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Commutative rings which are locally Noetherian

By

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It is well known that if R is a Noetherian commutative ring, then R_M, the localization of R at each maximal ideal M, is also Noetherian. Any almost Dedekind domain[1)] which is not Dedekind provides an example for which the converse fails [1, App. 1, Ex. 1]. We shall say that a ring R is *locally Noetherian* provided R_M is Noetherian for each maximal ideal M of R. The goal of this paper is to characterize those locally Noetherian rings which are also Noetherian.

Throughout this paper R will denote a commutative ring with identity. Our notation and terminology are essentially that of [1] with the following exception: if S is a multiplicatively closed subset of R and if A is an ideal of R, then we shall denote by AR_S the extension of A to R_S.

1. The Characterization Theorem

Let A be any ideal of R and let S be the set of elements of R which are not zero divisors modulo A. Then S is multiplicatively closed and $A \cap S = \emptyset$. Any prime ideal containing A which is maximal with respect to missing S is called a *maximal prime divisor* of A. A prime ideal P of R is called a *prime divisor* of A if there is a multi-

1) If D is an integral domain with identity, then D is said to be an *almost Dedekind domain* if D_M is a Noetherian valuation ring for each maximal ideal M of D [1, p. 408].

plicatively closed subset S_1 of R such that $A \cap S_1 = \emptyset$ and PR_{S_1} is a maximal prime divisor of AR_{S_1} [2, p. 19]. In the sequel the term prime divisor will be used only in the above sense. It is clear that any prime ideal of R which is minimal with respect to the property of containing A is a prime divisor of A and such prime ideals are called *minimal prime divisors* of A.

In order to prove our main theorem, we require some preliminary results.

Lemma 1.1. *Let R be a ring which satisfies the ascending chain condition (a. c. c.) for prime ideals and suppose that each finitely generated ideal of R has only finitely many minimal prime divisors. If P is any proper prime ideal of R, then P is the unique minimal prime divisor for some finitely generated ideal A of R.*

Proof. Let P be a proper prime ideal of R and let $x \in P - \{0\}$. By assumption, (x) has only finitely many minimal prime divisors—say $P_1, \cdots, P_n$, and since $P \supseteq (x)$, $P \supseteq P_i$ for some i. Suppose that $P \neq P_i$ for $1 \leq i \leq r$, where either $r = n$ or $r = n-1$, and let $x_i \in P - P_i$. If $A = (x, x_1, \cdots, x_r)$, then A has only finitely many minimal prime divisors—say $P_1', \cdots, P_m'$. Since $P_i' \supseteq A \supseteq (x)$, P_i' contains a minimal prime divisor of (x). If $P_i' \supseteq P$, then P is a minimal prime divisor of A and $P_i' = P$, since $P_i' \supseteq P \supseteq A$. If $P_i' \supseteq P_j$ for some j, $1 \leq j \leq r$, then the containment is proper for we have $x_j \in P_i' - P_j$. But R satisfies *a. c. c.* for prime ideals so it follows that, after finitely many repetitions of the above procedure, we may obtain a finitely generated ideal A_1 having unique minimal prime divisor P.

Corollary 1.2. *Let R be a ring which satisfies the a. c. c. for prime ideals. Then R is Noetherian if and only if each finitely generated ideal A of R has only finitely many minimal prime divisors and $\sqrt{A}$ is finitely generated.*

Proof. By Lemma 1.1, if P is a proper prime ideal of R, then P is the unique minimal prime divisor for some finitely generated ideal A of R. Thus, $P=\sqrt{A}$ and P is finitely generated. It follows that R is Noetherian [1, p. 26]. The converse is clear.

Lemma 1.3. *If R is a locally Noetherian ring, then R satisfies a. c. c. on prime ideals.*

Proof. Let $P_1 \subseteq P_2 \subseteq \cdots$ be a chain of prime ideals of R and set $P=U_{i=1}^{\infty} P_i$. Then P is a prime ideal of R and R_p is Noetherian [2, 6.5 and 6.11]. Since $P_1 R_p \subseteq P_2 R_p \subseteq \cdots \subseteq PR_p$ is a chain of prime ideals of R_p, there is an integer n such that $PR_p = P_{n+i} R_p$ for each nonnegative integer i. Hence, $P=P_{n+i}$ for each nonnegative integer i and the lemma follows.

We are now able to prove our main theorem.

