_p$ if $n+k\not\equiv -1(\mathrm{mod}\ p)$ by the same argument as the proof of Lemma 4. 1.

Case 4. $i=4k+3$ Consider the commutative diagram:

$$\begin{array}{ccccccc} \pi_{4n+4k+3}(Sp(n+k+1)) & \xrightarrow{p_{1*}} & \pi_{4n+4k+3}(Sp(n+k+1)/Sp(n)) & \rightarrow & \pi_{4n+4k+2}(Sp(n)) & \rightarrow & 0 \\ \| & & \downarrow p_* & & \downarrow & & \\ \pi_{4n+4k+3}(Sp(n+k+1)) & \xrightarrow{p_{2*}} & \pi_{4n+4k+3}(S^{4n+4k+3}) & \longrightarrow & \pi_{4n+4k+2}(Sp(n+k)) & \rightarrow & 0 \end{array}$$

where $\pi_{4n+4k+3}(Sp(n+k+1))\cong Z$ and $\pi_{4n+4k+2}(Sp(n+k))\cong Z_{(2n+2k+1)!}$, $n+$

k: even, $\pi_{4n+4k+2}(Sp(n+k))\cong Z_{2(2n+2k+1)!}$, $n+k$: odd (e.g. [5]), so the degree of p_{2*} is $(2n+2k+1)!$ or $2(2n+2k+1)!$. Moreover ${}^{p}\pi_{4n+4k+3}(Sp(n+k+1)/Sp(n))\cong Z$ by (1.5) and the proof of (0.3) shows that the degree of p_* is the product of $p^{x(2n+1,2k)}$ and an integer prime to p. Therefore

$$ {}^{p}\pi_{4n+4k+2}(Sp(n))\cong Z_{p^{N'}} \quad N'=\nu_p((2n+2k+1)!)-x(2n+1,2k). $$

By Proposition 2.4 and the definition of $N'(n,k)$ this proves Theorem 2.

Now let us show that ${}^{p}\pi_{2n+2k}(SU(n))$, $p(p-1)\leqq k<p^2-2$ $n+k\not\equiv -2 \pmod p$, has the direct summand Z_p or $Z_{p^{N'}}$. By the theorem of Harris (0.1), ${}^{p}\pi_i(Sp(n))$ is a direct summand of ${}^{p}\pi_i(SU(2n))$ and ${}^{p}\pi_i(SU(2n+1))$ so by Case 3, ${}^{p}\pi_{4n+4k}(SU(2n))$ and ${}^{p}\pi_{2(2n+1)+2(2k-1)}(SU(2n+1))$ have the direct summand Z_p in case of the question. By Case 4, ${}^{p}\pi_{2(2n+1)+4k}(SU(2n+1))$ has the direct summand $Z_{p^{N'(2n+1,2k)}}$ $+1,2k)$ and the exact sequence

$$ {}^{p}\pi_{4n+2(2k+1)}(SU(2n))\to{}^{p}\pi_{2(2n+1)+4k}(SU(2n+1))\to{}^{p}\pi_{4n+4k+2}(S^{4n+1}) $$

shows that if ${}^{p}\pi_{2(2n+1)+4k}(SU(2n+1))$ splits then ${}^{p}\pi_{4n+2(2k+1)}(SU(2n))$ splits in fact ${}^{p}\pi_{4n+4k+2}(S^{4n+1})=0$ for $2k+1<p^2-2$ and the direct summand comes from ${}^{p}\pi_{2(2n+1)+4k}(Sp(n))$. This completes the proof of (0.3).

§6. Proof of Proposition 2.2.

To prove Proposition 2.2 it is sufficient to show the following assertions.

Assertion 1. *Let a_i $(i=1,2,\cdots)$ be rational numbers such that*

$$ \sum_{i=1}^{\infty} a_i(e^x-1)^i=\sum_{k=0}^{\infty}\left(-\frac{1}{p}\right)^k\frac{(kp)!}{k!(kp-k+1)!}x^{kp-k+1} $$

then the denominators of a_i are prime to p for $i<p^2$.

(Remark. $a_{p^2}\equiv 1/p^2 \pmod{Q_p}$, $a_i\in Q_p$, $p^2<i<p^2+p$)

Assertion 2.

$$\left(\sum_{i=0}^{\infty}\left(-\frac{1}{p}\right)^i \frac{(ip)!}{i!(ip-i+1)!}x^{ip-i+1}\right)^n$$
$$=\sum_{k=0}^{\infty}\left(-\frac{1}{p}\right)^k \frac{n(n+kp-1)!}{k!(n+kp-k)!}x^{k(p-1)+n}.$$

