J. Math. Kyoto Univ.
10-1 (1970) 151-187

Certain numerical characters of singularities

By

Heisuke HIRONAKA*

(Communicated by Prof. Nagata, Oct. 28, 1969)

§ 1. Introduction

Samuel introduced numerical function $H_{X,x}(=H^{(1)}_{X,x}$ defined in § 3) associated with each point x of a possibly singular noetherian scheme X. For $m \gg 0$, its values $H_{X,x}(m)$ coincides with corresponding values of a polynomial $P_{X,x}$ with rational coefficients, whose degree is equal to $\dim_x X$. The coefficient in the highest degree term of $P_{X,x}$ is written as $e_x(X)/d!$ with $d = \dim_x X$, where $e_x(X)$ denotes the multiplicity of X at x as was so defined by Samuel. Many important works have been done on this number $e_x(X)$ by Samuel, Nagata, Serre (with a somewhat different approach) and some other prominent mathematicians, but almost always the function $H_{X,x}$ itself was treated only as a tool, or a background object, for the study of the multiplicity. A significant motivation that underwent this trend was the importance of the theory of intersection numbers. From the point of view of the theory of singularities for the sake of its own beauty, however, one discovers that the function $H_{X,x}$ along with many other numerical characters of singularities is a more natural and more useful object of study than just the multiplicity or just the polynomial $P_{X,x}$. For instance, $H_{X,x}(1) - H_{X,x}(0)$ is the dimension of Zariski tangent space of X at x (or the local imbedding dimension of X about x) and is not given in general by the polynomial $P_{X,x}$. One also observes, by abundance of examples, that a quadratic transformation or, more generally, a monoidal transformation permissible to a given situation may exhibit some of its effects on a

*) This work was partly supported by NSF GP 9152

singularity in terms of changes in some values of $H_{X,x}$ even if no improvement of singularity is seen in terms of $P_{X,x}$. We introduced another kind of numerical character $\nu_x^*(X, Z)$ when X is given as a subscheme of a regular scheme Z, and proved its importance in the resolution of singularities. Our study of the effects of permissible transformations to the characters $\nu_x^*(X, Z)$ was one of the key steps to the goal of resolution of singularities. This character $\nu_x^*(X, Z)$ (as is seen from the definition given in §3) differ from $H_{X,x}$ by the fact that $\nu_x^*(X, Z)$ reflects the multiplicative structure (or more exactly, its graded version) of the local ring $O_{X,x}$, while $H_{X,x}$ depends only on the filtered module structure of the local ring (i.e., the module structure of the associated graded algebra). These two characters $H_{X,x}$ and $\nu_x^*(X, Z)$ behave almost totally unrelatedly in general, but their close relation in special cases are seen through the notion of normal flatness and through permissible monoidal transformations. This was shown to some extent in [2] and [3], and will be shown more clearly in this paper.

This work is entirely devoted to the proofs of the four theorems stated in the beginning of §3. The first one of the four concerns itself with $H_{X,x}$ and is the main theorem of Bennett in [2]. The proof is substantially simplified. The second is the corresponding fact on $\nu_x^*(X, Z)$, and the third shows an important corelation between $H_{X,x}$ and $\nu_x^*(X, Z)$. The fourth theorem brings up to light a basic relation between the group of translations which map $C_{X,x}$ (=the tangential cone of X at x) into itself and the set of infinitely near singular points (having the same Samuel function) above x on the permissible monoidal transforms of X.

§2. Notation

Let D be a closed subscheme of a scheme X, say noetherian. Then $I_{X,D}$ denotes the ideal sheaf of D in the structure sheaf O_X of X. There is a natural structure of graded O_D-algebra in $\bigoplus_{d=0}[(I_{X,D})^d/(I_{X,D})^{d+1}]$ which we denote by $gr_D(X)$. We write $C_{X,D}$ for Spec $(gr_D(X))$ and call it the *normal cone* of X along D, when it is combined with the projection $C_{X,D}\to D$. The fibre of the normal cone above a point $x\in D$ will be denoted by $C_{X,D,x}$. In particular, if $D=x$, then we call $C_{X,x}(=C_{X,x,x})$ the *tangential cone* of X at x. $M_{X,x}$ denotes the maximal ideal of $O_{X,x}$ and $k(x)$ does the residue field $O_{X,x}/M_{X,x}$. Let $\mathrm{symm}_x(X)=\mathrm{symm}_{k(x)}M_{X,x}/M^2_{X,x}$, the symmetric tensor algebra over $k(x)$, which is a naturally graded polynomial ring. We write $E_{X,x}$ for Spec $(\mathrm{symm}_x(X))$, a vector space over $k(x)$, and call it *Zariski tangent space* of X at x. $C_{X,x}$ is naturally imbedded in $E_{X,x}$ as a cone through the origin (i.e., invariant under the non-zero scalar multiplications). There exists the maximal additive group subscheme $A_{X,x}$ of $E_{X,x}$ which leaves $C_{X,x}$ invariant, i.e., $C_{X,x}+A_{X,x}=C_{X,x}$ in the sense of the addition in $E_{X,x}$. $A_{X,x}$ is called the *tangent additive group* of X at x. There also exists the maximal vector subspace $T_{X,x}$ of $E_{X,x}$ which leaves $C_{X,x}$ invariant. In other words, $T_{X,x}$ is the maximal vector subspace of $E_{X,x}$ over $k(x)$ contained in $A_{X,x}$. We call $T_{X,x}$ the *tangent vector space* of X at x. We have $T_{X,x}\subset A_{X,x}\subset C_{X,x}\subset E_{X,x}$. Let Z be a regular scheme. Then for a point x of Z, we have $T_{Z,x}=A_{Z,x}=C_{Z,x}=E_{Z,x}$. If D is a regular subscheme of Z, then $C_{Z,D}$ is a locally trivial vector bundle over D. We then write $N_{Z,D}$ for $C_{Z,D}$ and call it the *normal bundle* of Z along D. Let us next consider the situation of $Z\supset X\supset D\ni x$, where Z and D are regular as above. We then have a commutative diagram

$$\begin{array}{ccccccc} 0\to & T_{D,x} & \to & T_{Z,x} & \to & N_{Z,D,x} & \to 0\\ & \| & & \cup & & \cup & \\ & T_{D,x} & \xrightarrow{\alpha} & C_{X,x} & \xrightarrow{\beta} & C_{X,D,x} & \end{array}$$

where the upper horizontal sequence is the well-known exact sequence of vector spaces over $k(x)$. It is known that X is normally flat along D at x if and only if $\alpha(T_{D,x})\subset T_{X,x}$ and β induces an isomorphism $C_{X,x}/\alpha(T_{D,x})\xrightarrow{\sim} C_{X,D,x}$. (This is clearly equivalent to saying that a splitting of the upper exact sequence induces a left inverse of α by which $C_{X,x}$ becomes a product of $T_{D,x}$ and $C_{X,D,x}$. For the equivalence of this statement and the normal flatness, see [3], Ch II, Th 2.). If A is a noetherian local ring and I an ideal in A, then we write $\nu_I(f)$ for the maximum (if it exists, and ∞ if not) of those integers d with $f\in I^d$. For $f\in O_{Z,x}$ and $M=M_{X,x}$, we sometimes write $\nu_x(f)$ for $\nu_M(f)$. The expression $\varphi=(f \bmod I)$ will be used to mean that φ is the residue class of f modulo I. A and I being as above, we let $\mathrm{in}_I(f)=(f \bmod I^{d+1})$ with $d=\nu_I(f)$. If H is an ideal we write $\mathrm{in}_I(H)$ for the homogeneous ideal in $gr_I(A)$ generated by those $\mathrm{in}_I(f)$ with $f\in H$. With $M=M_{Z,x}$, we sometimes write $\mathrm{in}_x(f)$ for $\mathrm{in}_M(f)$. We have a natural epimorphism of graded $k(x)$-algebras $gr_x(Z)\to gr_x(X)$. Its kernel is denoted by $\mathrm{in}_x(X, Z)$, or $\mathrm{in}_x(X)$ for short. We know that $\mathrm{in}_x(X, Z)=\mathrm{in}_M(H)$ with $M=M_{Z,x}$ and $H=I_{Z,X,x}$ (=the ideal of X in the local ring $O_{Z,x}$). Let $Z\supset D\ni x$ be as above, and let g: $Z'\to Z$ be the blowing-up with center D, i.e., $Z'=\mathrm{Proj}(\mathrm{pow}(P))$ with $P=I_{Z,D}$. Here pow(P) for an ideal P in O_Z denotes the graded O_Z-algebra $\bigoplus_{d=0}^{\infty}P^d$. Therefore we have a canonical identity $g^{-1}(D)=\mathrm{Proj}(gr_D(Z))$, i.e., the associated projective bundle of the vector bundle $N_{Z,D}$ over D. So $g^{-1}(x)$ is the associated projective space, say $\boldsymbol{P}^s$, of the vector space $N_{Z,D,x}$ where $s+1$ is the codimension of D in Z at x. Let G be the canonical image of the stalk $gr_D(Z)_x$ into $gr_x(Z)$, so that $\boldsymbol{P}^s=\mathrm{Proj}(G)$ and $N_{Z,D,x}=\mathrm{Spec}(G)$. We have defined an additive group subscheme $B_{\boldsymbol{P}^s,x'}$ of Spec(G), associated with each point x' of $\boldsymbol{P}^s$. (cf [4]) We define the *associated additive group* $B_{g,x'}$ of the blowing-up at $x'\in g^{-1}(x)$ to be the additive group subscheme of $T_{Z,x}$ such that $B_{g,x'}\supset T_{D,x}$ and $B_{g,x'}/T_{D,x}=B_{\boldsymbol{P}^s,x'}$. $U_{g,x'}$ denotes the ring of invariants of $B_{g,x'}$ in the polynomial ring $gr_x(Z)$. By the definition of $B_{g,x'}$, $U_{g,x'}$ is a graded $k(x)$-subalgebra of $gr_x(Z)$ and its homogeneous part of degree d consists of those $\varphi\in G_d$ such that the hypersurface F_φ in $\boldsymbol{P}^s$ defined by $(\varphi)G$ has the multiplicity$\geq d$ at x'.

§3. Effects of a permissible transformation

Let X be a noetherian scheme. We define Samuel functions $H^{(t)}_{X,x}$, $t\in Z_0$, of X at a point x as follows:

$$H^{(0)}_{X,x}(m)=\mathrm{rank}_{k(x)}(M^m_{X,x}/M^{m+1}_{X,x})$$

for every $m \in Z_0$, and then by induction on $t \in Z_0$,

$$H_{X,x}^{(t+1)}(m) = \sum_{i=0}^{m} H_{X,x}^{(t)}(i)$$

so that $H_{X,x}^{(t)}(m) = H_{X,x}^{(t+1)}(m) - H_{X,x}^{(t+1)}(m-1)$.
If X is given as a closed subscheme of a regular scheme Z, then we also define a sequence

$$\nu_x^*(X, Z) = (\nu_1, \nu_2, \cdots\cdots, \nu_m, \cdots)$$

by the following conditions

(i) $1 \leq \nu_1 \leq \nu_2 \leq \cdots\cdots$, where ν_i are either integers or ∞, and (ii) each integer ν is repeated among the ν_i exactly t times where $t = \operatorname{rank}_{k(x)}(\mathrm{in}_x(X, Z)_\nu / \mathrm{in}_x(X, Z)_{\nu-1}\, \mathrm{gr}_x(Z)_1)$.

Remark 1. A system of generators $\varphi = (\varphi_1, \cdots, \varphi_m)$ of a homogeneous ideal I in a graded algebra is called a *standard base* of I, if

(i) φ is a minimal base of I,

(ii) each φ_i is homogeneous for every i, and

(iii) $\deg \varphi_1 \leq \deg \varphi_2 \leq \cdots\cdots \leq \deg \varphi_m$

One can prove that if φ is any standard base of $\mathrm{in}_x(X, Z)$ in $\mathrm{gr}_x(Z)$ then $\nu_x^*(X, Z) = (\deg \varphi_1, \deg \varphi_2, \cdots, \deg \varphi_m, \infty, \cdots)$.

Now let us consider $x \in D \subset X \subset Z$, where Z is a regular scheme, X a closed subscheme of Z and D a closed regular subscheme of X such that

(2.1) X is normally flat along D at the point x.

Our basic geometric object then is the following

(2.2)
$$\begin{array}{ccc} x' \in X' & \xrightarrow{j} & Z' \\ \uparrow f & & \downarrow g \\ x \in X & \xrightarrow{i} & Z \end{array}$$

where g (resp. f) is the blowing-up with center D, i is the inclusion, j is the canonical imbedding (i.e., the unique morphism which makes the diagram commutative), and x' is any point of $f^{-1}(x)$. Under these circumstances, we shall prove the following four theorems:

THEOREM I. (Bennett) We have $H^{(d+1)}_{X',x'}(m) \leq H^{(1)}_{X,x}(m)$ for all $m \in Z_0$, where d is the dimension of the closure of x', i.e., $\mathrm{tr.deg}_{k(x)} k(x')$.

THEOREM II. We have $\nu^*_{x'}(X', Z') \leq \nu^*_x(X, Z)$ in the sense of lexicographical ordering.

THEOREM III. $\nu^*_{x'}(X', Z') = \nu^*_x(X, Z)$ if and only if $H^{(d+1)}_{X',x'} = H^{(1)}_{X,x}$ with the number d of TH I.

THEOREM IV. If the equality of TH III holds, then the tangential cone $C_{X,x}$ is invariant by the subgroup $B_{g,x'}$ of $T_{Z,x}$.

Remark 3. TH I has been proven by Bennett [2], Theorem (0). The proof presented here basically follows Bennett's approach but is substantially simplified. Bennett also proved the if-part of TH III under the assumption of $k(x') = k(x)$, [2] Ch II Prop (3.4).

We will prove the following inequalities and analyse the case of equality in each step. Note that THs I and II are included.

$$(4.1) \qquad H^{(1+d)}_{X',x'}(m) \leq H^{(2+d+s)}_{f^{-1}(x),x'}(m)$$
$$\leq H^{(1+s)}_{C_{X,D,x}}(m) = H^{(1)}_{X,x}(m)$$

for all $m \in Z_0$, where $s = \dim_x D$ and we write H_C for $H_{C,0}$ if C is a cone through the origin 0 in a vector space.

$$(4.2) \qquad \nu^*_{x'}(X', Z') \leq \nu^*_x(X, Z) \text{ and}$$
$$\nu^*_{x'}(f^{-1}(x), g^{-1}(x)) \leq \nu^*(C_{X,D,x}, N_{Z,D,x}) = \nu^*_x(X, Z)$$

where $\nu^*(C, N)$ denotes $\nu^*_0(C, N)$ if C is a cone through the origin 0 in a vector space N.

