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On the asymptotic behavior of eigenfunctions of second-order elliptic operators

By

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§1. Introduction and summary

The non-existence of positive eigenvalues with square integrable eigenfunctions of the Schrödinger operator $-\Delta+V$, where Δ is the Laplacian and V the multiplicative operator through the potential function $V(x)$, or some of its variants, has been investigated by many authors (see, e.g., [1]-[3], [5]-[16]). In studying this problem some authors assume, as seems physically natural, that $V(x)$ decreases to 0 at infinity.[1] But when a magnetic field is present so that the Schrödinger operator becomes $-(\partial_i+\sqrt{-1}\,b_i(x))(\partial_i+\sqrt{-1}\,b_i(x))+V$, where $\partial_i=\frac{\partial}{\partial x_i}$ and the usual summation convention is used here and in the sequel, it is not necessarily physically natural to assume that the vector potential $b(x)=(b_i(x))$ also vanishes at infinity. It might appear more natural to assume that the magnetic field obtained by taking the rotation of $b(x)$ (in the 3-dimensional case) diminishes at infinity. Our aim in the present paper may be said from the physical viewpoint

1) For the many-particle Schrödinger operator it cannot be expected that the potential decays uniformly at infinity. For the treatment of such potentials see Agmon [1] and Weidmann [14], [15].

to show the non-existence of positive eigenvalues of the Schrödinger operator with the magnetic field verifying the above-mentioned smallness condition at infinity.

More generally, we shall consider the eigenvalue problem associated with a second-order elliptic differential operator:

$$(1.1) \qquad [-(\partial_i+\sqrt{-1}b_i(x))a_{ij}(x)(\partial_j+\sqrt{-1}b_j(x))+c(x)]u=\lambda u,$$

where $(a_{ij}(x))$ is a positive definite matrix for each x (the precise conditions on the coefficients will be given later). We assume that $a_{ij}(x)$ tends to δ_{ij} (Kronecker's delta) as $|x|$ tends to ∞, though this is not absolutely necessary, and that $\partial_i b_j(x)-\partial_j b_i(x)$ and $c(x)$ tend to 0 as $|x|$ tends to ∞. Then our result will be a growth estimate at infinity of u and its derivatives, from which will follow the non-existence of positive eigenvalues to the eigenvalue problem (1.1).

Kato [7] considered the eigenfunction $u(x)$ as a function of $\tilde{x}=\frac{x}{|x|}$ with a parameter $r=|x|$ and quadratic functionals of $u(r\cdot)$ and its derivatives depending on r in the case when $a_{ij}\equiv\delta_{ij}$ and $b_i\equiv 0$. Investigating the asymptotic properties of these functionals by use of (1.1), he was able to show the non-existence of positive eigenvalues. The same goal was attained, on the other hand, by Roze [11] when $b_i\equiv 0$ through an extensive use of integration by parts, the starting step being to integrate by parts equation (1.1) multiplied by the function $|x|^{\alpha}x_i a_{ij}\partial_j u$ (this procedure in fact leads to Kato's functionals if one looks at the surface integrals obtained in its course). Our method may be said to be a compromise between them, introducing some relevant functionals by partial integration. One point which should be noted in our treatment may be that we try to regard the differential operation $\partial_i+\sqrt{-1}b_i$ as *one* entity as far as possible, not separating it unless necessary. What is needed then is an accumulation of shear computation and some ideas borrowed from Kato [7], for instance.

Here we shall list the notation which will be used freely in the sequel:

$x=(x_1, \cdots, x_n)$ is a position vector in R^n;

$|x|=(|x_1|^2+\cdots+|x_n|^2)^{1/2}$;

$\tilde{x}=\dfrac{x}{|x|} \qquad (|x|\neq 0)$;

$S_r=\{x\in R^n \mid |x|=r\}$ for $r>0$;

$B_{sr}=\{x\in R^n \mid s\leq|x|\leq r\}$ for $0<s<r$;

$E_r=\{x\in R^n \mid |x|>r\}$ for $r>0$;

$\partial_i=\dfrac{\partial}{\partial x_i} \qquad (i=1, \cdots, n)$;

$D_i=\partial_i+\sqrt{-1}b_i(x)$

$B_{ij}=B_{ij}(x)=D_iD_j-D_jD_i=\sqrt{-1}(\partial_i b_j(x)-\partial_j b_i(x))$;

$A=A(x)=(a_{ij}(x)) \qquad (i, j=1, \cdots, n)$;

$\langle f, g\rangle=f_i\bar{g}_i$ for $f=(f_1, \cdots, f_n)$ and $g=(g_1, \cdots, g_n)$;

$Du=(D_1u(x), \cdots, D_nu(x))$;

$ADu=(a_{1i}D_iu, \cdots, a_{ni}D_iu)$;

the dot $\cdot$ indicates the end of a differential operation as in $D_iu\cdot v=[D_iu(x)]v(x)$;