Lemma A. *Let $y=e^x-1$ and $f(y)$ be a formal power series with coefficient Q_p:*

$$f(y)=\sum_{\substack{n\not\equiv 0(p)\\ n>0}} \frac{(-1)^{n+1}}{n} y^n.$$

Then there exists a formal power series $a(y)=b_{p+1}y^{p+1}+b_{p+2}y^{p+2}+\cdots$ such that $b_{p+1}\in Q_p$ for *$i<p^2-p$, more precisely $a(y)\equiv\sum_{n=1}^{\infty}\frac{(1)^{n+1}}{np^2}y^{np^2}$* mod *$Q_p(v)$, and satisfies the relation:*

$$x=f+\frac{1}{p}(f^p+pa).$$

Proof.

$$f^p=\left(y-\frac{y^2}{2}+\cdots-\frac{y^{p-1}}{p-1}-\frac{y^{p+1}}{p+1}+\cdots\right)^p$$
$$=\sum_{\substack{n\not\equiv 0(p)\\ n>0}}(-1)^{n-1}\frac{y^{np}}{n^p}+ph(y) \quad \text{(multinomial expansion)}$$

where $h(y)=*y^{p+1}+*y^{p+2}+\cdots$ is a power series of y, $\frac{y^2}{2}, \cdots, \frac{y^k}{k}, \cdots (k\not\equiv 0(\text{mod } p))$ with integer coefficients. Let $g(y)=\sum_{n=1}^{\infty}\frac{(-1)^{np+1}}{np}y^{np}$ that is $x=\log(1+y)=f(y)+g(y)$, then for $n\not\equiv 0 \pmod p$ there exists integers c_n such that $n^{p-1}=pc_n+1$ and:

$$\frac{1}{p}\frac{y^{np}}{n^p}-\frac{y^{np}}{np}=\frac{y^{np}}{n^p p}(1-n^{p-1})=-\frac{c_n}{n^p}y^{np},$$

therefore if we set $a(y)=g-\frac{f^p}{p}=\sum_{\substack{n\not\equiv 0(p)\\ n<1}}(-1)^{n+1}\frac{c_n}{n^p}y^{np}+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{np^2}y^{np^2}-h(y)$ the lemma is proved.

Lemma B. *By the substitution $x=f+\frac{1}{p}(f^p+pa)$ we get*

$$\sum_{k=0}^{\infty}\left(-\frac{1}{p}\right)^k\frac{(kp)!}{k!(kp-k+1)!}x^{kp-k+1}=f+a+\sum_{\substack{i>0\\ ip\geq j\geq i}}\alpha_{ij}f^{ip-j}a^{j-i+1}$$

where $\alpha_{ij}\in Q_p$ *for* $1\leqq j-i+1<p,\ i<p+1$.

Proof.

$$x^{kp+k+1}=\sum_{\alpha=0}^{kp-k+1}\frac{1}{p^{\alpha}}\binom{kp-k+1}{\alpha}f^{kp-k+1-\alpha}\left(\sum_{\beta=0}^{\alpha}p^{\beta}\binom{\alpha}{\beta}f^{p(\alpha-\beta)}a^{\beta}\right)$$

$$=\sum_{\alpha=0}^{kp-k+1}\sum_{\beta=0}^{\alpha}\frac{1}{p^{\alpha-\beta}}\binom{kp-k+1}{\alpha}\binom{\alpha}{\beta}f^{(k+\alpha-\beta)p+1-k-\alpha}a^{\beta}.$$

We put $i=k+\alpha-\beta$, $j=k+\alpha-1$ then $\alpha=j-k+1$, $\beta=j-i+1$ and

$$x^{kp-k+1}=\sum_{j=k-1}^{kp}\sum_{i=k}^{j+1}\frac{1}{p^{i-k}}\binom{kp-k+1}{j-k+1}\binom{j-k+1}{j-i+1}f^{ip-j}\,a^{j-i+1}.$$

Now $x=f+\dfrac{f^p}{p}+a$,

$$-\frac{1}{p}x^p=-\frac{f^p}{p}-\frac{f^{2p-1}}{p}-f^{p-1}a-\cdots-\frac{a^p}{p},$$

$\cdots\cdots\cdots\cdots$,

hence $\displaystyle\sum_{k=0}^{\infty}\left(-\frac{1}{p}\right)\frac{(kp)!}{k!(kp-k+1)!}x^{kp-k+1}=\sum_{\substack{i\geq 0\\ ip\geq j\geq i-1}}\alpha_{ij}f^{ip-j}\,a^{j-i+1}$, where $\alpha_{0,-1}=\alpha_{0,0}=1$ and for $i\geqq 1$,

$$\alpha_{ij}=\sum_{k=1}^{i}\frac{(-1)^k(kp)}{p^i k(kp-k+1)}\binom{kp-k+1}{j-k+1}\binom{j-k+1}{j-i+1}$$

$$=\frac{1}{p^{i-1}}\sum_{k=1}^{i}\frac{(-1)^k(kp-1)!(kp-k+1)!(j-k+1)!}{(k-1)!(kp-k+1)!(j-k+1)!(kp-j)!(j-i+1)!(i-k)!}$$

$$=\frac{1}{p^{i-1}(j-i+1)!}\sum_{k=1}^{i}\frac{(-1)^k(kp-1)!}{(kp-j)!(k-1)!(i-k)!}$$

$$=\frac{1}{p^{i-1}(j-i+1)!(i-1)!}\sum_{k=1}^{i}(-1)^k(kp-1)(kp-2)\cdots(kp-j+1)\binom{i-1}{k-1}.$$