Example (4.2) Let $Z = \mathrm{Spec}(k[u, v])$ with $u = (u_0, u_1, u_2)$ and $v = (v_1, v_2)$, which is an affine 5-space over a field k. Let $X = \mathrm{Spec}(k[u, v]/(h_1, h_2)k[u, \mathrm{v}])$ where $h_1 = u_0 u_1 u_2 + u_0^5 + v_1^3$ and $h_2 = u_0 u_2^2 + u_0^5 + v_2^3$. Let x be the origin of the affine space, and let g: $Z' \to Z$ be the blowing-up with center x. The induced blowing-up f: $X' \to X$ is the same as before. Let $u_0' = u_0$, $u_i' = u_i/u_0$ and $v_i' = v_i/u_0$ for $i=1, 2$. Let x' be the origin of the affine space $\mathrm{Spec}(k[u', v']) \subset Z'$. Let us compare the numerical characters of X at x and X' at x'. We claim

(i) (h_1, h_2) is a standard base of $I_{Z,X,x}$ in $O_{Z,x}$ and $\nu_x^*(X, Z) = (3, 3, \infty, \dots)$.

(ii) Let $h_1' = u_1'u_2' + u_0'^2 + v_1'^3$ and $h_2' = u_2'^2 + u_0'^2 + v_2'^3$. Then (h_1', h_2') is a standard base of $I_{Z',X',x'}$ and $\nu_{x'}^*(X', Z') = (2, 2, \infty, \dots)$.

(iii) Let $\bar{u}_i'$(resp. $\bar{v}_i'$) be the class of u_i'(resp. v_i') mod $I_{Z',g^{-1}(x),x'} = (u_0')O_{Z',x'}$. Then $I_{g^{-1}(x),f^{-1}(x),x'}$ has a standard base $(\bar{h}_1', \bar{h}_2', \alpha)$ where $\bar{h}_1' = \bar{u}_1'\bar{u}_2' + \bar{v}_1'^3$, $\bar{h}_2' = \bar{u}_2'^2 + \bar{v}_2'^3$ and $\alpha = \bar{u}_2'\bar{v}_1'^3 - \bar{u}_1'\bar{v}_2'^3$. Hence $\nu_{x'}^*(f^{-1}(x), g^{-1}(x)) = (2, 2, 3, \infty, \dots)$.

This example shows that we cannot in general expect any inequalities between $\nu_{x'}^*(X', Z')$ and $\nu_{x'}^*(f^{-1}(x), g^{-1}(x))$. Accordingly, our proof of (4.2) is not completely parallel to that of (4.1). (In proving the claims of the above example, the key is the fact that in the case of a complete intersection, say $f_1 = f_2 = 0$, (f_1, f_2) is a standard base at a point if their initial forms at the point are relatively prime.)

The normal flatness assumption (2.1) implies that $C_{X,x}$ is invariant by $T_{D,x}$ and there exists a canonical isomorphism $C_{X,x}/T_{D,x} \xrightarrow{\sim} C_{X,D,x}$. The exact sequence of vector spaces $0 \to T_{D,x} \to T_{Z,x} \to N_{Z,D,x} \to 0$ splits over the base field $k(x)$, and a splitting of this sequence induces an isomorphism $C_{X,x} \xrightarrow{\sim} T_{D,x} \times C_{X,D,x}$. This implies the last equalities of (4.1) and (4.2), for obviously $H_{X,x}^{(1)} = H_{C_{X,x}}^{(1)}$ and $\nu_x^*(X, Z) = \nu^*(C_{X,x}, T_{Z,x})$. It is thus enough to prove the first two inequalities of (4.1) and (4.2) for THs I and II.

For a graded algebra G over a field K, we define $H_G^{(t)}$ by letting $H_G^{(0)}(m) = \mathrm{rank}_K G_m$. For a local ring A, we write $H_A^{(t)}$ for $H_G^{(t)}$ with $G = gr_M(A)$ where M denotes the maximal ideal of A. In other words, $H_A^{(t)} = H_{Y,y}^{(t)}$ with the unique closed point y of $Y = \mathrm{Spec}(A)$.

Proposition 5. (Bennett) Let A be a noetherian local ring with maximal ideal M. Let $z \in M$, $B = A/zA$ and $N = MB$. Then we have

$H_B^{(t+1)}(m) \geq H_A^{(t)}(m)$ for all integers $m \geq 0$ and $t \geq 1$.

Proof. The equality for $t=1$ and for all m is obviously enough. Let $A(m)=A/M^{m+1}$ and $B(m)=B/N^{m+1}$. We have $H_A^{(1)}(m)=\text{length}_A A(m)=\sum_{j=0}^{m} \text{length}_A\ z^j A(m)/z^{j+1}A(m)$. The multiplication by z^j in A induces an epimorphism $\lambda_{m,j}$: $A(m-j)/zA(m-j) \to z^j A(m)/z^{j+1}A(m)$. Hence $H_A^{(1)}(m) \leq \sum_{j=0}^{m} \text{length}_A A(m-j)/zA(m-j)$ which is equal to $H_B^{(2)}(m)$ because $A(m-j)/zA(m-j)=B(m-j)$.

Remark (5.1) This proof clearly shows that $H_B^{(2)}=H_A^{(1)}$ (which implies $H_B^{(t+1)}=H_A^{(t)}$ for all $t \geq 0$) if and only if all the $\lambda_{m,j}$ are bijective.

Remark (5.2) The first inequality of (4.1) follows from Prop 5. In fact, the ideal of $f^{-1}(D)$ in X' is invertible as $O_{X'}$-module, so that there exists $u_0 \in O_{X,x}$ with $I_{X',f^{-1}(D),x'}=u_0 O_{X',x'}$. D being regular, there are $u_i \in O_{X',x'}$, $1 \leq i \leq s$, such that $M_{D,x}=(u_1, \cdots, u_s)O_{D,x}$. Then $O_{f^{-1}(x),x'}=O_{X',x'}/(u_0, u_1, \cdots, u_s)O_{X',x'}$. Apply Prop 5 repeatedly $s+1$ times and get the first inequality of (4.1).

Proposition 6. For A, B and z of Prop 5, the following conditions are equivalent to one another:

(i) $H_B^{(t+1)}=H_A^{(t)}$ for all integers $t \geq 0$.

(ii) z is not a zero-divisor (and hence non-zero) in A and $M^{m+1} \cap zA=zM^m$ for all $m \geq 0$.

(iii) The image $\bar{z}$ of z in $gr_M(A)_1$ is not a zero-divisor in $gr_M(A)$.

(iv) $\bar{z}$ is not a zero-divisor in $gr_M(A)$ and the natural homomorphism $gr_M(A)/\bar{z}\, gr_M(A) \to gr_N(B)$ is bijective.

Proof. If z is a zero-divisor in A, then there exists $w \in M$, $\neq 0$, with $wz^j=0$ for some $j>0$. We can choose w and j in such a way that $w \notin zA$. Hence there exists an integer $m>j$ such that $w \notin zA+M^{m-j+1}$. If ω is the image of w in $A(m-j)/zA(m-j)$ then $\omega \neq 0$ and $\lambda_{m,j}(\omega)=0$. In view of (5.1), (i) implies that z is not a zero-divisor in A. We have

$$\operatorname{Ker}(\lambda_{m,j})=\lambda^{-1}(z^{j+1}A+M^{m+1})/zA+M^{m-j+1}$$

where λ: $A\to A$ is the multiplication by z^j. If z is not a zero-divisor in A, then for each $m\geq j>0$, $\operatorname{Ker}(\lambda_{m,j})=(0)\Leftrightarrow\lambda^{-1}(z^{j+1}A+M^{m+1})=zA+M^{m-j+1}\Leftrightarrow(z^{j+1}A+M^{m+1})\cap z^jA=z^{j+1}A+z^jM^{m-j+1}\Leftrightarrow M^{m+1}\cap z^jA=z^jM^{m-j+1}+M^{m+1}\cap z^{j+1}A$. The last equality for all $j\leq m$ with a fixed $m\geq 0$, implies $M^{m+1}\cap z^jA=z^jM^{m-j+1}+M^{m+1}\cap z^{j+a}A$ for all (j, a), $0\leq j\leq m$ and $1\leq a\leq m-j+1$, and so in particular (for $j+a=m+1$) $M^{m+1}\cap z^jA=z^jM^{m-j+1}$ for all j, $0\leq j\leq m$. Conversely, this equality clearly implies what was proven equivalent to $\operatorname{Ker}(\lambda_{m,j})=0$ for all j, $0\leq j\leq m$. We have thus (i)$\Leftrightarrow$(ii). Let us now assume (ii). Let m be an arbitrary integer ≥ 0 and φ an arbitrary element of $gr_M(A)_m$ such that $\bar{z}\varphi=0$. Let $f\in M^m$ be such that $\varphi=(f \bmod M^{m+1})$. Then $zf\in M^{m+2}\cap zA=zM^{m+1}$. Since z is not a zero-divisor in A, $f\in M^{m+1}$, i.e., $\varphi=0$. We have proven (ii)$\Rightarrow$(iii). Now assume (iii). Let f be any element of A with $zf=0$. If $f\neq 0$, then $j=\nu_M(f)<\infty$ and $\operatorname{in}_M(f)=(f \bmod M^{j+1})\neq 0$. But $\bar{z}\operatorname{in}_M(f)=(zf \bmod M^{j+2})=0$, contradictory to (iii). Hence z is not a zero-divisor in A. Let $f\in zA\cap M^{m+1}$. Write $f=zg$ with $g\in A$. Let $j=\nu_M(g)$. If $j<m$, then $\bar{z}\operatorname{in}_M(g)=(f \bmod M^{j+2})=0$, contradictory to (iii). Hence $j\geq m$, i.e., $f\in zM^m$. This shows (iii)$\Leftrightarrow$(ii). Our proof of Prop. 4 will be complete if we prove (ii)$\Rightarrow$(iv). So assume (ii). Then $gr_N(B)_j=N^j/N^{j+1}=M^j/M^{j+1}+(zA\cap M^j)=M^j/M^{j+1}+zM^{j-1}=(M^j/M^{j+1})/z(M^{j-1}/M^j)=gr_M(A)_j/\bar{z}\,gr_M(A)_{j-1}$ for all $j\geq 1$. This means (iv). Q.E.D.

If I is a homogeneous ideal in a naturally graded polynomial ring S over a field K, then $\nu^*(I, S)$ denotes the sequence $(\deg\varphi_1, \deg\varphi_2, \cdots, \deg\varphi_m, \infty, \infty, \cdots)$ with any minimal base $(\varphi_1, \varphi_2, \cdots, \varphi_m)$ of I with homogeneous φ_i and with $\deg\varphi_i\leq\deg\varphi_{i+1}$ for all $i\geq 1$. If R is a regular local ring with maximal ideal M, then $\nu^*(J, R)$ for an ideal J in R denotes $\nu^*(\operatorname{in}_M(J), gr_M(R))$ where $\operatorname{in}_M(J)$ is the kernel of the natural homomorphism $gr_M(R)\to gr_M(R/J)$. In other words, $\nu^*(J, R)=\nu^*_y(Y, W)$ where y is the unique closed point of $Y=\operatorname{Spec}(R/J)\subset W=\operatorname{Spec}(R)$.

Lemma 7. Let R be a regular local ring, M the maximal ideal of R and v an element of $M-M^2$. Let J be an ideal in R, $A=R/J$, $\bar{R}=R/vR$ and $\bar{A}=A/vA$. Then the equality $H_{\bar{A}}^{(1)}=H_A^{(0)}$ implies that a standard base of $\mathrm{in}_M(J)$ induces a standard base of $\mathrm{in}_{\bar{M}}(\bar{J})$ and in particular $\nu^*(\bar{J}, \bar{R})=\nu^*(J, R)$, where $\bar{J}=J\bar{R}$ and $\bar{M}=M\bar{R}$.

Proof. Let $t=\mathrm{in}_M(v)$. Pick any standard base $\varphi=(\varphi_1, \cdots, \varphi_m)$ of $\mathrm{in}_M(J)$ and let $d_i=\deg \varphi_i$. If $\bar{\varphi}_i=(\varphi_i \bmod (t)gr_M(J))$, then $\bar{\varphi}_i$ is homogeneous and of degree d_i unless $\bar{\varphi}_i=0$. Moreover, by Prop 6, $H_{\bar{A}}^{(1)} =H_A^{(0)}$ implies $\mathrm{in}_{\bar{M}}(\bar{J})=(\mathrm{in}_M(J) \bmod (t)gr_M(R))$ and hence $(\bar{\varphi}_1, \cdots, \bar{\varphi}_m) gr_{\bar{M}}(\bar{R})=\mathrm{in}_{\bar{M}}(\bar{J})$. It is therefore enough to prove that $\bar{\varphi}$ is a minimal base. If not, there exist homogeneous $\beta_i\in gr_M(R)$ such that $\deg \beta_i= d_j-d_i$ and $\bar{\varphi}_j-\sum_{i<j}\bar{\beta}_i\bar{\varphi}_i=0$. Namely $\varphi_j-\sum_{i<j}\beta_i\varphi_i=t\psi$ with $\psi\in gr_M(R)$. By Prop 6, t is not a zero-divisor in $gr_M(R)/\mathrm{in}_M(J)$. Hence $\psi\in\mathrm{in}_M(J)$, i.e., there exist homogeneous α_i of degrees $d_j-1-\deg \varphi_i$ in $gr_M(R)$ such that $\psi=\sum_{i<j}\alpha_i\varphi_i$. Then $\varphi_j=\sum_{i<j}(\beta_i+t\alpha_i)\varphi_i$, contrary to the minimality of φ. Q.D.E.

Let us now proceed to prove the second inequality of (4.1) (and hence TH. I). Let us choose a regular system of parameters $(x_0, x_1, \cdots, x_r, y_1, \cdots, y_s)$ of $O_{Z,x}$ such that $(x_0, \cdots, x_r)O_{Z,x}=I_{Z,D,x}$, the ideal of D in $O_{Z,x}$. Let $X_i=\mathrm{in}_M(x_i)$ and $Y_j=\mathrm{in}_M(y_j)$, where $M=M_{Z,x}$, so that $gr_x(Z)=K[X, Y]$ with $X=(X_0, \cdots, X_r)$ and $Y=(Y_1, \cdots, Y_s)$ where $K=O_{Z,x}/M_{Z,x}$. By means of the canonical morphism $T_{Z,x}\to N_{Z,D,x}$, we shall identify $N_{Z,D,x}$ with $\mathrm{Spec}(K[X])$. The normal flatness assumption (2.1) implies that $\mathrm{in}_x(X, Z)=Igr_x(Z)$ with $I=\mathrm{in}_x(X, Z)\cap K[X]$. We have $C_{X,D,x}=\mathrm{Spec}(K[X]/I)$ and $f^{-1}(x)=\mathrm{Proj}(K[X]/I)$. The following lemma proves the first inequality of (4.1) in the residually rational case, i.e., the case for $k(x')=k(x)$.