$\mathrm{Sp}\,A=\mathrm{Sp}\,A(x)=a_{ii}(x)$;

$\left[\int_{S_r}-\int_{S_s}\right]f\,dS=\int_{S_r}f\,dS-\int_{S_s}f\,dS$;

$\varepsilon_i(r)$ denotes a positive function for $r>0$ which tends to 0 as $r\to\infty \qquad (i=1, 2, \cdots)$;

L_2 denotes the class of square integrable functions, and thus $L_2(E_{R_0})$ is all L_2 functions over E_{R_0};

H_2 denotes the class of L_2 functions with distribution derivatives in L_2 up to the second order inclusive;

$L_{2,\mathrm{loc}}$ and $H_{2,\mathrm{loc}}$ denote the classes of locally L_2 and H_2 functions, respectively;

C^m denotes the class of m-times continuously differentiable functions.

Now let us state the conditions to be imposed on the coefficients of the differential operator appearing on the left side of (1.1).

We assume that there exists a positive constant R_0 *such that the following conditions are satisfied for* $|x|\geq R_0$:

(A1) *Each* $a_{ij}(x)$ *is a real-valued* C^1 *function and* $a_{ij}(x)=a_{ji}(x)$.

(A2) *There exist positive constants* C_1 *and* C_2 *such that* $0<C_1\leq C_2$ *and* $C_1|\xi|^2\leq a_{ij}(x)\xi_i\bar{\xi}_j\leq C_2|\xi|^2$ *for any complex n-vector* ξ.

(A3) $\partial_k a_{ij}(x)=o(|x|^{-1}) \qquad (|x|\to\infty)$.

(A4) $a_{ij}(x)\to\delta_{ij} \qquad$ *as* $|x|\to\infty$.

(B1) *Each* $b_i(x)$ *is a real-valued* C^1 *function.*

(B2) $B_{ij}(x)=o(|x|^{-1}) \qquad (|x|\to\infty)$.

(C1) $c(x)$ *is a complex-valued bounded function.*

(C2) $c(x)=o(|x|^{-1}) \qquad (|x|\to\infty)$.

(UC) *The unique continuation property holds.*[2)]

By a solution u of equation (1.1) is meant an $H_{2,\mathrm{loc}}$, hence $L_{2,\mathrm{loc}}$, function which satisfies (1.1) in the distribution sense in E_{R_0}.

Our principal aim is to prove the following theorem which restricts the asymptotic behavior of a non-trivial solution of (1.1).

Theorem 1.1. *If u is a not identically vanishing solution of* (1.1) *in* E_{R_0} *with a positive eigenvalue,* $\lambda>0$, *then we have for any* $\varepsilon>0$

$$\lim_{r\to\infty} r^{\varepsilon}\int_{S_r} <A\tilde{x},\ \tilde{x}>\{\lambda|u|^2+<ADu,\ Du>\}\,dS=\infty.$$

As a corollary to the above theorem we can obtain a theorem

2) See, e.g., Hörmander [4].

which asserts that there are no positive eigenvalues with L_2 eigenfunctions of the eigenvalue problem (1.1), i.e., we have the next

Theorem 1.2. *Let u be a solution of* (1.1) *with $\lambda>0$ which also belongs to $L_2(E_{R_0})$. Then $u\equiv 0$ in E_{R_0}.*

We shall prove Theorem 1.1 in § 2 and Theorem 1.2 in § 3. In the presentation of their proof we do not always follow a logical order. If we did, we could have started in § 2, for example, with Definition 2.8 instead of Definition 2.4, for the latter is a special case of the former. We also inserted lemmas and propositions that may seem more than necessary, for, logically viewed, many of them could have been included in the proof of other lemmas or propositions. However, for clarity's sake and in order to avoid too lengthy computation in a proof as well as possible, we thought it profitable to include seemingly even superfluous statements.

Finally we remark that by assuming that the coefficient $c(x)$ is of class C^1 it would be possible to obtain a result similar to those of Odeh [9] or Simon [12]. We did not enter into this problem, however.

§ 2. Proof of Theorem 1.1

Let u be a solution of (1.1):

$$-D_i a_{ij} D_j u + c(x) u = \lambda u. \tag{2.1}$$

If we introduce the function $v(x)$ by

$$v(x) = |x|^{\frac{n-1}{2}} u(x), \tag{2.2}$$

then the following result readily follows from (2.1).

