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A geometric meaning of a concept of isotropic Finsler spaces

By

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A concept of an isotropic Finsler space was introduced by H. Akbar-Zadeh [1]. In order to do so, he gave tensor equations, similar to the equation satisfied by the curvature tensor of a Riemannian space of constant curvature. The purpose of the present paper is to show a geometric meaning of the concept.

In a previous paper [4], we defined a lift $\overline{G}$ of a Finsler metric G on a differentiable n-manifold M to the tangent bundle $T(M)$ over M with reference to a non-linear connection N in $T(M)$ (on M), which is a direct generalization of the idea of S. Sasaki [7] in the case of a Riemannian metric. It seems, however, to me that the Riemannian metric $\overline{G}$ on $T(M)$ is not useful to consider a geometric meaning of the concept of isotropy. We shall introduce, in the following, another lift G^* of G to $T(M)$, and then it may be said that the isotropy is analogous, in a sense, to a concept of a space of constant curvature with respect to G^*.

1. Let $T(M)$ be the bundle of non-zero tangent vectors to a differentiable n-manifold M, and $\tau : T(M)\rightarrow M$ be the projection. A concept of a vertical lift is now well-known [5], and we denote by $l_y^v X$ the vertical lift of a tangent vector $X\in M_x$ (tangent space to M at a point x) to a point $y\in\tau^{-1}(x)$. If a vertical vector $\overline{X}\in T(M)_y^v$ (vertical subspace of the tangent space $T(M)_y$ to $T(M)$

at a point y) be given, there exists a unique tangent vector $X \in M_x$, such that $l_y^v X = \overline{X}$. Following to P. Dombrowski [2], we shall write $X = K\overline{X}$.

Let $F(M)$ be the Finsler bundle of M [6], that is, the bundle $\tau^{-1}L(M)$ induced from the bundle $L(M)$ of linear n-frames of M by the projection τ of $T(M)$. We consider the Finsler metric tensor field G, derived from a Finsler fundamental function in the usual way. G is thought of as a mapping $F(M) \to V^* \otimes V^*$ [5], where V^* is the dual space of a real vector n-space V, and hence $G(u)(v_1, v_2)$ ($v_1, v_2 \in V$, $u \in F(M)$) is a real number. It is remarked that the point u is a pair (y, z) of points $y \in T(M)$ and $z \in L(M)$, such that $\tau(y) = \pi(z)$, where $\pi : L(M) \to M$ is the projection. Therefore, if we consider two tangent vectors $X_1, X_2 \in M_x$ and a point $y \in \tau^{-1}(x)$, then the real number $\underline{G}(y; X_1, X_2) = G(u)(z^{-1}X_1, z^{-1}X_2)$ ($u = (y, z)$) is obtained, independent of the choice of a frame $z \in \pi^{-1}(x)$. In the following, we shall use the letter G itself, instead of $\underline{G}$. Thus, $G(y; X_1, X_2)$ is the scalar product of X_1 and X_2 with respect to the element of support y, in the sense of E. Cartan.

2. Assume that a non-linear connection N be given in $T(M)$ [5], and denote by $l_y X$ the horizontal lift of $X \in M_x$ to $y \in \tau^{-1}(x)$ with respect to N. We shall recall here a concept of the lifted Riemannian metric G of $\overline{G}$ with respect to N [4], which is a Riemannian metric on $T(M)$, defined by

$$\overline{G}(X_1, X_2)_y = G(y; \tau X_1, \tau X_2) + G(y; Kv'X_1, Kv'X_2)$$

where $X_i \in T(M)_y$ $(i = 1, 2)$, and $v'X_i$ are the vertical parts of X_i with respect to N.

Consider a frame $z \in \pi^{-1}(x)$ at a point $x \in M$, and then we obtain a frame $\Phi_N(u) = (l_y z, l_y^v z)$, $u = (y, z) \in F(M)$, at a point $y = \tau^{-1}(x)$ [4]. Let g_{ab}, $a, b = 1, \cdots, n$, be the components of G with reference to z, and then components $\overline{g}_{\alpha\beta}$, $\alpha, \beta = 1, \cdots, 2n$, of $\overline{G}$ with reference to $\Phi_N(u)$ are given by

$$\bar{g}_{ab}=g_{ab},\quad \bar{g}_{a(b)}=0,\quad \bar{g}_{(a)(b)}=g_{ab},$$
$$a, b=1, \cdots, n,\quad (a)=n+a,\quad (b)=n+b.$$