Lemma 8. Let I be a homogeneous ideal in a polynomial ring $K[X]$. Let $N=\mathrm{Spec}(K[X])$, $C=\mathrm{Spec}(K[X]/I)$, $E=\mathrm{Proj}(K[X])$ and $F=\mathrm{Proj}(K[X]/I)$. Let x' be any K-rational point of F. Then we have

(i) $H_C^{(t)}(m)\geq H_{F,x'}^{(t+1)}(m)$ for all $t\geq 0$ and all $m\geq 0$,

(ii) $\nu^*(C, N) \geq \nu^*_{x'}(F, E)$, and

(iii) the equalities of (i) hold if and only if that of (ii) does.

Proof. Since x' is K-rational, we may assume that $O_{E,x'} = K[T]_{(T)K[T]}$ where $T=(T_1, \cdots, T_r)$ with $T_i = X_i/X_0$. Let I' be the ideal of F in $K[T]$. We have $H^{(1)}_{F,x'}(m) = \text{length}(O_{F,x'}/M^{m+1}_{F,x'}) = \text{rank}_K (K[T]/(T)^{m+1}K[T]+I')$. For each m, let us define a homomorphism of K-modules α_m: $I_m \to I' + (T)^{m+1}K[T]/(T)^{m+1}K[T]$ by letting $\alpha_m(\varphi) = (\varphi/X_0^m \bmod (T)^{m+1}K[T])$. Since each φ/X_0^m is a polynomial of degree $\leq m$ in T, α_m is injective. Therefore, by the above equalities, we get $H^{(1)}_{F,x'}(m) \leq \text{rank}_K (K[T]/(T)^{m+1}K[T]) - \text{rank}_K I_m = \text{rank}_K(K[X]_m) - \text{rank}_K I_m = H^{(0)}_C(m)$. This proves (i) for $t=0$ and hence for all $t \geq 0$. To prove (ii) and (iii), let us consider the following condition:

$$(*) \quad (I \cap G)K[X] = I \text{ where } G = K[X_1, \cdots, X_r].$$

For simplicity, we write ν^*(resp. $\nu^{*\prime}$) for $\nu^*(C, N)$ (resp. $\nu^*_{x'}(F, E)$). We shall prove that $\nu^* \leq \nu^{*\prime} \Rightarrow (*)$, that the equalities of (i) $\Rightarrow (*)$ and that $(*) \Rightarrow$ both $\nu^* = \nu^{*\prime}$ and the equalities of (i). Note that (ii) and (iii) follow immediately from these implications. Let $\varphi = (\varphi_1, \cdots, \varphi_m)$ be any standard base of I, and let p be the largest integer $\leq$ m+1 such that $\varphi_i \in G$ for all $i < p$. Say $p < m+1$, and write $\varphi_p = \psi_p + \psi_{p-1}X_0 + \cdots + \psi_{p-j}X_0^j$ with $\psi_i \in G$ and $\psi_{p-j} \neq 0$. We want to prove that if $\nu^* = \nu^{*\prime}$ or the equalities of (i) hold, then $\psi_{p-j} \in I$ so that φ_p can be replaced by $\varphi_p - \psi_{p-j}X_0^j$. If this is so, then by induction we see that φ could be so chosen as to have $p = m+1$, i.e., $(*)$. In any case, we have an isomorphism of graded K-algebras ω: $G \xrightarrow{\sim} gr_{x'}(E)$, which sends X_i to the initial form of $T_i = X_i/X_0$. Note that $\omega(\varphi_i) = \text{in}_{M'}(\varphi_i/X_0^{d_i})$ for all $i<p$ and $\omega(\psi_{p-j}) = \text{in}_{M'}(\varphi_p/X_0^{d_p})$, where $M' = M_{E,x'}$ and $d_i = \deg \varphi_i$. We have $(\omega(\varphi), \omega(\psi_{p-j}))gr_{x'}(E) \subset \text{in}_{x'}(F, E)$, where $\omega(\varphi) = (\omega(\varphi_1), \cdots, \omega(\varphi_{p-1}))$. Now assume $\nu^* \not\leq \nu^{*\prime}$. Then $\omega(\varphi)$ extends to a standard base of $\text{in}_{x'}(F, E)$, and since $\deg \omega(\psi_{p-l}) = d_p - j < d_{p'}$, we must have $\omega(\psi_{p-j}) \in (\omega(\varphi))\, gr_{x'}(E)$, i.e., $\psi_{p-j} \in (\varphi_1, \cdots, \varphi_{p-1})G$. This means that φ_p could be replaced by $\varphi_p - \psi_{p-j}X_0^j$. Next, if $d < d_p$ we have $H_C^{(0)}(d)=\text{rank}_K(K[X]/(\varphi,\cdots,\varphi_{p-1})K[X])_d=\sum_{a=0}^{d}\text{rank}_K(G/(\varphi_1,\cdots,\varphi_{p-1})G)_a=\sum_{a=0}^{d}\text{rank}_K(gr_{x'}(E)/(\omega(\varphi))\,gr_{x'}(E))_a$

$$\geq\sum_{a=0}^{d}\text{rank}_K(gr_{x'}(E)/(\omega(\varphi),\omega(\psi_{p-j}))\,gr_{x'}(E))_a\geq H_{F,x'}^{(1)}(d)$$

Hence, if the equalities of (i) hold, then we must have $\omega(\psi_{p-j})\in(\omega(\varphi))gr_{x'}(E)$, i.e., once again $\psi_{p-j}\in(\varphi_1,\cdots,\varphi_{p-1})G$. We have thus proved the first two of the claimed implications. Now, conversely, let us assume $(*)$. Then the above ω gives rise to an isomorphism of $K[X_0]$-algebras from $K[X]/I$ to

$$(gr_{x'}(E)/\text{in}_{x'}(F,E))[X_0]$$

from which $\nu^*=\nu^{*\prime}$ and the equalities of (i) clearly follow. Q.E.D.

We are now interested in the case in which $k(x')$ is a non-trivial extension of $k(x)$. Let $\bar{u}$ be an element of $k(x')$ which is not in $k(x)$. Let $\bar{p}\in k(x)[U]$ be a monic irreducible polynomial for $\bar{u}$. Let $p\in O_{Z,x}[U]$ be a polynomial such that deg $p=$deg $\bar{p}$ and that $\bar{p}=(p \bmod M_{Z,x}[U])$. (In particular, if $\bar{u}$ is transcendental over $k(x)$, then $\bar{p}=0$ and hence $p=0$.) We then have a diagram

(9.1)
$$\begin{array}{ccc} Z' & \xleftarrow{j'} & \tilde{Z}' \\ g\downarrow & & \downarrow\tilde{g} \\ Z & \xleftarrow{j} & \tilde{Z} \end{array}$$

where $\tilde{Z}=\text{Spec}(O_{Z,x}[U]/(p)O_{Z,x}[U])$, $j\colon\tilde{Z}\to Z$ is the canonical morphism, $\tilde{g}$ is the blowing-up with center $\tilde{D}=j^{-1}(D)$ and j' is the unique morphism which makes the diagram commutative. Let $\tilde{x}$ be the generic point of $j^{-1}(x)=\text{Spec}(k(x)[\bar{u}])$ (which is either a line or a single point). The $k(x)$-homomorphism from $k(\tilde{x})$ to $k(x')$, which makes U correspond to $\bar{u}$, gives a point $\tilde{x}'$ of $\tilde{Z}'$ such that $\tilde{g}(\tilde{x}')=\tilde{x}$ and $j'(\tilde{x}')=x'$. Let $\tilde{X}=j^{-1}(X)$ and $\tilde{X}'$ the strict transform of $\tilde{X}$ by $\tilde{g}$, so that $\tilde{g}$ induces the blowing-up $\tilde{f}\colon\tilde{X}'\to\tilde{X}$ with center $\tilde{D}$. Moreover, the fact that j is flat implies that the diagram (9.1) is cartesian as well as the following

$$\begin{array}{ccc} X' & \xleftarrow{i'} & \tilde{X}' \\ f\downarrow & & \downarrow\tilde{f} \\ X & \xleftarrow{i} & \tilde{X} \end{array} \tag{9.2}$$

where i (resp. i') is induced by j (resp. j'). Since $p(U)$ remains irreducible mod $M_{Z,x}$, i induces isomorphisms

$$C_{\tilde{X},\tilde{x}} \tilde{\to} C_{X,x} \otimes_{k(x)} k(\tilde{x}) \text{ and} \tag{9.3}$$

$$C_{\tilde{X},\tilde{D},\tilde{x}} \tilde{\to} C_{X,D,x} \otimes_{k(x)} k(\tilde{x})$$

where the right hand side denotes the base field extensions. This implies $H^{(t)}_{\tilde{X},\tilde{x}} = H^{(t)}_{X,x}$ and $H^{(t)}_{\tilde{C}} = H^{(t)}_{C}$ for all $t \geq 0$ where $\tilde{C} = C_{\tilde{X},\tilde{D},\tilde{x}}$ and $C = C_{X,D,x}$. Therefore to prove

$$H^{(t+d)}_{X',x'}(m) \leq H^{(t)}_{X,x}(m), \tag{9.4}$$

$$\text{resp. } H^{(t+d+1)}_{f^{-1}(x),x'} \leq H^{(t)}_{C}(m)$$

we can do it in two steps, first $H^{(t+b)}_{\tilde{X}',\tilde{x}'}(m) \leq H^{(t)}_{\tilde{X},\tilde{x}}(m)$, resp. $H^{(t+b+1)}_{\tilde{f}^{-1}(\tilde{x}),\tilde{x}'}(m) \leq H^{(t)}_{\tilde{C}}(m)$, where $b = \mathrm{tr.deg}_{k(\tilde{x})} k(\tilde{x}')$, and then

$$H^{(t+d)}_{X',x'}(m) \leq H^{(t+b)}_{\tilde{X}',\tilde{x}'}(m) \tag{9.5}$$

$$\text{resp. } H^{(t+d+1)}_{f^{-1}(x),x'}(m) \leq H^{(t+b+1)}_{\tilde{f}^{-1}(\tilde{x}),\tilde{x}'}(m).$$

Thus by proving (9.5) and applying it repeatedly (necessarily a finite number of times), the proof of (9.4) can be reduced to the residually rational case. In particular, will then follow the second inequality of (9.4) (i.e., the same of (4.1)) and hence TH I.

Let us now propose to prove (9.5). Let $A = O_{X',x'}$ (resp. $O_{f^{-1}(x),x'}$) and $B = O_{\tilde{X}',\tilde{x}'}$ (resp. $O_{\tilde{f}^{-1}(\tilde{x}),\tilde{x}'}$), so that (9.5) can be written as

$$H^{(n+e)}_{A}(m) \leq H^{(n)}_{B}(m) \text{ where } e = d - b = \mathrm{tr.deg}_K K(\bar{u}) \tag{9.6}$$

$$\text{with } K = k(x).$$

Case 1. Assume that $\bar{u}$ is transcendental over K. Then $B = A[V]_{(M,V)A[V]}$ with a variable V over A. Then clearly the equality holds in (9.6), where $e=1$.

Case 2. Assume that $\bar{u}$ is separable algebraic over K. Then $\bar{p} = (p \bmod M[U])$ has no multiple root, so that the extension $A \to B$ (which is a localization of $A[U]/pA[U]$) is unramified and hence there exists an isomorphism $gr_N(B) \to gr_M(A)$, where M (resp. N) is the maximal ideal of A (resp. B). Again the equality holds in (9.6), where $e=0$.

Case 3. Assume that $\bar{u}$ is purely inseparable over K. We then have $B = A[U]/pA[U]$. Let us pick any element $u \in A$ such that $\bar{u} = (u \bmod M)$, and let $V = U - u$. Write $h(V)$ for $p(U)$. Then we have $B = A[V]/hA[V]$, where $h \equiv V^q \bmod M[V]$ with $q = [K(\bar{u}) : K] = \deg h$. Thus what we want is

Proposition 10. Let A be a noetherian local ring with maximal ideal M, and let $B = A[V]/hA[V]$ where $h = V^q + z_1 V^{q-1} + \cdots + z_q$ with $z_i \in M$ for all i, so that B is local. Then we have $H_B^{(n)}(m) \geq H_A^{(n)}(m)$ for all $n \geq 1$ and all $m \geq 0$.

Proof. Let $R = A[V]$, $R(m) = R/(M, V)^{m+1}R$ and $A(m) = A/M^{m+1}$ for each $m \geq 0$. We have $H_B^{(1)}(m) = \text{length } R(m)/hR(m)$

$$= \sum_{j=0}^{m} \text{length } \{V^j R(m)/V^j R(m) \cap hR(m) + V^{j+1} R(m)\}$$

The multiplication by z_q^j in R induces a homomorphism $\beta_{mj}: R(m) \to R(m+j)$ because $z_q \in M$ by assumption. We claim

$$(*) \quad \beta_{mj}(V^j R(m) \cap hR(m)) + V^{j+1} R(m+j)$$
$$= z_q^{j+1} V^j R(m+j) + V^{j+1} R(m+j)$$

In fact, let $w \in V^j R(m) \cap hR(m)$. We can write $w = (\sum_{i=0}^{m} d_i h V^i \bmod (M, V)^{m+1}R)$ where $d_i \in A$. If $\alpha_{nj}: A(n) \to R(n+j)$ is the A-homomorphism defined by $\alpha_{nj}(a \bmod M^{n+1})=(aV^j \bmod (M, V)^{n+j+1}R)$, α_{nj} induces an isomorphism $A(n) \xrightarrow{\sim} V^jR(n+j)/V^{j+1}R(n+j)$ and a direct sum decomposition $R(m)=\bigoplus_{j=0}^{m} Im(\alpha_{m-jj})$. Therefore $w \in V^jR(m)$ implies that

$$d_s z_q + d_{s-1} z_{q-1} + \cdots + d_0 z_{q-s} \equiv 0 \mod M^{m-s+1}$$

for all $s<j$, where $z_0=1$ and $z_i=0$ for $i<0$. Then by induction on s, $0 \leq s<j$, we get $d_s z_q^{s+1} \equiv 0 \mod M^{m+1}$ for every s. Moreover, if we write $w=(\sum_{i=j}^{m} e_i V^i \bmod (M, V)^{m+1}R)$ with $e_i \in A$, then $d_j z_q + d_{j-1} z_{q-1} + \cdots + d_0 z_{q-j} \equiv e_j \mod M^{m-j+1}$ and hence $d_j z_q^{j+1} \equiv z_q^j e_j \mod M^{m+1}$. This means

$$z_q^j(\sum_{i=j}^{m} e_i V^i) \equiv z_q^{j+1}(d_j V^j)$$

$$\mod \{(M, V)^{m+j+1}R + V^{j+1}R\}$$

In other words, $\beta_{mj}(w) \in z_q^{j+1} V^jR(m+j) + V^{j+1}R(m+j)$. This proves the first term of $(*)$ is included in the second. The reverse inclusion is clear because $\beta_{mj}(V^j hR(m))$ generates the second term of $(*)$ mod $V^{j+1}R(m+j)$. Now in view of the above equality for $H_B^{(1)}(m)$, $(*)$ implies $H_B^{(1)}(m) \geq \sum_{j=0}^{m} \text{length}\{z_q^j V^jR(m+j) + V^{j+1}R(m+j)/z_q^{j+1} V^jR(m+j) + V^{j+1}R(m+j)\} = \sum_{j=0}^{m} \text{length } z_q^j A(m)/z_q^{j+1}A(m) = H_A^{(1)}(m)$. Q.E.D.