Lemma 2.1. *$v(x)$ satisfies the equation*

(2.3) $\qquad -D_i a_{ij} D_j v + \frac{n-1}{|x|} < ADv, \tilde{x} > +$

$$+\left\{c(x)+\frac{n-1}{2|x|}\partial_i a_{ij}\cdot\tilde{x}_j-\frac{(n-1)(n+3)}{4|x|^2}<A\tilde{x},\tilde{x}>+\right.$$

$$\left.+\frac{n-1}{2|x|^2}Sp\,A-\lambda\right\}v=0.$$

What we want, in fact, to do for the present is to multiply (2.3) with $|x|^\alpha<\tilde{x}, ADv>$ and integrate over B_{sr}. We will be led to Definition 2.4 by collecting the resulting surface integrals, and the mentioned integration (by parts) over B_{sr} will be carried out in the proof of Lemma 2.5. We now prepare two computational propositions. Their proof can be obtained by (repeated, if necessary) application of integration by parts and will be omitted.

Proposition 2.2. *The following identity holds*:

$$Re\int_{B_{sr}}D_i a_{ij}D_j v\cdot|x|^\alpha<\tilde{x}, ADv>dx=$$

$$=\left[\int_{S_r}-\int_{S_s}\right]\left\{|<ADv,\tilde{x}>|^2-\right.$$

$$\left.-\frac{1}{2}<A\tilde{x},\tilde{x}><ADv,Dv>\right\}|x|^\alpha dS-$$

$$-\int_{B_{sr}}\left\{|ADv|^2-\frac{1}{2}(SpA+(\alpha-1)<A\tilde{x},\tilde{x}>)<ADv,Dv>+\right.$$

$$+(\alpha-1)|<ADv,\tilde{x}>|^2+Re\,a_{ij}D_jv\cdot x_k\partial_i a_{kl}\cdot\overline{D_l v}+$$

$$\left.+Re\,a_{ij}D_jv\cdot x_k a_{kl}\overline{B_{il}v}-\frac{1}{2}x_k\partial_l a_{ij}a_{kl}\cdot D_iv\cdot\overline{D_jv}\right\}|x|^{\alpha-1}dx.$$

Proposition 2.3 *Let $f(x)$, $x\in E_{R_0}$, and $g(t)$, $t\geq R_0$, be real-valued C^1 functions. Then we have the following identity for $R_0\leq s<r$*:

$$Re\int_{B_{sr}}f(x)g(|x|)v|x|^\alpha<\tilde{x}, ADv>dx=$$

$$=\frac{1}{2}\left[\int_{S_r}-\int_{S_s}\right]<A\tilde{x},\tilde{x}>fg|x|^\alpha|v|^2dS-$$

$$-\frac{1}{2}\int_{B_{sr}}\{[(\alpha-1)<A\tilde{x},\ \tilde{x}>+SpA+x_i\partial_j a_{ij}]fg+$$
$$+x_i\partial_j f\cdot a_{ij}g+|x|fg'<A\tilde{x},\ \tilde{x}>\}|v|^2|x|^{\alpha-1}dx.$$

Let us now define a functional of $v(r\cdot)$ and $Dv(r\cdot)$, a motivation of which will be seen, as we remarked before, in the proof of Lemma 2.5.

Definition 2.4. *Let α be real. We put for $r>R_0$*

$$F(r,\ \alpha)=r^{\alpha}\int_{S_r}J_1(x,\ v,\ Dv)dS,^{3)}$$

$$J_1(x,\ v,\ Dv)=|<ADv,\ \tilde{x}>|^2-\frac{1}{2}<A\tilde{x},\ \tilde{x}><ADv,\ Dv>+$$
$$+\frac{(n-1)(n+3)}{8}\frac{1}{|x|^2}|<A\tilde{x},\ \tilde{x}>|^2|v|^2-$$
$$-\frac{n-1}{4|x|^2}SpA<A\tilde{x},\ \tilde{x}>|v|^2+\frac{\lambda}{2}<A\tilde{x},\ \tilde{x}>|v|^2.$$

Lemma 2.5. *We have*

$$\frac{dF(r,\alpha)}{dr}=r^{\alpha-1}\int_{S_r}\{J_2(x,\ Dv)+(K_1(x)+K_2(x))|v|^2+ \tag{2.4}$$
$$+J_3(x,\ v,\ Dv)\}\,dS,^{4)}$$

$$J_2(x,\ Dv)=|ADv|^2-$$
$$-\frac{1}{2}(SpA+(\alpha-1)<A\tilde{x},\ \tilde{x}>)<ADv,\ Dv>+$$
$$+(\alpha+n-2)|<ADv,\ \tilde{x}>|^2,$$

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3) This surface integral certainly makes sense, because the solutions u and v are seen to be in $C^1(E_{R_0})$. The latter fact can be obtained through an integral representation of u in terms of a fundamental solution for the differential equation (1.1). The same remark will apply to Definition 2.8.