Next, consider a local coordinate (x^i), $i=1, \cdots, n$, of $x\in M$, and then we have a local coordinate $(x^\lambda)=(x^i, y^i)$, $\lambda=1, \cdots, 2n$, of $y=\tau^{-1}(x)$, such that $y=y^i(\partial/\partial x^i)_x$. Then, the lift $l_y X$ of $X=X^i(\partial/\partial x^i)_x\in M_x$ is expressed by

$$l_y X=X^i\left(\frac{\partial}{\partial x^i}-F_i^{\,j}(x, y)\frac{\partial}{\partial y^j}\right),$$

where functions $F_i^{\,j}$, $i, j=1, \cdots, n$, are called the parameters of the non-linear connection N. Let g_{ij} be the components of G with reference to (x^i), and then the components $\bar{g}_{\lambda\mu}$, $\lambda, \mu=1, \cdots, 2n$, of $\bar{G}$ with reference to $(x^\lambda)=(x^i, y^i)$ are given by

$$\bar{g}_{ij}=g_{ij}+g_{kl}F_i^{\,k}F_j^{\,l}\quad \bar{g}_{i(j)}=g_{jk}F_i^{\,k},\quad \bar{g}_{(i)(j)}=g_{ij},$$
$$i, j, k, l=1, \cdots, n,\quad (i)=n+i,\quad (j)=n+j.$$

3. Now, we shall introduce another lift G^* of a Finsler metric G to $T(M)$ by the equation

$$G^*(X_1, X_2)_y=G(y;\ \tau X_1, \tau X_2)+G(y;\ \tau X_1, Kv'X_2)$$
$$+G(y;\ Kv'X_1, \tau X_2)+G(y;\ Kv'X_1, Kv'X_2). \tag{1}$$

The symmetric tensor field G^* of $(0, 2)$-type on $T(M)$ will be called the *natural lift* of G with respect to the non-linear connection N. It follows from (1) that

$$G^*(l_y X_1, l_y X_2)_y=G(y;\ X_1, X_2),$$
$$G^*(l_y X_1, l_y^v X_2)_y=G(y;\ X_1, X_2),\qquad X_1, X_2\in M_{\tau(y)}, \tag{2}$$
$$G^*(l_y^v X_1, l_y^v X_2)_y=G(y;\ X_1, X_2),$$

which shows that, in terms of the above frame $\Phi_N(u)$, the components $g^*_{\alpha\beta}$ of G^* are given by

$$g^*_{ab}=g_{ab},\quad g^*_{a(b)}=g_{ab},\quad g^*_{(a)(b)}=g_{ab}. \tag{3}$$

Next, we consider the components $g^*_{\lambda\mu}$ of G^* with reference to the local coordinate $(x^\lambda)=(x^i, y^i)$. If we pay attention to the fact that $Kv'(\partial/\partial x^i)_y = K(F_i^j(\partial/\partial y^j)_y) = F_i^j(\partial/\partial x^j)_x$, $x=\tau(y)$, we obtain

$$\begin{aligned} g^*_{ij} &= G^*((\partial/\partial x^i)_y, (\partial/\partial x^j)_y) \\ &= G(y;\ (\partial/\partial x^i)_x, (\partial/\partial x^j)_x) + G(y;\ (\partial/\partial x^i)_x, F_j^k(\partial/\partial x^k)_x) \\ &\quad + G(y;\ F_i^k(\partial/\partial x^k)_x, (\partial/\partial x^j)_x) + G(y;\ F_i^k(\partial/\partial x^k)_x, F_j^l(\partial/\partial x^l)_x) \\ &= g_{ij} + g_{ik}F_j^k + g_{kj}F_i^k + g_{kl}F_i^kF_j^l, \end{aligned}$$

and the similar way leads us to the equations

$$\begin{aligned} & g^*_{ij} = g_{ij} + g_{ik}F_j^k + g_{kj}F_i^k + g_{kl}F_i^kF_j^l, \\ (4)\qquad & g^*_{i(j)} = g_{ij} + g_{kj}F_i^k, \\ & g^*_{(i)(j)} = g_{ij}\,. \end{aligned}$$

It should be noticed here that the natural lift G^* is not Riemannian, but quasi-Riemannian metric with vanishing determinant on $T(M)$. The following will be easily verified.