Remark (**10.1**) To prove (4.1) (and hence TH. I), we need only the second inequality of (9.5) for $t \geq 1$. Hence it is enough to have the inequalities of Prop 10 for $n \geq 2$. This is substantially easier to prove, than Prop 10. In fact, let S be the localization of $A[V]$ by the maximal ideal $(M, V)A[V]$. As V is an indeterminate over A, we get $H_A^{(2)}=H_S^{(1)}$. By Prop 5, $H_S^{(1)}(m) \leq H_{S/hS}^{(2)}(m)$ so that $H_A^{(2)}(m) \leq H_B^{(2)}(m)$ for all $m \geq 0$.

Problem (10.2) Does the inequality of Prop 10 hold in general for every flat local extension $A \rightarrow B$ of noetherian local rings?

Proposition 11. Under the assumption of Prop 10, assume $q>1$.

Then the following conditions are equivalent to one another:

(i) $H_B^{(n)}=H_A^{(n)}$ for any one (and hence for all) $n\geq 0$.

(ii) The class (z_q mod M^2) is not a zero-divisor in $gr_M(A)$.

(iii) (z_q mod M^2) is not a zero-divisor in $gr_M(A)$ and generates the kernel of the natural epimorphism $gr_{(M,V)R}(R)\to gr_N(B)$ where $R=A[V]$ and N is the maximal ideal of B.

Proof. As was shown in (10.1), we have $H_A^{(2)}(m)=H_S^{(1)}(m)\leq H_{S/hS}^{(2)}(m)=H_B^{(2)}(m)$ for all $m\geq 0$. Hence $H_A^{(2)}=H_B^{(2)}$ if and only if $H_S^{(1)}=H_{S/hS}^{(2)}$. By Prop 6, this last equality holds if and only if (h mod P^2) is not a zero-divisor in $gr_P(S)$ where $P=(M, V)S$, the maximal ideal of S. As $q\geq 2$ and $z_i\in M$ for all i, we get (h mod P^2)$=$(z_q mod P^2). There exists a canonical isomorphism $gr_M(A)[V]\xrightarrow{\sim} gr_P(S)$ which sends V to $\mathrm{in}_P(V)$. This isomorphism maps (z_q mod M^2) to (z_q mod P^2). It is therefore clear that (z_q mod P^2) is not zero-divisor in $gr_P(S)$ if and only if (z_q mod M^2) is not a zero-divisor in $gr_M(A)$. We have thus proven (i)$\Leftrightarrow$(ii). Furthermore we have a natural epimorphism $gr_P(S)\to gr_N(B)$. If (ii) holds, then its kernel is generated by (h mod P^2) as was proven in Prop 6. This means (iii) by what was shown above. Q.E.D.

Lemma 12. In addition to the assumption of Prop 10, assume further that $A=E/I$ with a regular local ring E and an ideal I in E. Let g be any monic polynomial in $E[V]$ such that $h=(g \bmod I[V])$, and let $F=E[V]/gE[V]$ so that $B=F/J$ with $J=IF$. If $H_A^{(t)}=H_B^{(t)}$ for any one (and hence all) $t\geq 0$, then

(i) F is a regular local ring,

(ii) $\nu^*(I, E)=\nu^*(J, F)$, and

(iii) a standard base of I induces a standard base of J.

Proof. Let P be the maximal ideal of E. We can then write $g=V^q+w_1V^{q-1}+\cdots+w_q$ with all $w_i\in P$, $1\leq i\leq q$. This implies that F is local. By Prop 11, the assumption $H_A^{(t)}=H_B^{(t)}$ implies that (z_q mod M^2) is not a zero-divisor in $gr_M(A)$. In particular z_q is not in M^2 and hence w_q not in P^2. This implies that g is not in $(P, V)^2E[V]$

and hence F is regular. Now to prove (ii) and (iii), let T be the localization of $E[V]$ with respect to $(P, V)E[V]$, and let $L=(P, V)\,T$, the maximal ideal. V being an indeterminate, we have a natural injection $gr_P(E)\rightarrow gr_L(T)$ and $\mathrm{in}_L(IT)=\mathrm{in}_P(I)gr_L(T)$. This shows that every standard base of I induces a standard base of IT and $\nu^*(IT, T)=\nu^*(I, E)$. Let $S=T/IT$. Then clearly $H_S^{(0)}=H_A^{(1)}$ and hence $H_S^{(0)}=H_B^{(1)}$ by the assumption of Lemma. As $B=S/gS$ and $F=T/gT$, it then follows by Lemma 7 that every standard base of IT induces a standard base of J and $\nu^*(J, F)=\nu^*(IT, T)$. There follow (ii) and (iii). Q.E.D.

Remark (12.2) If the equality holds instead of the first (resp. the second) inequality of (4.1), then the corresponding equality holds for ν^*'s. (cf (4.2).)

Proof. Let $(y_1, \cdots, y_s)$ be a system of elements in $O_{Z,x}$ which induces a regular system of parameters of $O_{D,x}$. Let x_0 be a generator of the principal ideal $I_{Z',g^{-1}(D),x'}$. Let $R=O_{Z',x'}$, $R_0=R/x_0R$ and $R_i=R/(x_0, y_1, \cdots, y_i)R$ for $1\leq i\leq s$. Let $A=O_{X',x'}$, $A_0=A/x_0A$ and $A_i=A/(x_0, y_1, \cdots, y_i)A$ for $1\leq i\leq s$. By Prop 5, if the equality holds instead of the first inequality of (4.1), then $H_A^{(0)}=H_{A_0}^{(1)}$ and $H_{A_i}^{(0)}=H_{A_{i+1}}^{(1)}$ for all $i\geq 0$. Hence by applying Lemma 7 repeatedly $(s+1)$ times, we get $\nu^*(J, R)=\nu^*(J_0, R_0)$ and $\nu^*(J_i, R_i)=\nu^*(J_{i+1}, R_{i+1})$ for all $i\geq 0$, where $J=I_{Z',X',x'}$ and $J_i=JR_i$. Thus $\nu^*(J, R)=\nu^*(J_s, R_s)$ which is the first corresponding equality for ν^*'s. Next assume the equality in the place of the second inequality of (4.1). In the residually rational case, (iii) of Lemma 8 proves the corresponding equality of (4.2). If $k(x')\neq k(x)$, we repeat the technique of base extension of the type (9.1) and (9.2). The given extension $k(x)\rightarrow k(x')$ is attained by a finite number of successive simple extensions either transcendental, separable or purely inseparable, i.e., either case 1, case 2 or case 3 in the paragraph of (9.6). Therefore, by Lemma 8 and Prop 10, the above assumption implies that the second inequality of (9.5) can be replaced by an equality. In the purely inseparable case, this implies $\nu_{x'}^*(f^{-1}(x), g^{-1}(x))$

$=\nu^*_{\tilde{x}'}(\tilde{f}^{-1}(\tilde{x}),\ \tilde{g}^{-1}(\tilde{x}))$ by Lemma 12. In the other two cases, this equality is automatic due to the unramifiedness. Hence the question is reduced to the residually rational case. Q.E.D.

We are now interested in proving the inequalities of (4.2) (and TH II). This will be done not by the technique of successive base extensions of the type (9.1), but by making use of certain special properties of the additive subgroup $B_{g,x'}$ of the tangent space $T_{Z,x}$. As a matter of fact, TH IV will be simultaneously proven. (cf. §1.)

The basic assumptions (2.1) and (2.2) remain valid throughout. We have defined two additive subgroups $A_{X,x}$ and $B_{g,x'}$ of $T_{Z,x}$. Let $gr_x(Z, D)$ be the graded algebra of the fibre $N_{Z,D,x}$ of the normal bundle $N_{Z,D}\to D$ at the point $x\in D$. We shall identify $gr_x(Z, D)$ with its image by the monomorphism $gr_x(Z, D)\to gr_x(Z)$ associated with the natural epimorphism of vector spaces $T_{Z,x}\to N_{Z,D,x}$. $T_{D,x}$ is the kernel of this epimorphism and, by definition, is contained in $B_{g,x'}$. Hence the ring of invariants $U_{g,x'}$ of $B_{g,x'}$ in $gr_x(Z)$ is contained in $gr_x(Z,D)$. To make it explicit, let us choose

(13.1) a regular system of parameters of $O_{Z,x}$, say $(x_0, x_1,\cdots, x_r, y_1, \cdots, y_s)$, such that $(x_0, x_1, \cdots, x_r)O_{Z,x}=I_{Z,D,x}$ and $(x_0)O_{Z',x'}=I_{Z',g^{-1}(D),x'}$.

Let K be the residue field of $O_{Z,x}$. Let $X_i=\mathrm{in}_M(x_i)$ and $Y_j=\mathrm{in}_M(y_j)$, so that $gr_x(Z)=K[X_0, X_1, \cdots, X_r, Y_1, \cdots, Y_s]$ and $gr_x(Z, D)=K[X_0, X_1, \cdots, X_r]$. $U_{g,x'}$ is then the graded K-subalgebra of the polynomial ring $K[X]$ whose homogeneous part of degree d is $\{\varphi\in K[X]_d \mid \nu_{x'}(\varphi/X_0^d)\geq d\}$, where $g^{-1}(x)$ is identified with $\mathrm{Proj}(K[X])$ and x' is viewed as a point of $\mathrm{Proj}(K[X])$. Then, after a suitable permutation on $(x_1, \cdots, x_r)$ (not involving x_0), we may assume that

(13.2) $U_{g,x'}=K[\sigma_1, \sigma_2, \cdots, \sigma_e]$ where $\sigma_i=X_i^{q_i}+\sum_{j=i+1}^{r+1} c_{ij}X_j^{q_i}$ with $1\leq i\leq e$, $c_{ij}\in K$, $X_{r+1}=X_0$ and $1\leq q_1\leq\cdots\leq q_e$.

If the characteristic p of K is positive, then all the q_i are necessarily powers of p. If $p=0$, then $q_i=1$ for all i. (cf. [4].)

By the above assumption on x_0, the point x' belongs to the affine piece Spec$(K[T])$ of Proj$(K[X])$, where $T_i=X_i/X_0$, $1\leq i\leq r$. Let S be the localization of $K[T]$ at the point x', and N its maximal ideal. Let S_i be the localization of $K[T_{e-i+1}, \cdots, T_r]$ dominated by S, and N_i its maximal ideal, where $0\leq i\leq e$.

Lemma 14. We have a natural isomorphism $S_0/N_0 \xrightarrow{\sim} S/N$.

Proof. Let $\tau_j=\sigma_j/X_0^{q_j}$. Then S_{i+1} is a localization of $S_i[T_{e-i}]$ by a prime ideal containing N_i and τ_{e-i}. Since τ_{e-i} is a purely inseparable polynomial in T_{e-i}, such a prime ideal is maximal and unique. Hence we have an isomorphism from $S_i[T_{e-i}]/(N_i, \tau_{e-i})S_i[T_{e-i}]$ to $S_{i+1}/(N_i, \tau_{e-i})S_{i+1}$. Call this ring Q_{i+1}. Since $S/N_{i+1}S$ is regular, a regular system of parameters ω of S_{i+1} extends to a regular system of parameters (ω, z) of S. Then $\bar{S}=S/(z)S$ is unramified and flat over S_{i+1}. Hence $N_{i+1}^{\alpha}\subset N^{\alpha}\cap S_{i+1}\subset N^{\alpha}\bar{S}\cap S_{i+1}=N_{i+1}^{\alpha}\bar{S}\cap S_{i+1}=N_{i+1}^{\alpha}$, so that $N_{i+1}^{\alpha}=N^{\alpha}\cap S_{i+1}$. This implies $\nu_{N_{i+1}}(\tau_{e-i})=\nu_N(\tau_{e-i})$ which is deg $\sigma_{e-i}=q_{e-i}$ because $\sigma_{e-i}\in U_{g,x'}$. On the other hand, $\mathrm{rank}_{S_i/N_i}(Q_{i+1})=\deg\tau_{e-i}=q_{e-i}$. Therefore the residue field of Q_{i+1} should be a trivial extension of S_i/N_i. Q.E.D.

By this lemma, there exist $\beta_i\in S_0$ for each i, $1\leq i\leq e$, such that $T_i-\beta_i\in N$. Let us fix a regular system of parameters $\omega=(\omega_1, \cdots, \omega_{r'-e})$ of S_0, where $r'=r-d=\dim S$. It follows that

(14.1) (ω, z) is a regular system of parameters of S where $z_i=T_i-\beta_i$, $1\leq i\leq e$.

Let E be the localization of $S_0[\tau]$ dominated by S, where $\tau_i=\sigma_i/X_0^{q_i}$, $1\leq i\leq e$. Let $\Delta=\{b=(b_1, \cdots, b_e)\in Z_0^e \mid 0\leq b_i<q_i\}$. Let $\hat{S}_i$(resp. $\hat{E}$) be the completion of S_i(resp. E). Then by the structure theorem of Cohen, there exists and we fix once for all a subfield F of $\hat{S}_0$ such that

(14.2) $\hat{E}=F[[\omega, \tau]]$, $\hat{S}=F[[\omega, z]]$ and $\hat{S}$ is a free $\hat{E}$-module with a free base $\{z^b \mid b\in\Delta\}$.

Let I be the homogeneous ideal of the cone $C_{X,D,x}$ in $K[X]$, and let I' be the ideal of $f^{-1}(x)$ in $K[T]$, which is generated by φ/X_0^d for all $d \in Z_0$ and all $\varphi \in I_d$ (=the homogeneous part of degree d of I). By definition, $C_{X,x}$ is invariant by $B_{g,x'}$ if and only if the ideal of $C_{X,x}$ in $gr_x(Z)$ is generated by elements of $U_{g,x'}$. By the normal flatness assumption (2.1), the ideal of $C_{X,x}$ in $gr_x(Z)$ is generated by the ideal of $C_{X,D,x}$ in $gr_x(Z, D) = K[X]$. It follows that

(14.3) $C_{X,x}$ is invariant by $B_{g,x'}$ if and only if there exists a standard base of the ideal I consisting of elements in $U_{g,x'}$.