4) What we actually prove in Lemma 2.5 and Lemma 2.6, respectively, are that $F(r,\alpha)$ is absolutely continuous and (2.4) and $dF(r,\alpha)/dr\geq 0$ hold almost everywhere, respectively. The same remark will apply to Lemmas 2.9 and 2.10.

$$K_1(x)=\frac{\lambda}{2}(Sp\,A+(\alpha-1)<A\tilde{x},\ \tilde{x}>),$$

$$K_2(x)=\frac{\lambda}{2}x_i\partial_j a_{ij}+$$

$$+\frac{n-1}{|x|^2}\Big\{\frac{n+3}{8}(Sp\,A+(\alpha-1)<A\tilde{x},\ \tilde{x}>)<A\tilde{x},\ \tilde{x}>+$$

$$+\frac{n+3}{4}(|A\tilde{x}|^2-2|<A\tilde{x},\ \tilde{x}>|^2)-$$

$$-\frac{1}{4}(Sp\,A+(\alpha-1)<A\tilde{x},\ \tilde{x}>)Sp\,A+$$

$$+\frac{1}{2}Sp\,A<A\tilde{x},\ \tilde{x}>+\frac{n+3}{8}x_i\partial_j a_{ij}\cdot<A\tilde{x},\ \tilde{x}>+$$

$$+\frac{n+3}{8}x_k\partial_l a_{ij}\cdot\tilde{x}_i\tilde{x}_j a_{kl}-\frac{1}{4}Sp\,Ax_i\partial_j a_{ij}-\frac{1}{4}a_{ij}x_i\partial_j Sp\,A\Big\},$$

$$J_3(x,\ v,\ Dv)=Re\,a_{ij}D_jv\cdot x_k\partial_i a_{kl}\cdot\overline{D_lv}+Re\,a_{ij}D_jv\cdot a_{kl}x_k\overline{B_{il}v}-$$

$$-\frac{1}{2}x_k\partial_l a_{ij}a_{kl}\cdot D_iv\cdot\overline{D_jv}+$$

$$+Re\Big(|x|c+\frac{n-1}{2|x|}x_j\partial_i a_{ij}\Big)v<x,\ ADv>.$$

Proof. By Lemma 2.1 we have for $R_0\leq s<r$

$$Re\int_{B_{sr}}\Big\{-D_ia_{ij}D_jv+\frac{n-1}{|x|}<ADv,\ \tilde{x}>+cv+\frac{n-1}{2|x|}\partial_i a_{ij}\cdot x_jv-$$

$$-\frac{(n-1)(n+3)}{4|x|^2}<A\tilde{x},\ \tilde{x}>v+$$

$$+\frac{n-1}{2|x|^2}Sp\,Av-\lambda v\Big\}|x|^\alpha<\tilde{x},\ ADv>dx=0.$$

Apply Proposition 2.2 to the first term of the integrand; leave the second, third and fourth terms untouched; and to the fifth, sixth and last terms apply Proposition 2.3 with $f(x)=<A\tilde{x},\ \tilde{x}>$ and $g(t)=-\frac{(n-1)(n+3)}{4t^2}$, with $f(x)=Sp\,A$ and $g(t)=\frac{n-1}{2t^2}$, and with $f(x)$

$=1$ and $g(t)=-\lambda$, respectively. Then, gathering the resulting surface integrals in the left-hand side and the volume integrals in the right-hand, we obtain

$$F(r,\alpha)-F(s,\alpha)=\int_{B_{sr}}[\text{integrand of (2.4)}]|x|^{\alpha-1}dx$$

whence follows the desired relation by differentiation in r. Q.E.D.

Lemma 2.6. *Given any* ε, $0<\varepsilon<1$, *there exists an* $R_1\geq R_0$ *such that*

$$\frac{dF(r,\alpha)}{dr}\geq 0$$

for $r\geq R_1$ *and for any* α *satisfying* $1-n+\varepsilon\leq\alpha\leq 3-n-\varepsilon$.

Proof. We have Lemma 2.5, and therefore what we have to do is to estimate (2.4) from below. Let us first estimate $J_3(x, v, Dv)$. By (A2), (A3), (A4), (B2) and (C2) it follows that there is a function $\varepsilon_1(r)$ verifying

$$J_3(x, v, Dv)\geq-\varepsilon_1(|x|)\{|ADv|^2+|v|^2\}. \tag{2.5}$$

Similarly, the following inequalities are seen to hold:

$$K_1(x)\geq\frac{\lambda}{2}\delta-\varepsilon_2(|x|), \tag{2.6}$$

$$K_2(x)\geq-\varepsilon_3(|x|), \tag{2.7}$$

where we have put

$$\delta=n+\alpha-1.$$

In order to get an estimate of $J_2(x, Dv)$ we note the inequalities

$$|\langle ADv, Dv\rangle-|ADv|^2|\leq\varepsilon_4(|x|)|ADv|^2, \tag{2.8}$$

$$|\langle ADv, \tilde{x}\rangle|\leq|ADv|. \tag{2.9}$$

(2.8) follows from (A4), and (2.9) from the fact that $\langle ADv, \tilde{x}\rangle$ is the $\tilde{x}$-component of the vector ADv. In view of (2.8) and (2.9), using