Proposition 1. *With respect to the natural lift G^* of a positive-definite Finsler metric G, (1) the null vector on $T(M)$ is expressed by $l_yX - l^v_yX$, $X\in M_{\tau(y)}$, (2) a horizontal vector l_yX_1, $X_1\in M_{\tau(y)}$, is orthogonal to a vertical vector $l^v_yX_2$, $X_2\in M_{\tau(y)}$, if and only if X_1 is orthogonal to X_2 with respect to the original Finsler metric G and the element of support y.*

4. Let (Γ, N) be a Finsler connection of M, which has been fully treated in a previous paper [6]. Γ is a connection in the Finsler bundle $F(M)$ with the base space $T(M)$ and the structure group $GL(n, R)$, and N is a non-linear connection in the tangent bundle $T(M)$. Further, we defined [4] the bundle mapping $\Phi_N : F(M) \to L(T(M))$ (the bundle of linear $2n$-frames over $T(M)$) by $\Phi_N(u) = (l_yz, l^v_yz)$, $u=(y, z)$. Hence, a linear connection $\Gamma' = \Phi_N(\Gamma)$ is induced in $L(T(M))$, which was called the linear connection of Finsler type.

Here, we shall write down the parameters $\Gamma'^\lambda_{\mu\nu}$ of the connection Γ' in terms of a local coordinate $(x^\lambda)=(x^i, y^i)$. Let $F_{j\,k}^{\,i}$, $F_j^{\,i}$ and $C_{jk}^{\,i}$

be the parameters of the Finsler connection (Γ, N), where $F_j^{\ i}$ are parameters of the non-linear connection N, and functions $F_{jk}^{\ i}(x, y)$, $C_{jk}^{\ i}(x, y)$ are such that the respective lifts of $((\partial/\partial x^i) - F_i^{\ j}(\partial/\partial y^j))_y$ and $(\partial/\partial y^j)_y$ to a point $u = (y, z) = (x^i, y^i, z_a^i)$ with respect to Γ are given by

$$\frac{\partial}{\partial x^i} - F_i^{\ j}\frac{\partial}{\partial y^j} - z_a^k F_{ki}^{\ j}\frac{\partial}{\partial z_a^j} \quad \text{and} \quad \frac{\partial}{\partial y^i} - z_a^k C_{ki}^{\ j}\frac{\partial}{\partial z_a^j}.$$

Then, $\Gamma'^{\lambda}_{\mu\nu}$ are given by

$$\begin{aligned}
&\Gamma'^{i}_{jk} = \Gamma^{i}_{jk} (= F_{jk}^{\ i} + C_{jl}^{\ i} F_k^{\ l}),\\
&\Gamma'^{(i)}_{jk} = \frac{\partial F_j^{\ i}}{\partial x^k} + F_j^{\ l}\Gamma^{i}_{lk} - F_l^{\ i}\Gamma^{l}_{jk},\\
&\Gamma'^{i}_{(j)k} = 0, \quad \Gamma'^{i}_{j(k)} = C_{jk}^{\ i}, \quad \Gamma'^{(i)}_{(j)k} = \Gamma^{i}_{jk},\\
&\Gamma'^{(i)}_{j(k)} = \frac{\partial F_j^{\ i}}{\partial y^k} + F_j^{\ l} C_{lk}^{\ i} - F_l^{\ i} C_{jk}^{\ l},\\
&\Gamma'^{i}_{(j)(k)} = 0, \qquad \Gamma'^{(i)}_{(j)(k)} = C_{jk}^{\ i},\\
&i, j, k, l = 1, \cdots, n, \ (i) = n+i, \ (j) = n+j, \ (k) = n+k.
\end{aligned}$$

5. Now, we consider a pair (G^*, Γ') of the natural lift G^* of a Finsler metric G, and the linear connection Γ' of Finsler type derived from a Finsler connection (Γ, N). It follows from (3) that the Finsler decomposition $(G_{11}^*, G_{12}^*, G_{22}^*)$ of G^* [5] is given by

$$G_{11}^* = G_{12}^* = G_{22}^* = G.$$

Therefore, the Finsler decomposition of the covariant derivative $\Delta' G^*$ of G^* with respect to the connection Γ' is given by

$$\begin{aligned}
(\Delta' G^*)_{111} &= (\Delta' G^*)_{121} = (\Delta' G^*)_{221} = \Delta^h G,\\
(\Delta' G^*)_{112} &= (\Delta' G^*)_{122} = (\Delta' G^*)_{222} = \Delta^v G,
\end{aligned}$$

where Δ^h and Δ^v denote h- and v-covariant derivatives with respect to the Finsler connection (Γ, N) respectively [3]. Thus, we obtain

Proposition 2. *The linear connection Γ' of Finsler type derived from a Finsler connection (Γ, N) is metrical with respect to the natural lift G^* of a Finsler metric G, if and only if (Γ, N) is metrical with respect to G.*

6. With respect to the metric G^*, we shall denote by $|X|^*$ the length of a tangent vector $X \in T(M)_y$, and by $(X_1, X_2)^*$ the inner product of tangent vectors $X_1, X_2 \in T(M)_y$.