If $\varphi = (\varphi_1, \cdots, \varphi_m)$ is a standard base of I, then clearly I' is generated by $\psi = (\psi_1, \cdots, \psi_m)$ with $\psi_i = \varphi_i / X_0^{d_i}$, where $d_i = \deg \varphi_i$. We ask if φ can be so chosen that ψ is a standard base of I'.

Lemma 15. Let $(\varphi_1, \cdots, \varphi_a)$ be a system of elements of $U_{g,x'}$, which is a standard base of the ideal it generates in the graded algebra $K[X]$. Let $\psi_i = \varphi_i / X_0^{d_i}$ with $d_i = \deg\, \varphi_i$, and let $\bar{\psi}_i = \mathrm{in}_N(\psi_i) \in gr_N(S) = gr_{x'}(g^{-1}(x))$. Then $(\bar{\psi}_1, \cdots, \bar{\psi}_a)$ is a standard base of the ideal it generates in $gr_N(S)$, and $\deg \bar{\psi}_i = d_i$ for all i.

Proof. Let K'' be an algebraically closed field containing K. Let S'' be any localization of $K''[T]$ which dominates S, and let N'' be the maximal ideal of S''. Let U'' be the graded K''-subalgebra of $K''[X]$ such that its homogeneous part of degree $d \geq 0$ is $\{\varphi \in K''[X]_d \mid \nu_{N''}(\varphi / X_0^d) \geq d\}$. Clearly $U'' \supset U_{g,x'}$ and hence $\varphi_i \in U''$ for all i. Since K'' is algebraically closed, U'' is generated by its linear homogeneous part. Hence we can find a free base $(X''_0, X''_1, \cdots, X''_r)$ of $K''[X]_1$ such that $X''_0 = X_0$ and $U'' = K''[X''_c, X''_{c+1}, \cdots, X''_r]$. $(0 < c \leq r)$ Let $T''_i = X''_i / X''_0$, $1 \leq i \leq r$. We have an isomorphism of graded K''-algebras α: $U'' \xrightarrow{\sim} K''[T''_c, \cdots, T''_r]$ such that $\alpha(X''_i) = T''_i$, $c \leq i \leq r$. We have $T''_i \in N''$ for $c \leq i \leq r$, and hence $(T''_c, \cdots, T''_r)$ extends to a regular system of parameters of S''. Hence we get a monomorphism of graded L-algebras β: $L[T''_c, \cdots, T''_r] \to gr_{N''}(S'')$ such that $\beta(T''_i) = \mathrm{in}_{N''}(T''_i)$ for $c \leq i \leq r$, where $L = S''/N''$ which is an extension of K'' in a natural way. By assumption, $(\varphi_1, \cdots, \varphi_a)$ is a standard base of the ideal it generates in $K[X]$. In short, we say that it has SB property in $K[X]$. Neither coefficient field extensions nor adjunctions of extra indeterminates do affect this property. Therefore, $(\varphi_1, \cdots, \varphi_a)$ has SB property in $K''[X]$ and hence in U''. Since $\psi_i = \alpha(\varphi_i)$, $(\psi_1, \cdots, \psi_a)$ has SB property in $K''[T''_c, \cdots, T''_r]$ and hence in $L[T''_c, \cdots, T''_r]$. As ψ_i is a homogeneous polynomial of degree d_i in T''_j, $c \leq j \leq r$, we see that $\beta(\psi_i) = \mathrm{in}_{N''}(\psi_i)$ for every i. As $gr_{N''}(S'')$ is a polynomial ring over $Im(\beta)$, $(\bar{\psi}''_1, \cdots, \bar{\psi}''_a)$ has SB property in $gr_{N''}(S'')$ where $\bar{\psi}''_i = \mathrm{in}_{N''}(\psi_i)$. We also see $\nu_{N''}(\psi_i) = \deg \bar{\psi}''_i = d_i$ for every i. The inclusion $S \rightarrow S''$ induces $N^d/N^{d+1} \rightarrow N''^d/N''^{d+1}$ for every $d \geq 0$ and hence a homomorphism of graded algebras λ: $gr_N(S) \rightarrow gr_{N''}(S'')$. For every $\psi \in S$, $\lambda(\mathrm{in}_N(\psi))$ is either zero or equal to $\mathrm{in}_{N''}(\psi)$. This second case holds if $\nu_N(\psi) = \nu_{N''}(\psi)$. Since $\sigma_i \in U_{g,x'}$, $d_i \leq \nu_N(\psi_i) \leq \nu_{N''}(\psi_i) = d_i$ and hence $\nu_N(\psi_i) = \nu_{N''}(\psi_i)$. Thus we have $\lambda(\bar{\psi}_i) = \bar{\psi}''_i$ for all i. As λ is a homomorphism of graded algebras, it is then easy to see that if $(\bar{\psi}''_1, \cdots, \bar{\psi}''_a)$ has SB property in $gr_{N''}(S'')$ then $(\bar{\psi}_1, \cdots, \bar{\psi}_a)$ has the same in $gr_N(S)$. Q.E.D.

In what follows, we will work with a fixed presentation of $U_{g,x'}$, as was given in (13.2). In particular, each $\varphi \in U_{g,x'}$ will be viewed as a polynomial in $\sigma = (\sigma_1, \cdots, \sigma_e)$. We define the *leading exponent symbol* $ex_\sigma(\varphi)$ for $\varphi \in U_{g,x'}$ to be the largest $A = (a_1, \cdots, a_e) \in Z_0^e$, in the lexicographical ordering, among those for which the coefficient of σ^A in the polynomial expression of φ is not zero. If H is an ideal in $K[\sigma]$, we define $ex_\sigma(H) = \{ex_\sigma(\varphi) \mid \varphi \in H\}$. Remark that if L is any field containing K then $ex_\sigma(H) = ex_\sigma(HL[\sigma])$.

Remark (16.1) In general, the field of (14.2) cannot be so chosen as to contain the subfield K of S_0. However, we have a monomorphism ε_0: $K \rightarrow F$ which, when followed by the natural homomorphism $F \rightarrow S_0/N_0$, yields the canonical homomorphism $K \rightarrow S_0/N_0$. It extends to a monomorphism ε: $K[\tau] \rightarrow F[\tau]$ defined by $\varepsilon(a) = \varepsilon_0(a)$ for $a \in K$ and $\varepsilon(\tau_j) = \tau_j$ for all j. Note that if φ is a homogeneous polynomial of degree d in $K[X]$ and if $\varphi \in U_{g,x'}$, then $\psi = \varphi/X_0^d \in K[\tau]$ and we have

$$\nu_{\hat{N}}(\psi - \varepsilon(\psi)) > d \quad \text{where } \hat{N} = N\hat{S}.$$

In fact, if $\varphi = \sum c_A \sigma^A$ where $c_A \in K$ and the summation extends to certain $A \in Z_0^e$, then $\psi = \sum c_A \tau^A$ and $\varepsilon(\psi) = \sum \varepsilon_0(c_A)\tau^A$. The above inequality then follows, because $c - \varepsilon_0(c) \in \hat{N}$ for all $c \in K$.

Remark (16.2) Let $\psi = \sum \psi(A, B, C)\tau^A z^B \omega^C$ with $\psi(A, B, C) \in F$, where the summation extends to all (A, B, C) with $A \in Z_0^e$, $B \in \Delta$ and $C \in Z_0^{r'-e}$. We will write Aq for $(a_1 q_1, \cdots, a_e q_e)$ if $A = (a_1, \cdots, a_e)$. Then we claim $\nu_{\hat{N}}(\psi) = \min_{A,B,C} \{|Aq| + |B| + |C| \mid \psi(A, B, C) \neq 0\}$, where $\hat{N} = N\hat{S}$. (Note that $\nu_{\hat{N}}(\psi) = \nu_N(\psi)$ if $\psi \in S$.) In fact, let v be this minimum. Clearly $\nu_{\hat{N}}(\psi) \geq v$ because $\nu_N(\tau_1) \geq q_i$ for all i. Let $(\bar{D}, \bar{C})$ be the largest one, in the lexicographical ordering, among those $(Aq + B, C)$ such that $|Aq| + |B| + |C| = v$ and $\psi(A, B, C) \neq 0$. Write $\psi = \sum h(D, C) z^D w^C$ with $h(D, C) \in F$, where the summation extends to all (D, C) with $D \in Z_0^e$ and $C \in Z_0^{r'-e}$. In view of (13.2) and (14.2), we have $\tau_i - z_i^{q_i} \in \hat{S}_0[[z_{i+1}, \cdots, z_e]] = F[[\omega, z_{i+1}, \cdots, z_e]]$. Therefore $h(\bar{D}, \bar{C}) = \sum_0 \psi(A, B, \bar{C})$ where $\sum_0$ dentes the summation for those (A, B) with $Aq + B = \bar{D}$. Notice, however, if B and B' are in Δ, then $Aq + B = A'q + B'$ implies $A = A'$ and $B = B'$. Hence $\sum_0$ does not have any more than one term $\psi(A, B, \bar{C})$, which is not zero by the choice of $(\bar{D}, \bar{C})$. So $h(\bar{D}, \bar{C}) \neq 0$ and, by (14.2), $\nu_{\hat{N}}(\psi) \leq |\bar{D}| + |\bar{C}| = v$.

Lemma 17. Let $\varphi_j \in K[X]_{d_j} \cap U_{g,x'}$ for $1 \leq j \leq a'$. Let $\psi_j = \varphi_j / X_0^j$. Let t'' be an integer > 0 and $b_j \in \hat{S}$, $1 \leq j \leq a'$, such that $\nu_{\hat{N}}(b_j) \geq t'' - d_j$ for all j and $\nu_{\hat{N}}(\sum_{j=1}^{a'} b_j \psi_j) > t''$. Then there exist $c_{kj} \in K[X]_{d(k)-d_j} \cap U_{g,x'}$ and $b'_k \in \hat{N}^{t''-d(k)}$ for a finite set of indices k, where $d(k) \in Z_0$, such that $\sum_{j=1}^{a'} c_{kj} \varphi_j = 0$ for all k and $\nu_{\hat{N}}(b_j - b''_j) > t'' - d_j$ for all j, where $b''_j = \sum_k (c_{kj} / X_0^{d(k)-d_j}) b'_k$. In particular $\sum_{j=1}^{a'} b''_j \psi_j = 0$.

Proof. Write $b_j = \sum b_j(A', B, C)\tau^{A'} z^B \omega^C$ with $b_j(A', B, C) \in F$ and $\sum$ extending to all (A', B, C) such that $A' \in Z_0^e$, $B \in \Delta$ and $C \in Z_0^{r'-e}$. By (16.2), the assumptions imply that $b_j(A', B, C) = 0$ if $|A'q| + |B| + |C| < t'' - d_j$ and that

$$\nu_{\hat{N}}(\textstyle\sum_{**} b_j(A', B, C)\psi_j \tau^{A'} z^B \omega^C) > t''$$

where $\sum_{**}$ denotes the summation for all those (j, A', B, C) such that

$1 \leq j \leq a'$ and $|A'q|+|B|+|C|=t''-d_j$. Hence, by (16.1),

$$\nu_{\hat{N}}(\sum_{**} b_j(A', B, C)\varepsilon(\psi_j)\tau^{A'}z^B\omega^C) > t''$$

But $\varepsilon(\psi_j)$ is of the form $\sum c_{A''}\tau^{A''}$ where $c_{A''} \in F$ and the summation extends to those $A'' \in Z_0^e$ with $|A''q|=d_j$. Hence, by (16.2), the above inequality implies that

$$\sum_{**} b_j(A', B, C)\varepsilon(\psi_j)\tau^{A'}z^B\omega^C = 0$$

By (14.2), this implies

$$\sum^{(B)} b_j(A', B, C)\varepsilon(\psi_j)\tau^{A'}w^C = 0 \text{ for all } B \in \Delta$$

where $\sum^{(B)}$ denotes the summation for all (j, A', C) with $1 \leq j \leq a'$ and $|A'q|+|B|+|C|=t''-d_j$. Let $\{u_t\}_{t \in \Gamma}$ be a free base of F as $\varepsilon_0(K)$-module, and write

$$b_j(A', B, C) = \sum_{t \in \Gamma} \varepsilon_0(b_j(A', B, C, t))u_t$$

where $b_j(A', B, C, t) \in K$. Then, in view of the preceeding equality,

$$\sum_{t \in \Gamma} u_t \sum^{(B)} \varepsilon_0(b_j(A', B, C, t))\varepsilon(\psi_j)\tau^{A'}\omega^C = 0$$

This implies $\sum_{j=1}^{a'} \sum^{(j,B,C)} \varepsilon(b_j(A', B, C, t)\psi_j\tau^{A'}) = 0$ for all (B, C, t), where $\sum^{(j,B,C)}$ is the summation for all $A' \in Z_0^e$ with $|A'q|+|B|+|C|=t''-d_j$. Hence

$$\sum_{j=1}^{a'} \sum^{(j,B,C)} b_j(A', B, C, t)\psi_j\tau^{A'} = 0$$

and so $\sum_{j=1}^{a'} \sum^{(j,B,C)} b_j(A', B, C, t)\varphi_j\sigma^{A'} = 0$ for all (B, C, t). Let us write k for (B, C, t) for short. Then let $d(k)=t''-|B|-|C|$ and $c_{kj}=\sum^{(j,B,C)} b_j(A', B, C, t)\sigma^{A'}$ and $b'_k=u_t z^B\omega^C$. Note that $c_{kj}=b'_k=0$ except for finitely many k's. Now the equalities $\sum_{j=1}^{a'} c_{kj}\varphi=0$ are clear. Moreover, since $b_j(A', B, C, t)-\varepsilon_0(b_j(A', B, C, t)) \in \hat{N}$ for all (j, A', B, C, t) by (16.1), we get $\nu_{\hat{N}}(b_j-b''_j) > t''-d_j$ for every j. Q.E.D.

Lemma 18. Let $(\varphi_1, \cdots, \varphi_a)$ be a system of homogeneous elements of $K[X]$. Let $d_i = \deg \varphi_i$, and $\psi_i = \varphi_i / X_0^{d_i}$. Assume

(i) $\nu_N(\psi_i) \geq d_i$, i.e., $\varphi_i \in U_{g,x'}$ for all $i < a$, and

(ii) $\nu_N(\psi_a - \sum_{j=1}^{a-1} b_j \psi_j) \geq t$ with an integer $t > 0$ and $b_j \in S$.