(A4) again, and choosing η such that $\varepsilon \leq \frac{\eta}{2} < \delta < \eta \leq 2 - \varepsilon$, we obtain

$$\begin{aligned} J_2(x, Dv) &\geq \left(1 - \frac{\eta}{2}\right)(|ADv|^2 - |<ADv, \tilde{x}>|^2) + \\ &+ \left(\delta - \frac{\eta}{2}\right)|<ADv, \tilde{x}>|^2 + \left(\frac{\eta}{2} - \frac{\delta}{2}\right)|ADv|^2 - \\ &- \varepsilon_5(|x|)|ADv|^2 \geq \frac{1}{2}(\eta - \delta)|ADv|^2 - \varepsilon_5(|x|)|ADv|^2. \end{aligned} \tag{2.10}$$

Combining (2.5), (2.6), (2.7) and (2.10) with (2.4), and taking note of the property of the ε functions, we get to the assertion of the lemma. Q.E.D.

Our next task is to estimate from below the functional $F(r, \alpha)$ itself when $v[u]$ is a non-trivial solution of (2.3) [(2.1)]. To this end we further introduce the function $w_m(x)$ for real m by

$$w_m(x) = e^{m\sqrt{|x|}} v(x). \tag{2.11}$$

The equation fulfilled by $w_m(x)$ can be easily derived from (2.3) and (2.11), that is, we have

Lemma 2.7. *$w_m(x)$ satisfies the equation*

$$\begin{aligned} &-D_i a_{ij} D_j w_m + \left(\frac{n-1}{|x|} + \frac{m}{\sqrt{|x|}}\right) < ADw_m, \tilde{x}> - \\ &- \frac{1}{4|x|}\left(m^2 + \frac{m(2n+1)}{\sqrt{|x|}} + \frac{(n-1)(n+3)}{|x|}\right) < A\tilde{x}, \tilde{x}> w_m + \\ &+ \frac{Sp\, A}{2|x|}\left(\frac{n-1}{|x|} + \frac{m}{\sqrt{|x|}}\right) w_m + \\ &+ \left\{c + \frac{1}{2}\left(\frac{m}{\sqrt{|x|}} + \frac{n-1}{|x|}\right)\partial_i a_{ij} \cdot \tilde{x}_j\right\} w_m - \lambda w_m = 0. \end{aligned} \tag{2.12}$$

We shall multiply (2.12) with $|x|^\alpha < \tilde{x}, ADw_m>$ and integrate by parts over B_{sr}, which will be an essential part of the proof of Lemma 2.9. The resulting surface integrals plus an additional surface integral form the following functional of $w_m(r\cdot)$ and $Dw_m(r\cdot)$.

Definition 2.8. *Let m, α and β be real. We put for $r>R_0$*

$$F(r, \alpha, \beta, m)=r^{\alpha}\int_{S_r}\{J_1(x, w_m, Dw_m)+K_3(x, \alpha, \beta, m)|w_m|^2\}\,dS,$$

where J_1 is given in Definition 2.4, *and*

$$K_3(x, \alpha, \beta, m)=\frac{1}{8|x|}\left\{m^2+\frac{m(2n+1)}{\sqrt{|x|}}\right\}<A\tilde{x}, \tilde{x}>^2-$$

$$-\frac{m\,Sp\,A}{4|x|^{3/2}}<A\tilde{x}, \tilde{x}>-\frac{1}{2}|x|^{\beta-\alpha}<A\tilde{x}, \tilde{x}>.$$

Lemma 2.9. *We have*

$$\frac{dF(r, \alpha, \beta, m)}{dr}=r^{\alpha-1}\int_{S_r}\Big\{J_2(x, Dw_m)+J_4(x, w_m, Dw_m, m)+$$

$$+\frac{m^2}{|x|}(K_4(x)+K_5(x))|w_m|^2+\frac{m}{|x|^{3/2}}(K_6(x)+K_7(x))|w_m|^2+ \tag{2.13}$$

$$+(K_1(x)+K_2(x))|w_m|^2+J_3(x, w_m, Dw_m)+$$

$$+J_5(x, w_m, Dw_m, \alpha, \beta)\Big\}\,dS,$$

where J_2, J_3, K_1 and K_2 are given in Lemma 2.5, *and*

$$J_4(x, w, Dw, m)=m\sqrt{|x|}\,|<ADw, \tilde{x}>|^2+$$

$$+\frac{m}{2}\sqrt{|x|}\,Re\,\tilde{x}_j\partial_i a_{ij}\cdot w_m<\tilde{x}, ADw_m>,$$

$$K_4(x)=\frac{1}{4}|A\tilde{x}|^2+\frac{\alpha-4}{8}<A\tilde{x}, \tilde{x}>^2+\frac{1}{8}\,Sp\,A<A\tilde{x}, \tilde{x}>,$$