Theorem 1.4. *The following conditions are equivalent in a locally Noetherian ring R.*

(1) *Each finitely generated ideal of R may be expressed as a finite intersection of primary ideals of R.*

(2) *Each finitely generated ideal of R has only finitely many prime divisors.*

(3) *Each finitely generated ideal A of R has only finitely many minimal prime divisors and $\sqrt{A}$ is finitely generated.*

(4) *R is Noetherian.*

Proof. It is clear that $(4) \Rightarrow (1) \Rightarrow (2)$. That (3) and (4) are equivalent is an immediate consequence of Corollary 1.2 and Lemma 1.3. We now show that $(2) \Rightarrow (4)$. Thus, let P be a proper prime ideal of R. By Lemma 1.1 and Lemma 1.3, P is the unique minimal prime divisor of some finitely generated ideal A of R. If $p_1, \cdots, p_r \in R$ are such that $PR_p = (p_1, \cdots, p_r) R_p$, then P is the unique minimal prime divisor of $A_1 = A + (p_1, \cdots, p_r)$ and $A_1 R_p = PR_p$. Suppose that $P, P_1, \cdots, P_n$

are the prime divisors of A_1 and let $y_{i_1}, \cdots, y_{im_i} \in R$ be such that $PR_{p_i} = (y_{i1}, \cdots, y_{im_i}) R_{p_i}$. For $1 \leq i \leq n$, P_i is not a prime divisor of $A_2 = A_1 + \sum_{i=1}^{n} (y_{i1}, \cdots, y_{im_i})$, since $P_i R_{p_i}$ is not a prime divisor of $A_2 R_{p_i} = PR_{p_i}$. Suppose that Q is any proper prime ideal of R which contains A_2 and suppose that $Q \not\supseteq P_i$ for any i. But $Q \supseteq A_2 \supseteq A_1$, so $Q \supseteq P$ and it follows that PR_Q is the unique prime divisor of $A_1 R_Q$. Thus, $A_1 R_Q$ is PR_Q-primary [2, p. 20] and, consequently, if $(A_1 R_Q)^{ec}$ is the extension and contraction of $A_1 R_Q$ with respect to $(R_Q)_{PR_Q}$, then $A_1 R_Q = (A_1 R_Q)^{ec} = [A_1 (R_Q)_{PR_Q}]^c = (A_1 R_p)^c = (PR_Q)^c = PR_Q$ [2, p. 17]. But $A_1 R_Q \subseteq A_2 R_Q \subseteq PR_Q$ so we have $A_2 R_Q = PR_Q$. Hence, QR_Q is not a prime divisor of $A_2 R_Q$ and it follows that Q is not a prime divisor of A_2. Therefore, if $P, P'_1, \cdots, P'_m$ are the prime divisors of A_2, then for $1 \leq i \leq m$, $P'_i \supset P_j$ for some j, $1 \leq j \leq n$. Since R satisfies *a. c. c.* for prime ideals, a finite number of repetitions of the above procedure yields a finitely generated ideal A_3 such that $A_3 R_p = PR_p$ and P is the unique prime divisor of A_3. Then A_3 is P-primary [2, p. 20] so $A_3 R_p = PR_p$ implies that $A_3 = P$. Thus, P is finitely generated and the theorem follows.

In general a ring can satisfy any of the conditions (1), (2) and (3) of Theorem 1.4 without being Noetherian. In fact, if $X = \{X_i\}_{i=1}^{\infty}$ is a countable collection of indeterminates over the field K, then $K[X]$ is a non-Noetherian ring in which each of these conditions is satisfied. To see this, let $A = (f_1, \cdots, f_r)$ be a finitely generated ideal of $K[X]$. For some integer n we have $f_1, \cdots, f_r \in D_n$, where $D_n = K[X_1, \cdots, X_n]$. Since D_n is Noetherian, $A' = (f_1, \cdots, f_r) D_n$ has a finite irredundant primary representation in D_n—say $A' = Q_1 \cap \cdots \cap Q_m$, and $\sqrt{A'}$ is finitely generated. Hence, $A = A'[X]$ has the finite irredundant primary representation $A = Q_1[X] \cap \cdots Q_m[X]$ and $\sqrt{A} = (\sqrt{A'})[X]$ is finitely generated.

No two of the conditions (1), (2) and (3) are necessarily equivalent in an arbitrary ring R. Any rank one nondiscrete valuation ring satisfies property (1) but not (3). For an example of a ring in which (2) holds but neither (1) nor (3) holds, consider any rank two valuation ring V. Property (1) holds only if principal ideals of V are primary, but if P is the minimal prime ideal of V and $x \in P-\{0\}$, (x) is not primary [1, p. 173]. If M is the maximal ideal of V and $y \in M-P$, then either $\sqrt{(y)}=M$ is not finitely generated or $\sqrt{(x)}=P$ is not finitely generated [1, p. 73].

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