Then there exist $c_j \in K[X]_{d_a - d_j}$ such that if $\varphi''_a = \varphi_a - \sum_{j=1}^{a-1} c_j \varphi_j$ then $\nu_N(\varphi''_a / X_0^{d_a}) \geq t$. In particular,

(1) if $t = d_a$ then $\varphi''_a \in U_{g,x'}$ and

(2) if $t > d_a$ then $\varphi_a = \sum_{j=1}^{a-1} c_j \varphi_j$

Proof. Each element $\varphi \in K[X]$ will be written in the form $\sum \varphi(A, B) \sigma^A Y^B$ where $Y = (X_1, \cdots, X_e)$, $\varphi(A, B) \in K[X_{e+1}, \cdots, X_{r+1}]$ and the summation extends to all (A, B) with $A \in Z_0^e$ and $B \in \varDelta$. (Recall $X_{r+1} = X_0$.) The existence and uniqueness of such an expression are due to (13.2). By a lexicographical descending induction, it can be easily shown that there exist $c_j \in K[X]_{d_a - d_j}$ such that $(\varphi_a - \sum_{j=1}^{a-1} c_j \varphi_j)(A, B) = 0$ for all (A, B) with $A \in \Lambda = ex_\sigma((\varphi_1, \cdots, \varphi_{a-1}) K[\sigma])$. Hence we shall assume that $\varphi_a(A, B) = 0$ for all (A, B) with $A \in \Lambda$. We claim that under this additional assumption, $\varphi''_a = \varphi_a$ has the property of Lemma 18. Let $\psi_a(A, B) = \varphi_a(A, B) / X_0^{d(A,B)}$ with $d(A, B) = \deg \varphi_a(A, B)$, and write $\psi_a = \sum_* \psi_a(A, B) \tau^A V^B$ where $V = (T_1, \cdots, T_e)$ and $\sum_*$ symbolizes the summation for all (A, B) with $A \in Z_0^e - \Lambda$ and $B \in \varDelta$. With z of (14.1), we substitute $z + \beta$ for V and obtain $\psi_a = \sum_* g(A, B) \tau^A z^B$ with $g(A,B) \in S_0$. Here it should be noted that the range of the summation is not affected. In view of (14.2), we can write $g(A, B) = \sum g(A, B, C) \omega^C$ with $g(A, B, C) \in F$, where the summation extends to all $C \in Z_0^{r'-e}$. Let $t' = \nu_N(\psi_a)$. Then by (ii), we have $\nu_N(\sum_{j=1}^{a-1} b_j \psi_j) \geq \min(t, t')$. Therefore, by Lemma 17, we may assume that $\nu_N(b_j) \geq \min(t, t') - d_j$ for all j. Let us write $b_j = \sum_{**} b_j(A, B, C) \tau^A z^B \omega^C$, where $b_j(A, B, C) \in F$ and $\sum_{**}$ denotes the summation for all (A, B, C) with $A \in Z_0^e$, $B \in \varDelta$ and $C \in Z_0^{r'-e}$. Suppose $t' < t$. Then, in view of (16.2), (ii) implies

$$\nu_N(\sum' b_j(A'', B, C) \psi_j \tau^{A'} z^B \omega^C - \sum'' g(A, B, C) \tau^A z^B \omega^C) > t'$$

where $\sum'$ (resp. $\sum''$) denotes the summation for all (j, A', B, C) (resp. all (A, B, C)) with $1 \leq j < a$ and $|A'q| + |B| + |C| = t' - d_j$ (resp. with $|Aq| + |B| + |C| = t'$). It then follows that

$$\sum' b_j(A', B, C)\varepsilon(\psi_j)\tau^{A'}z^B\omega^C - \sum'' g(A, B, C)\tau^A z^B \omega^C = 0$$

Pick any (B, C) for which there exists at least one A with $|Aq| + |B| + |C| = t'$ and $g(A, B, C) \neq 0$. Then with this (B, C),

$$\sum'^{*} b_j(A', B, C)\varepsilon(\psi_j)\tau^{A'} = \sum''^{*} g(A, B, C)\tau^A$$

where $\sum'^{*}$ (resp. $\sum''^{*}$) denotes the summation for those (j, A') (resp. those A) with $1 \leq j < a$ and $|A'q| + |B| + |C| = t' - d_j$ (resp. with $|Aq| + |B| + |C| = t'$). This equality shows that there exists at least one (A, B, C) with $g(A, B, C) \neq 0$ and $A \in ex_\tau((\varepsilon(\psi_1), \cdots, \varepsilon(\psi_{a-1}))F[\tau]) = ex_\tau((\varepsilon(\psi_1), \cdots, \varepsilon(\psi_{a-1}))\varepsilon(K[\tau]) = ex_\sigma((\varphi_1, \cdots, \varphi_{a-1})K[\sigma]) = \Lambda$. Then $g(A, B) \neq 0$ and, since it is an S_0-linear combination of certain $\psi_a(A, B'')$ with the same A, there exists at least one (A, B'') with $\psi_a(A, B'') \neq 0$ and $A \in \Lambda$. But $\psi_a(A, B'') \neq 0$ if and only if $\varphi_a(A, B'') \neq 0$. This contradicts the assumption we made on φ_a and hence proves $t' \geq t$. This means that $\varphi_a'' = \varphi_a$ has the required property. Now (1) follows the definition of $U_{g,x'}$. As for (2), since φ_a'' is homogeneous of degree d_a, $\nu_N(\varphi_a''/X_0^{d_a}) > d_a$ is possible only if $\varphi_a'' = 0$. (See Lemma 15 for instance.) Q.E.D.

Recall that $N_{Z,D,x} = \mathrm{spec}(K[X])$, $I =$ the homogeneous ideal of $C_{X,D,x}$ in $K[X]$, Spec $(K[T]) =$ an affine neighborhood of x' in $g^{-1}(x)$ and $I' =$ the ideal of $f^{-1}(x)$ in $K[T]$. Therefore the following corollary implies that if $\nu_{x'}^*(f^{-1}(x), g^{-1}(x)) \geq \nu^*(C_{X,D,x'} N_{Z,D,x})$ then $C_{X,x}$ is invariant by the subgroup $B_{g,x'}$ of $T_{Z,x}$.

Corollary (18.1) If $\nu^*(I, K[X])$, then there exists a standard base of I consisting of elements in $K[\sigma]$ and moreover $\nu^*(I'S, S) = \nu^*(I, K[X])$.

Proof. Let us pick any standard base $(\varphi_1, \cdots, \varphi_m)$ of I. Then I'

is generated by $\psi_i=\varphi_i/X_0^{d_i}$ with $d_i=\deg\varphi_i$, $1\leq i\leq m$. Let a be the largest integer, >0 and $\leq m+1$, such that $\varphi_i\in U_{g,x'}=K[\sigma]$ for all $i<a$. If $a=m+1$, there is nothing to prove. Assume $a\leq m$. By Lemma 15, we have $\nu_N(\psi_i)=d_i$ for all $i<a$ and $(\psi_1,\cdots,\psi_{a-1})$ is a standard base of the ideal it generates in S. In view of the assumption, this shows $\nu^*(I'S, S)_i=\nu^*(I, K[X])_i$ for all $i<a$, where $\nu^*(\ )_i$ denotes the i th number in the sequence $\nu^*(\ )$, and moreover that $(\psi_1,\cdots,\psi_{a-1})$ extends to a standard base of $I'S$. Hence there exist $b_j\in S$ such that $\nu_N(\psi_a-\sum_{j=1}^{a-1}b_j\psi_j)\geq\nu^*(I'S, S)_a\geq\nu^*(I, K[X])_a=\deg\varphi_a=d_a$. Therefore, by Lemma 16, there exist $c_j\in K[X]_{d_a-d_j}$ such that $\varphi_a-\sum_{j=1}^{a-1}c_j\varphi_j\in U_{g,x'}$. This element of $U_{g,x'}$ can replace φ_a from the beginning. By repeating this process, we get a standard base of I consisting of elements in $U_{g,x'}$. Finally the asserted equality of the corollary follows by Lemma 15. Q.E.D.

The assumption (2.1) implies that there exists

(19.1) a standard base $(f_1,\cdots,f_m)$ of $I_{Z,X,x}$ such that $f_i\in(I_{Z,D,x})^{d_i}$ for all i, where $d_i=\nu_x(f_i)=\nu_x^*(X, Z)_i$.

Recall that we have chosen a regular system of parameters (x, y) of $O_{Z,x}$ as is given in (13.1). For each f_i of (19.1), we let $g_i=f_i/x_0^{d_i}$. It is known that $(g_1,\cdots,g_m)$ generates the ideal $I_{Z',X',x'}$. We are now interested in the question when and how $(g_1,\cdots,g_m)$ can be modified to yield a standard base of $I_{Z',X',x'}$. In what follows, we will use the following notation:

(19.2) $$R=O_{Z,x},\ M=M_{Z,x},\ R'=O_{Z',x'}\ \text{and}\ M'=M_{Z',x'}$$

The notation of the paragraphs of (13.1), (13.2), (14.1) and (14.2) will be also used.

Lemma 20. Let $\varphi_i=\mathrm{in}_M(f_i)$, $1\leq i\leq m$. Let a be an integer, $1\leq a\leq m$, and assume that for all $i<a$

(i) $\nu_{x'}^*(X', Z')_i=\nu_x^*(X, Z)_i$

(ii) $\varphi_i\in U_{g,x'}$ and

(iii) there exist $w'_i \in (x_0, y)(g_1, \cdots, g_{i-1})O_{Z',x'}$ such that $\nu_{x'}(g''_i) = d_i$ with $g''_i = g_i - w'_i$.

Assume moreover

(iv) $\nu^*_{x'}(X', Z')_a \geq \nu^*_x(X, Z)_a$

Then we can find $w_a \in \sum_{i=1}^{a-1} f_i(I_{Z,D,x})^{d_a-d_i}$ and $w'_a \in (x_0, y)(g_1, \cdots, g_{a-1})O_{Z',x'}$ such that

(1) $\nu_x(f''_a) = d_a$ and $\mathrm{in}_M(f''_a) \in U_{g,x'}$ where $f''_a = f_a - w_a$ and

(2) $\nu_{x'}(g''_a) = d_a$ with $g''_a = g_a - (w_a/x_0^{d_a}) - w'_a$.

Moreover we have

(3) $\nu^*_{x'}(X', Z')_a = \nu^*_x(X, Z)_a$

Proof. By (19.1) we have $\varphi_i \in K[X]$ for all i. Let $\psi_i = \varphi_i / X_0^{d_i} \in S$. We have $S = R'/(x_0, y)R'$ and $\psi_i = (g_i \bmod (x_0, y)R')$ for all i. Then by (iii), $\psi_i = (g''_i \bmod (x_0, y)R')$ for all $i < a$. By Lemma 15, (ii) implies that $(\bar{\psi}_1, \cdots, \bar{\psi}_{a-1})$ is a standard base of the ideal it generates in $gr_N(S)$ where $\bar{\psi}_i = \mathrm{in}_N(\psi_i)$. Therefore, by (i), (iii) and (iv), $(\bar{g}''_1, \cdots, \bar{g}''_{a-1})$ with $\bar{g}''_i = \mathrm{in}_{M'}(g''_i)$ is a standard base of the ideal it generates in $gr_{x'}(Z')$. Hence by (iv), there exist $h_i \in R'$ such that

$$\nu_{M'}\Big(g_a - \sum_{i=1}^{a-1} h_i g_i''\Big) \geq d_a \tag{20.1}$$

Let $b_i = (h_i \bmod (x_0, y)R')$. It then follows that

$$\nu_N\Big(\psi_a - \sum_{i=1}^{a-1} b_i \psi_i\Big) \geq d_a \tag{20.2}$$

Hence, by Lemma 18, there exist $c_j \in K[X]_{d_a-d_j}$ such that $\varphi''_a = \varphi_a - \sum_{j=1}^{a-1} c_j \varphi_j \in U_{g,x'}$. There exist $C_j \in (I_{Z,D,x})^{d_a-d_j}$ such that $c_l = (C_j \bmod M^{d_a-d_j+1})$. Let $w_a = \sum_{j=1}^{a-1} C_j f_j$, which belongs to $\sum_{i=1}^{a-1} f_i(I_{Z,D,x})^{d_a-d_i}$. With this w_a, (1) is clear. Next, to find w'_a, we replace f_a by $f''_a = f_a - w_a$ and assume $\varphi_a \in U_{g,x'}$. Under this assumption, we shall prove that

(20.3) we can choose h_i of (20.1) from $(x_0, y)R'$.

Given h_i of (20.1), define b_i as before. Let $d'=\min_{1\leq i\leq a-1}\{\nu_N(b_i)+d_i\}$. If $d'\geq d_a$, then pick $h_{i0}\in R'$ such that $\nu_{M'}(h_{i0})\geq d_a-d_i$ and $b_i=(h_{i0} \bmod (x_0, y)R')$. Then (20.1) is not affected if we replace h_i by h_i-h_{i0} for all i. So (20.3) is valid. Now assume $d'<d_a$. We shall then find h_{i0} in such a way that the replacement of h_i by h_i-h_{i0} increase d' without affecting (20.1). Write $b_i=\sum b_i(A, B, C)\tau^A z^B \omega^C$ with $b_i(A, B, C)\in F$, where the summation is taken for all (A, B, C) with $A\in Z_0^e$, $B\in\Delta$ and $C\in Z_0^{r'-e}$ Since $\varphi_a\in U_{g,x'}$, $\nu_N(\psi_a)\geq d_a$. Hence (20.2) implies

$$\nu_N(\textstyle\sum_\Lambda b_i(A, B, C)\psi_i\tau^A z^B\omega^C)>d'$$

where $\sum_\Lambda$ denotes the summation for all $(i, A, B, C)\in\Lambda=\{(i, A, B, C)\mid 1\leq i<a, |Aq|+|B|+|C|=d'-d_i\}$. By (16.1) and (16.2), we get

$$\textstyle\sum_\Lambda b_i(A, B, C)\varepsilon(\psi_i)\tau^A z^B\omega^C=0$$

This implies that if $\Lambda(B)=\{(i, A, C)\mid(i, A, B, C)\in\Lambda\}$

$$\textstyle\sum_{\Lambda(B)} b_i(A, B, C)\varepsilon(\psi_i)\tau^A\omega^C=0 \text{ for all } B\in\Delta.$$

Let $\{u_t\}_{t\in\Gamma}$ be a free base of F as $\varepsilon(K)$-module, and write

$$b_i(A, B, C)=\textstyle\sum_{t\in\Gamma}\varepsilon(b_i(A, B, C, t))\,u_t$$

with $b_i(A, B, C, t)\in K$. Since all $\psi_i\in K[\tau]$ for $i<a$, we get

$$\textstyle\sum_{\Lambda(B)} b_i(A, B, C, t)\psi_i\tau^A\omega^C=0 \text{ for all } (B, t)\in\Delta\times\Gamma.$$

This implies that if $\Lambda(B, C)=\{(i, A)\mid(i, A, B, C)\in\Lambda\}$

$$\textstyle\sum_{\Lambda(B,C)} b_i(A, B, C, t)\varphi_i\sigma^A=0 \tag{20.4}$$

for all $(B, C, t)\in\Delta\times Z_0^{r'-e}\times\Gamma$. Pick $p_i(A, B, C, t)\in R$ such that $b_i(A, B, C, t)=(p_i(A, B, C, t) \bmod M)$. In what follows, P will denote $I_{Z,D,x}$. Let us pick $s_i\in P^{q_i}$ such that $\sigma_i=(s_i \bmod M^{q_i+1})$, $1\leq i\leq e$, and let