$$K_5(x)=\frac{1}{8}<A\tilde{x}, \tilde{x}>x_i\partial_j a_{ij}+\frac{1}{8}\,\partial_l a_{ij}\cdot\tilde{x}_i\tilde{x}_j x_k a_{kl},$$

$$K_6(x)=\frac{2n+1}{4}|A\tilde{x}|^2+\left(\frac{\alpha(2n+1)}{8}-\frac{9n}{8}-\frac{9}{16}\right)<A\tilde{x}, \tilde{x}>^2+$$

$$+\frac{n-\alpha+3}{4}\,Sp\,A<A\tilde{x}, \tilde{x}>-\frac{1}{4}(Sp\,A)^2,$$

$$K_7(x)=\frac{2n+1}{8}<A\tilde{x},\ \tilde{x}>x_i\partial_j a_{ij}-\frac{1}{4}\ a_{ij}\,x_i\partial_j\,Sp\,A-$$

$$-\frac{1}{4}\ Sp\,A x_i\partial_j a_{ij}+\frac{2n+1}{8}\ a_{ij}\,x_i\partial_j a_{kl}\cdot\tilde{x}_k\,\tilde{x}_l,$$

$$J_5(x, w, Dw, \alpha, \beta)=-Re\,w<\tilde{x},\ ADw>|x|^{\beta-\alpha+1}-$$

$$-\frac{1}{2}\ ((\beta-1)<A\tilde{x},\ \tilde{x}>+Sp\,A+x_i\partial_j a_{ij})\,|x|^{\beta-\alpha}|w|^2.$$

Proof. We first note the following identity which follows from Proposition 2.3 with $f=g\equiv 1$ and $\alpha=\beta$:

$$(2.14)\quad Re\int_{B_{sr}} w|x|^{\beta}<\tilde{x},\ ADw>dx=$$

$$=\frac{1}{2}\left[\int_{S_r}-\int_{S_s}\right]<A\tilde{x},\ \tilde{x}>|w|^2|x|^{\beta}ds-$$

$$-\frac{1}{2}\int_{B_{sr}}\{(\beta-1)<A\tilde{x},\ \tilde{x}>+Sp\,A+x_i\partial_j a_{ij}\}|w|^2|x|^{\beta-1}dx.$$

By Lemma 2.7 and (2.14) with $w=w_m$ we have for $R_0\leq s\leq r$

$$Re\int_{B_{sr}}\Big\{-D_i a_{ij} D_j w_m+\Big(\frac{m}{\sqrt{|x|}}+\frac{n-1}{|x|}\Big)<ADw_m,\ \tilde{x}>-$$

$$-\frac{1}{4|x|}\Big(m^2+\frac{m(2n+1)}{\sqrt{|x|}}+\frac{(n-1)(n+3)}{|x|}\Big)<A\tilde{x},\ \tilde{x}>w_m+$$

$$+\frac{1}{2|x|}\Big(\frac{m}{\sqrt{|x|}}+\frac{n-1}{|x|}\Big)\ Sp\,Aw_m+cw_m+$$

$$+\frac{1}{2}\Big(\frac{m}{\sqrt{|x|}}+\frac{n-1}{|x|}\Big)\ \partial_i a_{ij}\cdot\tilde{x}_j w_m-\lambda w_m\Big\}\,|x|^{\alpha}<\tilde{x},\ ADw_m>dx-$$

$$-\int_{B_{sr}}\Big\{Re\,|x|^{\beta}w_m<\tilde{x},\ ADw_m>+$$

$$+\frac{1}{2}\ ((\beta-1)<A\tilde{x},\ \tilde{x}>+Sp\,A+x_i\partial_j a_{ij})\,|w_m|^2|x|^{\beta-1}\Big\}\,dx+$$

$$+\frac{1}{2}\left[\int_{S_r}-\int_{S_s}\right]<A\tilde{x},\ \tilde{x}>|w_m|^2|x|^{\beta}dS=0.$$

Now we can proceed as in the proof of Lemma 2.5: Apply Proposition 2.2 to the first term; to the third, fourth and seventh terms apply Propositon 2.3 with $f(x)=<A\tilde{x},\tilde{x}>$ and $g(t)=-\frac{1}{4t}\Big(m^2+\frac{m(2n+1)}{\sqrt{t}}+\frac{(n-1)(n+3)}{t}\Big)$, with $f(x)=Sp\,A$ and $g(t)=\frac{1}{2t}\Big(\frac{m}{\sqrt{t}}+\frac{n-1}{t}\Big)$, and with $f(x)=1$ and $g(t)=-\lambda$, respectively; and leave the remaining terms unchanged. Then collecting the resulting surface integrals we obtain

$$F(r,\alpha,\beta,m)-F(s,\alpha,\beta,m)=\int_{B_{sr}}[\text{integrand of (2.13)}]|x|^{\alpha-1}dx,$$

whence follows the desired relation by differentiation in r. Q.E.D.