Definition. Let R' be the curvature tensor of the linear connection Γ' of Finsler type on $T(M)$, derived from a Finsler connection (Γ, N). Then, (Γ, N) is called (1) *h-*, (2) *hv-*, (3) *v-isotropic with respect to a Finsler metric* G at a point $y \in T(M)$, if there exists a scalar K such that

$$R_*(X_1, X_2, X_1, X_2) = K[(|X_1|^* \cdot |X_2|^*)^2 - ((X_1, X_2)^*)^2]$$

holds good for any tangent vectors $X_1, X_2 \in T(M)_y$, where R_* is the covariant curvature tensor constructed from R' and G^*, and (1) X_1, X_2 are horizontal, (2) X_1 is horizontal and X_2 is vertical, (3) X_1, X_2 are vertical, with respect to the non-linear connection N respectively.

Thus, R_* is defined by $R_*(X_1, X_2, X_3, X_4) = (X_2, R'(X_1, X_3, X_4))^*$. In terms of an n-frame $z \in \pi^{-1}\tau(y)$ on M and the $2n$-frame $\Phi_N(u)$, $u = (y, z)$, on $T(M)$, the components of the curvature tensor R' are expressed [4]

$$R'^{a}_{b.cd} = R'^{(a)}_{(b).cd} = R^{a}_{b.cd}, \qquad R'^{(a)}_{b.cd} = R'^{a}_{(b).cd} = 0,$$

$$R'^{a}_{b.c(d)} = R'^{(a)}_{(b).c(d)} = P^{a}_{b.cd}, \qquad R^{(\prime a)}_{b.c(d)} = R'^{a}_{(b).c(d)} = 0,$$

$$R'^{a}_{b.(c)(d)} = R'^{(a)}_{(b).(c)(d)} = S^{a}_{b.cd}, \quad R'^{(a)}_{b.(c)(d)} = R'^{a}_{(b).(c)(d)} = 0,$$

where $R^{a}_{b.cd}$, $P^{a}_{b.cd}$, $S^{a}_{b.cd}$ are components of the curvature tensors of (Γ, N). It then follows from (3) that

$$R_{*abcd} = R_{abcd}, \qquad R_{*(a)(b)(c)(d)} = S_{abcd},$$

$$R_{*a(b)c(d)} = R_{*(a)b(c)d} = P_{abcd}.$$

On the other hand, $X \in T(M)_y$ is horizontal, if and only if the components of X are $(X^1, \cdots, X^n, 0, \cdots, 0)$ in terms of a frame $\Phi_N(u)$. Therefore, (Γ, N) is h-isotropic, if

$$[R_{abcd} - K(g_{ac}g_{bd} - g_{ad}g_{bc})] X_1^a X_2^b X_1^c X_2^d = 0$$

is satisfied for any X_1^a, X_2^a, $a=1, \cdots, n$. Thus we know that the concept of h-isotropy coincides with that of partial isotropy due to H. Akbar-Zadeh [1]. Next, observe that $X \in T(M)$, is vertical, if and only if the components of X are $(0, \cdots, 0, X^{(1)}, \cdots, X^{(n)})$ in terms of a frame $\Phi_N(u)$. Thus, (Γ, N) is hv-isotropic, if

$$[P_{abcd} - K(g_{ac}g_{bd} - g_{ad}g_{bc})] X_1^a X_2^{(b)} X_1^c X_2^{(d)} = 0$$

is satisfied for any X_1^a, $X_2^{(a)}$, $a=1, \cdots, n$. Finally, (Γ, N) is v-isotropic, if

$$[S_{abcd} - K(g_{ac}g_{bd} - g_{ad}g_{bc})] X_1^{(a)} X_2^{(b)} X_1^{(c)} X_2^{(d)} = 0$$

is satisfied for any $X_1^{(a)}$, $X_2^{(a)}$, $a=1, \cdots, n$.

Remark. If we treat the lifted Riemannian metric $\overline{G}$, then the covariant curvature tensor $\overline{R}$ constructed from R' and $\overline{G}$ is given by

$$\overline{R}_{abcd} = R_{abcd}, \quad \overline{R}_{a(b)c(d)} = 0, \quad \overline{R}_{(a)(b)(c)(d)} = S_{abcd},$$

which is the inconvenient circumstances for our purpose.

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