(20.5) $$p(B, C, t)=\sum_{A(B,C)} p_i(A, B, C, t) f_i s^A$$

Let us fix (B, C, t) arbitrarily for a while, and let $d''=d'-|B|-|C|$. Clearly $p(B, C, t)\in P^{d''}$ and by (20.4), $p(B, C, t)\in M^{d''+1}$. Hence $p(B, C, t)\in MP^{d''}$. Let us write I for $I_{Z,X,x}$. Since $f_i\in I$ for all i, we have

(20.6) $$p(B, C, t)\in MP^{d''}\cap I$$

By the normal flatness (2.1), we have $\mathrm{in}_P(I)=(\Phi_1, \cdots, \Phi_m)\, gr_P(R)$ with $\Phi_i=\mathrm{in}_P(f_i)$ and $M gr_P(R)\cap \mathrm{in}_P(I)=M\ \mathrm{in}_P(I)$. (cf. [3] Ch II, Lemma 7.) Therefore, (20.6) implies that there exist $p_j(B, C, t)\in MP^{d''-d_j}$ such that $p(B, C, t)-\sum_{j=1}^{a-1} p_j(B, C, t) f_j$ belongs to $P^{d''+1}$. Hence, again by the above fact about $\mathrm{in}_P(I)$, we find $q_j(B, C, t)\in P^{d''-d_j+1}$ such that

$$p(B, C, t)=\sum_{j=1}^{a-1} p_j(B, C, t) f_j+\sum_{j=1}^{m} q_j(B, C, t) f_j$$

Let $p'(B, C, t)=p(B, C, t)/x_0^{d''}$, $p_j'(B, C, t)=p_j(B, C, t)/x_0^{d''-d_j}$ and $q_j'(B, C, t)=q_j(B, C, t)/x_0^{d''-d_j}$. Then $p_j'(B, C, t)\in MR'=(x_0, y)R'$ and $q_j'(B, C, t)\in PR'=(x_0)R'$. Hence

$$\begin{aligned} p'(B, C, t)&=\sum_{j=1}^{a-1} p_j'(B, C, t)\, g_j+\sum_{j=1}^{m} q_j'(B, C, t)\, g_j \\ &\in (x_0, y)(g_1, \cdots, g_m)R' \\ &\subset (x_0, y)\{(g_1'', \cdots, g_{a-1}'')\, R'+M'^{d_a}\} \end{aligned}$$

Hence there exists $p_j''(B, C, t)\in (x_0, y)\, R'$ such that

(20.7) $p'(B, C, t)-p''(B, C, t)\in (x_0, y)\, M'^{d_a}$, where $p''(B, C, t)=\sum_{j=1}^{a-1} p_j''(B, C, t)\, g_j''$.

Let $\hat{R}'$ be the completion of R', and choose representatives $\tilde{u}_t$, $\tilde{z}_i$, $\tilde{\omega}_j$ in $\hat{R}'$ of the elements u_t, z_i, ω_j of $\hat{S}$, where $t\in\Gamma$, $1\leq i\leq e$, $1\leq j\leq r'-e$. Let $\Lambda(i)=\{(A, B, C)\mid (i, A, B, C)\in\Lambda\}$ and

(20.8) $$h_i'=\sum_{t\in\Gamma} \tilde{u}_t \sum_{\Lambda(i)} p_i(A, B, C, t)\, \tilde{\tau}^A \tilde{z}^B \tilde{\omega}^C$$

where $\tilde{\tau}_j = s_j/x_0^{q_j}$, $1 \leq j \leq e$. Then, in view of (iii), we can find $h_i^* \in R'$, $1 \leq i < a$, such that

$$(20.9) \qquad h_i^* \equiv h_i' \bmod (x_0, y)R' \text{ and } \sum_{i=1}^{a-1} h_i' g_i = \sum_{i=1}^{a-1} h_i^* g_i''$$

By (20.5), $p'(B, C, t) = \sum_{A(B,C)} p_i(A, B, C, t) g_i \tilde{\tau}^A$ and hence by (20.8)

$$(20.10) \qquad \sum_{i=1}^{a-1} h_i^* g_i'' = \sum_{t \in \Gamma} \tilde{u}_t \sum{}^* p'(B, C, t) \tilde{z}^B \tilde{\omega}^C$$

where $\sum^*$ denotes the summation for all (B, C) with non-empty $\Lambda(B, C)$. Let us define $h_{i0} = h_i^* - h_i''$ with

$$(20.11) \qquad h_i'' = \sum_{t \in \Gamma} \tilde{u}_t \sum{}^* p_i''(B, C, t) \tilde{z}^B \tilde{\omega}^C$$

which is clearly in $(x_0, y)R'$. By (20.9), $(h_i - h_{i0} \bmod (x_0, y)R') = (h_i - h_i' \bmod (x_0, y)R')$, which by (20.8) is equal to

$$b_i - \sum_{t \in \Gamma} u_t \sum_{A(i)} b_i(A, B, C, t) \tau^A z^B \omega^C$$

$$= b_i - \sum_{A(i)} b_i(A, B, C) \tau^A z^B \omega^C$$

$= \sum_{\Omega(i)} b_i(A, B, C) \tau^A z^B \omega^C$, where $\Omega(i) = \{(A, B, C) \mid |Aq| + |B| + |C| > d' - d_i\}$. This shows that, if $b_{i0} = (h_{i0} \bmod (x_0, y)R')$

$$(20.12) \qquad \min_{1 \leq i \leq a-1} \{\nu_N(b_i - b_{i0}) + d_i\} > d'$$

Moreover $g_a - \sum_{i=1}^{a-1} (h_i - h_{i0}) g_i'' = (g_a - \sum_{i=1}^{a-1} h_i g_i'') + \sum_{i=1}^{a-1} h_{i0} g_i''$. But $\sum_{i=1}^{a-1} h_{i0} g_i''$

$$= \sum_{i=1}^{a-1} h_i^* g_i'' - \sum_{i=1}^{a-1} h_i'' g_i''$$

$$= \sum_{t \in \Gamma} \tilde{u}_t \sum{}^* (p'(B, C, t) - p''(B, C, t)) \tilde{z}^B \tilde{\omega}^C$$

by (20.10), (20.11) and (20.7). Hence by (20.7), we get $\nu_N(\sum_{i=1}^{a-1} h_{i0} g_i'') \geq d_a$ and so

$$\nu_N(g_a - \sum_{i=1}^{a-1} (h_i - h_{i0}) g_i'') \geq d_a$$

This combined with (20.12) proves that by replacing h_i by $h_i - h_{i0}$, we can increase the number d' without affecting (20.1). We have seen that this suffices for (20.3). Now, with h_i of (20.3), let $w'_a = \sum_{i=1}^{a-1} h_i g''_i$ which is in $(x_0, y)(g''_1, \cdots, g''_{a-1}) R' = (x_0, y)(g_1, \cdots, g_{a-1}) R'$. We have shown that (2) holds with this w'_a. As before, $(\bar{g}''_1, \cdots, \bar{g}''_a)$ is a standard base of the ideal it generates in $gr_{M'}(R')$. Hence (iv) implies (3).

Q.E.D.

Having proven Lemma 20 as an inductive procedure, we can easily deduce the following proposition.

Proposition 21. Let the basic assumptions be those of (2.1) and (2.2), and choose (x, y) according to (13.1). If $\nu^*_{x'}(X', Z') \geq \nu^*_x(X, Z)$, then we can choose a standard base $(f_1, \cdots, f_m)$ of the ideal $I_{Z,X,x}$ such that for every i, $1 \leq i \leq m$,

(i) $f_i \in (I_{Z,D,x})^{d_i}$ with $d_i = \nu_x(f_i) = \nu^*_x(X, Z)_i$

(ii) $\mathrm{in}_x(f_i) \in U_{g,x'}$, and

(iii) if $g_j = f_j / x_0^{d_j}$, $1 \leq j \leq m$, then there exist $w'_i \in (x_0, y)(g_1, \cdots, g_{i-1}) O_{Z',x'}$ such that $\nu_{x'}(g''_i) = d_i$ with $g''_i = g_i - w'_i$.

Moreover, we then have

(iv) $(g''_1, \cdots, g''_m)$ is a standard base of $I_{Z',X',x'}$, and

(v) $\nu^*_{x'}(X', Z') = \nu^*_x(X, Z)$.

Here note that (iv) is a consequence of (i), (ii) and (iii). In fact, if $\varphi_i = \mathrm{in}_x(f_i)$ and $\psi_i = (g''_i \bmod (x_0, y) O_{Z',x'})$, then $\psi_i = \varphi_i / X_0^{d_i}$ for all i. Thanks to Lemma 15, $\deg \bar{\psi}_i = d_i$ and $(\bar{\psi}_1, \cdots, \bar{\psi}_m)$ is a standard base of the ideal it generates in $gr_{x'}(g^{-1}(x))$, where $\bar{\psi}_i = \mathrm{in}_{x'}(\psi_i)$. Hence by (iii), $(\bar{g}''_1, \cdots, \bar{g}''_m)$ is a standard base of the ideal it generates in $gr_{x'}(Z')$. Hence (iv) follows from the assumed inequality of the proposition. Finally (v) is immediate from (iii) and (iv).

We are now interested in the implication: $\nu^*_{x'}(X', Z') = \nu^*_x(X, Z) \Rightarrow H^{(d+1)}_{X',x'} = H^{(1)}_{X,x}$ as was asserted in *TH III*. To prove this, we need some elementary facts about regular sequences in a graded algebra.

Remark (22.1) Let $A = \bigoplus_{i \geq 0} A_i$ be a graded A_0-algebra, and let

$v=(v_0, v_1, \cdots, v_s)$ be a system of homogeneous elements in A, which are either of positive degrees or in the Jacobson radical of A_0. We then know that the following conditions are equivalent to each other ([1], Proposition (2.8)):

(i) v is a regular sequence for A,

(ii) we have $\mathrm{Ker}(\alpha)=Im(\beta)$ for the A-homomorphisms defined as follows:

$$\bigoplus_{0\leq i<j\leq s} Ae_{ij} \xrightarrow{\beta} \bigoplus_{0\leq i\leq s} Ae_i \xrightarrow{\alpha} A$$

where the left end (resp. the middle) is a free A-module with a free base $\{e_{ij}\}$ (resp. $\{e_i\}$) and α (resp. β) is defined by $\alpha(e_i)=v_i$(resp. $\beta(e_{ij}) =v_je_i-v_ie_j$).

Remark (22.2) Let $G=L[V]$ be a polynomial ring over a field L, naturally graded, and let H be a homogeneous ideal in G. Let $V_{(i)}$ denote the subsystem $(V_0, V_1, \cdots, V_i)$ of V. Then the following conditions are equivalent for each pair $(\bar{d}, s)\in Z_0^2$:

(i) $(V_{(s)})G\cap H_d\subset(V_{(s)})H$ for every $d\leq\bar{d}$.

(ii) $(V_{(i)})G\cap H_d\subset(V_{(i)})H$ for every $d\leq\bar{d}$ and $i\leq s$.

(iii) We have $\mathrm{Ker}(\alpha)_{d'}=Im(\beta)_{d'}$ for all $d'<\bar{d}$, where α and β are the homomorphisms of (22.1) defined for $A=G/H$ and $v_i=(V_i \bmod H)$. ($\bigoplus_i Ae_i$ is graded by $(\bigoplus_i Ae_i)_n=\bigoplus_i(A_n)e_i$.)

Proof. Pick any $\pi=\sum_{i=0}^{s-1} g_iV_i\in H_d$. Then, assuming (i), we shall prove $\pi\in(V_{(s-1)})H$. Note that if this is done, the implication (i)$\Rightarrow$(ii) follows immediately. For $d=0$, the assertion is clear. We use an induction on d. Say $d>0$. By (i) there exist $h_i\in H_{d-1}$, $0\leq i\leq s$, such that $\sum_{i=0}^{s-1} g_iV_i=\sum_{i=0}^{s-1} h_iV_i+h_sV_s$. Then clearly $h_s\in(V_{(s-1)})G$. So $h_s\in(V_{(s-1)})G\cap H_{d-1}\subset(V_{(s-1)})H$ by induction assumption on the degree. Hence $h_s=\sum_{i=0}^{s-1} h_i'V_i$ with $h_i'\in H$ and $\pi=\sum_{i=0}^{s-1}(h_i+h_i'V_s)V_i\in(V_{(s-1)})H$. Next we shall prove (i)$\Rightarrow$(iii). Take any $\lambda=\sum_{i=0}^{s} a_ie_i\in\mathrm{Ker}(\alpha)_{d'}-Im(\beta)_{d'}$, if this were not empty. Take any representative $g_i\in G_{d'}$ for each a_i, so that $\sum_{i=0}^{s} g_iV_i\in H$. Then by (i) there exist $h_i\in H_{d'}$ such that $\sum_{i=0}^{s}(g_i-h_i)V_i$

$=0$. G being a polynomial ring, we find $g_{ij}\in H_{d'-1}$ for $0\leq i, j\leq s$ such that $g_{ij}=-g_{ij}$, $g_{ii}=0$ for all i, j and $g_i-h_i=\sum_{j=0}^{s}g_{ij}V_j$. Let $b_{ij}=(g_{ij} \bmod H)$. Then $\beta(\bigoplus_{0\leq i\leq j\leq s}b_{ij}e_{ij})=\lambda$, which is absurd. The implication (iii)$\Rightarrow$(i) is immediate from the following diagram:

$$\begin{array}{ccccc}\bigoplus_{i,j}Ge_{ij} & \longrightarrow & \bigoplus_i Ge_i & \longrightarrow & G\\ \downarrow & & \downarrow & & \downarrow\\ \bigoplus_{i,j}Ae_{ij} & \xrightarrow{\beta} & \bigoplus_i A_{ei} & \xrightarrow{\alpha} & A\end{array}$$

where the vertical arrows are the natural homomorphisms with kernels generated by H and the horizontal arrows increase degrees by one. Also (ii)$\Rightarrow$(i) is trivial. Q.E.D.

Remark (22.3) The assumption being the same as that of (22.2), let us define

$$\bigoplus_{0\leq i<j\leq s}Ge_{ij}\xrightarrow{\beta'}\bigoplus_{0\leq i\leq s}Ge_i\xrightarrow{\alpha'}G$$

as before by letting $\beta'(e_{ij})=V_je_i-V_ie_j$ and $\alpha'(e_i)=V_i$. If the conditions (i)$-$(iii) of (22.2) are satisfied, then we have

(iv) Ker $(\alpha')\cap\bigoplus_i(H_{d'})e_i=\beta'(\bigoplus_{ij}(H_{d'-1})e_{ij})$
for all $d'\leq\bar{d}$.