Lemma 2.10. *There exist an $R_2\geq R_0$ and an $m_0\geq 0$ such that for any $r\geq R_2$ and any $m\geq m_0$ we have*

$$\frac{dF\Big(r,\frac{5}{2}-n,\frac{7}{4}-n,m\Big)}{dr}\geq 0.$$

Proof. We put in Lemma 2.9 $\alpha=\frac{5}{2}-n$ and $\beta=\frac{7}{4}-n$, and then note the inequality $1-n<\frac{5}{2}-n<3-n$ so that the various estimates worked out in the proof of Lemma 2.6 are applicable also in the present case. We thus have for J_2, K_1, K_2 and J_3 of (2.13)

$$J_2(x,Dw_m)\geq\frac{1}{2}\Big(\eta-\frac{3}{2}\Big)|ADw_m|^2-\varepsilon_5(|x|)|ADw_m|^2 \tag{2.15}$$

$$[(2.10)\text{ with }\delta=\frac{3}{2}\text{ and }0<\frac{\eta}{2}<\frac{3}{2}<\eta<2],$$

$$K_1(x)\geq\frac{3}{4}\lambda-\varepsilon_2(|x|), \tag{2.16}$$

$$K_2(x)\geq-\varepsilon_3(|x|), \tag{2.17}$$

$$J_3(x,w_m,Dw_m)\geq-\varepsilon_1(|x|)\{|ADw_m|^2+|w_m|^2\}. \tag{2.18}$$

Let us now estimate the rest of (2.13). Using (A4) we have

(2.19) $$\frac{m^2}{|x|} K_4(x) \geq \frac{m^2}{16|x|} (1-\varepsilon_6(|x|)).$$

Using (A3) we have

(2.20) $$\frac{m^2}{|x|} K_5(x) \geq -\frac{m^2}{|x|} \varepsilon_7(|x|).$$

Similarly

(2.21) $$\frac{m}{|x|^{3/2}} K_6(x) \geq -\frac{m}{|x|^{3/2}} \varepsilon_8(|x|),$$

(2.22) $$\frac{m}{|x|^{3/2}} K_7(x) \geq -\frac{m}{|x|^{3/2}} \varepsilon_9(|x|),$$

(2.23) $$J_5(x, w_m, Dw_m, \alpha, \beta) \geq -\frac{3-\gamma}{4} \lambda |w_m|^2 - $$

$$-\frac{1}{3-\gamma} \frac{1}{\lambda} |x|^{1/2} |<ADw_m, \tilde{x}>|^2 -$$

$$-\frac{3}{8|x|^{3/4}} |w_m|^2 - \frac{1}{|x|^{3/4}} \varepsilon_{10}(|x|) |w_m|^2 \qquad (0<\gamma<3),$$

where in (2.23) we have used the inequality

$$|ab| \leq \frac{\varepsilon}{2} |a|^2 + \frac{1}{2\varepsilon} |b|^2 \qquad (\varepsilon>0).$$

Let us employ (A3) and the above inequality to obtain

(2.24) $$J_4(x, w_m, Dw_m, m) \geq m(1-\varepsilon_{11}(|x|)) |x|^{1/2} |<ADw_m, \tilde{x}>|^2 -$$

$$-m|x|^{-3/2} \varepsilon_{12}(|x|) |w_m|^2.$$

Now let $m_0 = \frac{2}{(3-\gamma)\lambda}$ and choose R_2 sufficiently large. Then in view of (2.15) through (2.24) [compare, especially, (2.23) with (2.16) and (2.24)] the assertion of the lemma follows without difficulty. Q.E.D.

Remark 2.11. The special choice $\alpha = \frac{5}{2} - n$ and $\beta = \frac{7}{4} - n$ made in the above lemma has no particular meaning. It is enough to have the assertion of the lemma valid for *some* α and β. This remark will also apply to the following two lemmas.

Lemma 2.12. *Assume that the solution* u *(and hence* v*) is not identically equal to* 0 *in* E_{R_0}. *Then there exist constants* R_3 *and* m_1 *such that* $R_3 \geq R_2$ *and* $m_1 \geq m_0$, *and*

$$F\left(R_3, \frac{5}{2}-n, \frac{7}{4}-n, m_1\right) > 0.$$

Proof. Consider $F\left(r, \frac{5}{2}-n, \frac{7}{4}-n, m\right)$ and express w_m therein in terms of v using (2.11). Then we obtain

$$(2.25)\quad F\left(r, \frac{5}{2}-n, \frac{7}{4}-n, m\right) = e^{2m\sqrt{r}} \int_{S_r} \left\{ \frac{m^2}{4|x|} |<A\tilde{x}, \tilde{x}>|^2 |v|^2 + mJ_6(x, v, Dv) + J_7(x, v, Dv) \right\} dS,$$

where it is not difficult to see that J_6 and J_7 are quadratic in v and Dv, and independent of m, though we do not give their explicit expressions, because they are not needed in what follows. It now suffices to show that there is an $r \geq R_2$ such that the inequaliy

$$(2.26)\qquad \int_{S_r} <A\tilde{x}, \tilde{x}>^2 |v(x)|^2 dS > 0$$

holds. For, then, we can choose for such an r a sufficiently large m so that we may have (2.25) positive.