Now let α_r be the coefficient of x^r in $(x-1)(x-2)\cdots(x-j+1)$ then:

$$\sum_{k=1}^{i}(-1)^k(kp-1)(kp-2)\cdots(kp-j+1)\binom{i-1}{k-1}$$

$$=\sum_{k=1}^{i}(-1)^k\sum_{r=0}^{j-1}\alpha_r k^r p^r\binom{i-1}{k-1}=\sum_{r=0}^{j-1}\alpha_r p^r\sum_{k=1}^{i}(-1)^k k^r\binom{i-1}{k-1}.$$

By considering the r-th derivative of $(1-e^t)^n=\Sigma(-1)^k\binom{n}{k}e^{kt}$ at $t=0$, it is easily shown that $\sum_{k=0}^{n}(-1)^k k^r\binom{n}{k}=0$ for $r<n$. Therefore

$\sum_{k=1}^{i}(-1)^k k^r\binom{i-1}{k-1}$ vanishes for $r<i-1$ and

$$\alpha_{ij}=\frac{1}{(j-i+1)!(i-1)!}\sum_{r=j-1}^{j-1}\alpha_r p^{r-i+1}\sum_{k=1}^{i}(-1)^k k^r\binom{i-1}{k-1}$$

hence $\alpha_{ij}=0$ for $j-i<0$ and the denominators of α_{ij} are prime to p for $j-i+1<p, i-1<p$.

Proof of Assertion 1. By Lemma A the coefficient of y^r in $f^{ip-j}a^{j-i+1}$, $ip-j>0$, is zero for $r<ip-j+(j-i+1)(p+1)=(j-i+1)p+(i-1)(p-1)+p$, so by Lemma B the denominators of the coefficients of y^r in $\alpha_{ij}f^{ip-j}a^{j-i+1}$ is prime to p for $r<p^2+p$ and one in a is same for $r<p^2$. Thus Assertion 1 is proved.

Proof of Assertion 2. Let us define $b_{n,k}(n, k\geqq 0)$ as folldws

$$b_{0,0}=1$$

$$b_{n,k}=\frac{n(n+kp-1)!}{k!(n+kp-k)!}\qquad (n,k)\neq(0,0)$$

then $b_{n+1,k}-b_{n,k}=b_{n+p,k-1}(k\geqq 1)$, since

$$\begin{aligned}b_{n+1,k}-b_{n,k}&=\frac{(n+1)(n+kp)!}{k!(n+kp-k+1)!}-\frac{n(n+kp-1)!}{k!(n+kp-k)!}\\&=\frac{\{(n+1)(n+kp)-n(n+kp-k+1)\}(n+kp-1)!}{k!(n+kp-k+1)!}\\&=\frac{k(n+p)(n+kp-1)!}{k!(n+kp-k+1)!}=b_{n+p,k-1}.\end{aligned}$$

Next we will show that $b_{n+1,k}=\sum_{i=0}^{k}b_{n,i}b_{1,k-i}$.

We prove the formula by induction on n and k. The cases $k=0$ or $n=0$ are trivial and

$$\begin{aligned}b_{n+1,k}&=b_{n,k}+b_{n+p,k-1}\\&=\sum_{i=0}^{k}b_{n-1,i}b_{1,k-i}+\sum_{j=0}^{k-1}b_{n+p-1,j}b_{1,k-j-1}\\&=b_{n-1,0}b_{1,k}+\sum_{i=1}^{k}b_{n-1,i}b_{1,k-1}+\sum_{j=0}^{k-1}(b_{n,j+1}-b_{n-1,j+1})b_{1,k-(j+1)}\\&=b_{n,0}b_{1,k}+\sum_{j=0}^{k-1}b_{n,j+1}b_{1,k-(j+1)}\\&=\sum_{i=0}^{k}b_{n,i}\,b_{1,k-1}.\end{aligned}$$

Therefore $\sum_{k\geqq 0} b_{n+1,k}x^k=(\sum_{i\geqq 0} b_{n,i}\, x^i)\,(\sum_{j\geqq 0} b_{1,j}\, x^j)$ which implies the formula $(\sum_{i\geqq 0} b_{1,i}x^i)^n=\sum_{k\geqq 0} b_{n,k}x^k$ inductively, and Assertion 2 follows immediately.

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