Proof. Take any $\tau=\sum_{i=0}^{s}h_ie_i$ with $h_i\in H_{d'}$ such that $\alpha'(\tau)=0$. Let t be the smallest integer such that $h_i=0$ for all $i>t$. We shall prove $\tau\in\beta'(\bigoplus_{ij}(H_{d'-1})e_{ij})$ by induction on t. This is trivial for $t=0$. Say $t>0$. Then by (ii), $h_t\in(V_{(t-1)})G\cap H_{d'}\subset(V_{(t-1)})H$. Write $h_t=\sum_{i=0}^{t-1}h_{it}V_i$ with $h_{it}\in H_{d'-1}$. Then $\tau+\beta'(\sum_{i=0}^{t-1}h_{it}e_{it})=\sum_{i=0}^{t-1}h'_ie_i$ where $h'_i=h_i+h_{it}V_t\in H_{d'}$. Apply the induction assumption to this element. Q.D.E.

Back to the notation of (13.1), (13.2), (14.1), (14.2) and (19.2), let us put $X'_0=\mathrm{in}_{M'}(x_0)$ and $Y'_j=\mathrm{in}_{M'}(y_j)$, $1\leq j\leq s$.

Lemma 23. Under the assumptions of Proposition 21, let us assume that $\nu^*_{x'}(X', Z')=\nu^*_x(X, Z)$. Then (X'_0, Y') is a regular sequence for $gr_{x'}(X')$, where $gr_{x'}(X')$ is viewed as $gr_{x'}(Z')$-module in a natural way.

Proof. Let $G=gr_{x'}(Z')$ and $V=(V_0, \cdots, V_s)$ where $V_0=X'_0$ and $V_i=Y'_i$, $1\leq i\leq s$. We shall prove the following statement by induction on d.

(23.1. d) $(V)G\cap H_{d'}\subset(V)H$ for all $d'\leq d$, where $H=(\bar{g}''_1, \cdots, \bar{g}''_m)G$ with $\bar{g}''_i=\mathrm{in}_{M'}(g''_i)$.

This is trivial for $d=0$. Now take any $d>0$ and any $\bar{g}\in(V)G\cap H_d$. Write $\bar{g}=\sum_{i=1}^{m}\bar{h}_i\bar{g}''_i$ with $\bar{h}_i\in G_{d-d_i}$. Pick one representative $h_i\in M'^{d-d_i}$ for each $\bar{h}_i$ and let $b_i=(h_i \bmod (x_0, y)R')$. Since $\bar{g}\in(V)G$ and $\psi_i=(g''_i \bmod (x_0, y)R')$, $\nu_N(\sum_{i=1}^{m}b_i\psi_i)>d$. Moreover, clearly $\nu_N(b_i)\geq d-d_i$ for all i. Hence by Lemma 17, there exist $c_{kj}\in K[X]_{d(k)-d_j}\cap U_{g,x'}$ and $b'_k\in\hat{N}^{d-d(k)}$ such that

(23. i) $\quad\sum_{j=1}^{m}c_{kj}\varphi_j=0$ for all k and $\nu_{\hat{N}}(b_j-b''_j)>d-d_j$ where

$$b''_j=\textstyle\sum_k(c_{kj}/X_0^{d(k)-d_j})b'_k.$$

Pick a representative $C_{kj}\in P^{d(k)-d_j}$ for each c_{kj}, where $P=I_{Z,D,x}$, and let $C_k=\sum_{j=1}^{m}C_{kj}f_j$ which belongs to $P^{d(k)}$. (23. i) implies $C_k\in P^{d(k)}\cap M^{d(k)+1}=MP^{d(k)}$. By (19.1), $\mathrm{in}_P(I)=(\Phi_1, \cdots, \Phi_m)G$ with $\Phi_j=\mathrm{in}_P(f_i)$ and $Mgr_P(R)\cap\mathrm{in}_P(I)=M\,\mathrm{in}_P(I)$, where $I=I_{Z,X,x}$. Therefore, by the same argument used in the paragraph of (20.6)-(20.7), we get

(23. ii) $$C_k\in\sum_{j=1}^{m}MP^{d(k)-d_j}f_j$$

Pick a representative $B'_k\in\hat{M}'^{d-d(k)}$ for each b'_k, and let $h''_j=\sum_k(C_{kj}/x_0^{d(k)-d_j})B'_k$. Then $\sum_{j=1}^{m}h''_j g_j=\sum_k(C_k/x_0^{d(k)})B'_k$ which, by (23. ii), belongs to $M(g_1, \cdots, g_m)\hat{R}'$. As $M\hat{R}'=(x_0, y)\hat{R}'$,

(23. iii) $$\sum_{j=1}^{m}h''_j g''_j=\sum_{j=1}^{m}h''_j(g''_j-g_j)+\sum_{j=1}^{m}h''_j g_j\in(x_0, y)(g''_1, \cdots, g''_m)\hat{R}'$$

(See (iii) of Prop 21.) We have $b''_j=(h''_j \bmod (x_0, y)\hat{R}')$ and the inequality of (23. i) implies that there exist

(23. iv) $h_i^* \in (x_0, y)R'$, $1 \leq i \leq m$, such that $h_i - h_i'' - h_i^*$ belong to $\hat{M}'^{d-d_i+1}$ for all i.

This implies $\nu_{\hat{M}'}(\sum_{i=1}^{m}(h_i - h_i'' - h_i^*)g_i'') > d$. Hence if we write g^* for $\sum_{i=1}^{m}(h_i'' + h_i^*)g_i''$ then

(23. v) $$\bar{g} = \bar{g}^* \text{ with } \bar{g}^* = (g^* \bmod M'^{d+1})$$

Let us write y_0 for x_0. We have $g^* \in (y_0, y)(g_1'', \cdots, g_m'')R'$ by (23. iii) and (23.iv). Let us pick an expression $g^* = \sum_{j=0}^{s} g_j^* y_j$ with $g_j^* \in (g_1'', \cdots, g_m'')R'$ in such a way that the number $d^* = \min_{0 \leq j \leq s}\{\nu_{M'}(g_j^*)\}$ takes its maximum. Let $\bar{g}_j^* = (g_j^* \bmod M'^{d^*+1})$. If $d^* > d-1$, then $\bar{g}^* = 0$. If $d^* = d-1$, then $\bar{g}^* = \sum_{j=0}^{s} \bar{g}_j^* V_j$. In these two cases, we get $\bar{g}^* \in (V)H$, which is by (22. v) what we want for (23.1. d). Suppose $d^* < d-1$. Then we get $\sum_{j=0}^{s} \bar{g}_j^* V_j = 0$ because $\nu_{M'}(g^*) \geq d$. By induction assumption, (23.1. d) holds for $d' = d^* + 1$ and hence (iv) of (22.3) does for $d' = d^*$. This means that there exist $\bar{g}_{ij}^* \in H_{d^*-1}$, $0 \leq i < j \leq s$, such that $\bar{g}_j^* = \sum_{i=0}^{s} V_i \bar{g}_{ij}^*$ where $\bar{g}_{ij}^* = -\bar{g}_{ij}^*$ and $\bar{g}_{ii}^* = 0$ for all (i, j). For each (i, j) with $0 \leq i < j \leq s$, pick $g_{ij}^* \in M'^{d^*-1} \cap (g_1'', \cdots, g_m'')R'$ which represents $\bar{g}_{ij}^*$. Then $g^* = \sum_{j=0}^{s} g_j^* y_j = \sum_{j=0}^{s} g_j^* y_j - \sum_{0 \leq i,j \leq s} y_i g_{ij}^* y_j = \sum_{j=0}^{s} (g_j^* - \sum_{i=0}^{s} y_i g_{ij}^*) y_j$, where $g_{ij}^* = -g_{ji}^*$ for all (i, j) and $g_{ii}^* = 0$ for all i. But this contradicts the maximality of d^*, because $\bar{g}_j^* = \sum_{i=0}^{s} V_i \bar{g}_{ij}^*$ and hence $\nu_{M'}(g_j^* - \sum_{i=0}^{s} y_i g_{ij}^*) > d^*$ for all j. This completes the proof of (23.1.d) for all d. Now having (23.1. d) for all $d \geq 0$, we get (iii) of (22.2) for all $d' \geq 0$. Hence by (22.1), V is a regular sequence for G/H. This is what Lemma 23 asserts. Q.E.D.

Corollary (23.2) Under the assumptions of Lemma 23, we have $H_{X',x'}^{(t)} = H_{f^{-1}(x),x'}^{(t+s+1)}$ for all $t \geq 0$.

Let Q be a graded algebra over a field L. Let Q' be a graded free Q-module with a free base $\{\lambda_B\}_{B \in \Delta}$ where λ_B is homogeneous of degree

d_B for every $B \in \Delta$. Let J be a homogeneous ideal in Q. Then we

(24.1) $$\operatorname{rank}_L (Q'/JQ')_m = \sum_{B\in\Delta} \operatorname{rank}_L (Q/J)_{m-d_B}$$

for all m, where $(\ )_i$ denotes the homogeneous part of degree i. This equality is clear from the fact that $(Q'/JQ')_m$ admits a direct sum decomposition $\sum_{B\in\Delta} \lambda_B (Q/J)_{m-d_B}$.

Lemma 24. Under the assumptions of Lemma 23, we have $H^{(t+d+1)}_{f^{-1}(x),x'} = H^{(t)}_{C_{X,D,x}}$ for all $t \geq 0$.

Proof. Let $(f_1, \cdots, f_m)$ and $(g''_1, \cdots, g''_m)$ be the systems given in Prop 21. If $\varphi_i = \operatorname{in}_M(f_i)$, then $\varphi_i \in U_{g,x'}$ and $\psi_i = \varphi_i / X_0^{d_i} = (g''_i \bmod (x_0, y)R')$ for all i. Following the notation of (14.1) and (14.2), let $L = S/N$, $\bar{z}_i = \operatorname{in}_N(z_i)$ and $\bar{\tau}_i = \operatorname{in}_N(\tau_i)$ for $1 \leq i \leq e$. Also let $\bar{\omega}_j = \operatorname{in}_N(\omega_j)$, $1 \leq j \leq r'-e$, so that $gr_N(S) = L[\bar{z}, \bar{\omega}]$. Let $Q' = gr_N(S)$ and $Q = L[\bar{\tau}, \bar{\omega}]$. Then Q' is a free Q-module with a free base $\{\bar{z}^B\}_{B\in\Delta}$. The $\bar{\tau}_i$ are homogeneous of degrees q_i, and the natural grading in Q' induces a grading in Q by $Q_m = Q'_m \cap Q$. Let $\bar{\psi}_j = \operatorname{in}_N(\psi_j)$, which belongs to $L[\bar{\tau}]$ for all j. Let $J_0 = (\bar{\psi}_1, \cdots, \bar{\psi}_m) L[\bar{\tau}]$ and $J = J_0 Q$. By definition, $H^{(0)}_{f^{-1}(x),x'}(m) = \operatorname{rank}_L (Q'/JQ')_m$, which by (24.1), $= \sum_{B\in\Delta} \operatorname{rank}_L (Q/J)_{m-|B|} = \sum_{B\in\Delta} H^{(r'-e)}_{L[\bar{\tau}]/J_0}(m-|B|)$. This then is equal to $\sum_{B\in\Delta} H^{(r'-e)}_{K[\sigma]/I_0}(m-|B|)$ with $I_0 = (\varphi) K[\sigma]$, because, K being naturally imbedded in L, $\bar{\psi}_i$ is the image of φ_i by the K-homomorphism $K[\sigma] \to L[\bar{\tau}]$ which sends σ_i to $\bar{\tau}_i$ for all i. On the other hand, $\{Y^B\}_{B\in\Delta}$ with $Y = (X_1, \cdots, X_e)$ is a free base of $K[X]$ as a graded module overe $K[\sigma, X_{e+1}, \cdots, X_r, X_0]$. Hence for the same reason as above, $\sum_{B\in\Delta} H^{(r+1-e)}_{K[\sigma]/I_0}(m-|B|) = H^{(0)}_{K[X]/I_0 K[X]}(m) = H^{(0)}_{C_{X,D,x}}(m)$. As $d+1 = r+1-r'$ by definition, the equality of Lemma 24 follows. Q.E.D.

Results up to here include implicitly all the proofs of the four theorems stated in the beginning. We want to summarize and make them explicit by following their logics once again as follows:

Proof of TH I. Thanks to the assumption (2.1), as is explained in the paragraph immediately following Example (4.2), it is enough to prove the two inequalities of (4.1). The first of these two follows Prop 5 as was shown in Remark (5.2). The second inequality was proven in Lemma 8 in the case of $k(x)=k(x')$. As for the general case, it is reduced to the special case of the arguments given in the paragraphs of (9.1)-(9.6), Prop 10 and Remark (10.1). (As was explained there, the main difficulty for the general case is due to the possible inseparablity of the extension $k(x)\rightarrow k(x')$. Prop 10 is the key to overcome such a difficulty.)

Proof of TH II. By Prop 21 (especially (v)), $\nu_x^*(X, Z)<\nu_{x'}^*(X', Z')$ is impossible.

Proof of TH III. Suppose $H_{X',x'}^{(d+1)}=H_{X,x}^{(1)}$. Then, as was seen in the proof of TH I, all the equalities of (4.1) must hold. Then by Remark (12.2), the if-part of TH III is proven. Next assume $\nu_{x'}^*(X', Z')=\nu_x^*(X, Z)$. Then by Corollary (23.2), we get $H_{X',x'}^{(0)}=H_{f^{-1}(x),x'}^{(s+1)}$. Moreover by Lemma 24, $H_{f^{-1}(x),x'}^{(d+1)}=H_C^{(0)}$ with $C=C_{X,D,x}$. By the normal flatness (2.1), $H_C^{(s)}=H_{X,x}^{(0)}$. Hence $H_{X',x'}^{(d)}=H_{X,x}^{(0)}$.

Proof of TH IV. If $\nu_{x'}^*(X', Z')=\nu_x^*(X, Z)$, then by Prop 21 ((ii) especially) we have a standard base of the ideal of $C_{X,x}$ in $gr_x(Z)$, which consists of elements of $U_{g,x'}$. This means that $C_{X,x}$ is invariant by $B_{g,x'}$ as was seen in (14.3).

References

[1] M. Auslander-D. Buchsbaum, "codimension and multiplicity", Annals of Math. (**68**), 1958, pp. 625-657.

[2] B. Bennett, "On the characteristic functions of a local ring", to appear in Annals. of Math.

[3] H. Hironaka, "Resolution of singularities of an algebraic variety over a field of characteristic zero: I-II, "Annals of Math. (**79**), 1964, pp. 109-326.

[4] H. Hironaka, "Additive groups associated with points of a projective space", accepted by Annals of Math.

[5] M. Nagata, "Theory of multipliplicity in general local rings" Proc. Int. Symp., Tokyo-Nikko, 1955, pp. 191-226.

HARVARD UNIVERSITY