Let us prove that there is an $r \geq R_2$ verifying (2.26). Suppose that this is not the case. Then, since A is positive definite as (A2) shows, $v(x)$ and hence $u(x)$ must vanish identically in E_{R_2}, which with (UC) leads to $u(x) \equiv 0$ in E_{R_0}. This is a contradiction.

Q.E.D.

Lemma 2.13. *Let* R_3 *and* m_1 *be as in Lemma* 2.12. *Then we have*

$$F\left(r, \frac{5}{2}-n, \frac{7}{4}-n, m_1\right)>0 \qquad \textit{for } r \geq R_3.$$

Proof. Since $R_3 \geq R_2$, the assertion is an immediate consequence of Lemmas 2.10 and 2.12. Q.E.D.

Here we insert a computational proposition that will be employed in the proof of Lemma 2.15 concerning the monotonicity or non-monotonicity of the integral $\int_{S_r} <A\tilde{x}, \tilde{x}>^2 |u|^2 dS$ at infinity.

Proposition 2.14. *The following identity holds for* $r > R_0$:

$$Re\, r^\theta \int_{S_r} <A\tilde{x}, \tilde{x}><ADv, \tilde{x}>\bar{v}\, dS=$$

$$=\frac{1}{2}\frac{d}{dr} r^\theta \int_{S_r} <A\tilde{x}, \tilde{x}>^2 |v|^2 dS-$$

$$-\frac{1}{2} r^{\theta-1} \int_{S_r} \{(Sp\, A+(\theta-1)<A\tilde{x}, \tilde{x}>+x_i\partial_j a_{ij})<A\tilde{x}, \tilde{x}>+$$

$$+x_k\partial_l a_{ij}\cdot\tilde{x}_i\tilde{x}_j a_{kl}+2|A\tilde{x}|^2-2<A\tilde{x}, \tilde{x}>^2\} |v|^2 dS.$$

Proof. Put in Proposition 2.3 $f(x)=<A\tilde{x}, \tilde{x}>$, $g=1$ and $\alpha=\theta$. The obtained result can be differentiated with respect to r to yield the above identity. Q.E.D.

Lemma 2.15. *Suppose that the function*

$$r^{1-n}\int_{S_r} <A\tilde{x}, \tilde{x}>^2 |v|^2 dS=\int_{S_r} <A\tilde{x}, \tilde{x}>^2 |u|^2 dS$$

is not monotone increasing in $r \in (R, \infty)$ *for any* $R \geq R_0$. *Then there exists an* $R_4 \geq R_3$ *such that for any real* α *we have*

$$F(R_4, \alpha)>0. \tag{2.27}$$

Proof. It suffices to show the existence of R_4 which satisfies

(2.27) for some α, for we have from Definition 2.4

$$F(R_4, \alpha)=R_4{}^{\alpha-\alpha'}F(R_4, \alpha').$$

Now let us consider $F\left(r, \frac{5}{2}-n, \frac{7}{4}-n, m_1\right)>0$ and express w_{m_1} therein in terms of v. Then we have, noting $\beta-\alpha=-\frac{3}{4}$,

$$F\left(r, \frac{5}{2}-n, \frac{7}{4}-n, m_1\right)=$$

$$=e^{2m_1\sqrt{r}}r^{\frac{5}{2}-n}\int_{S_r}\left\{J_1(x, v, Dv)-\frac{<A\tilde{x}, \tilde{x}>}{2|x|^{3/4}}|v|^2\right\}dS$$

$$+\frac{m_1}{2}e^{2m_1\sqrt{r}}r^{2-n}Re\int_{S_r}<A\tilde{x}, \tilde{x}><ADv, \tilde{x}>\bar{v}\,dS+ \tag{2.28}$$

$$+e^{2m_1\sqrt{r}}r^{\frac{5}{2}-n}\int_{S_r}\left\{m_1\left(\frac{2n+1}{8|x|^{3/2}}<A\tilde{x}, \tilde{x}>^2-\right.\right.$$

$$\left.\left.-\frac{Sp\,A}{4|x|^{3/2}}<A\tilde{x}, \tilde{x}>\right)+m_1{}^2\frac{<A\tilde{x}, \tilde{x}>^2}{4|x|}\right\}|v|^2dS.$$

Let us apply Proposition 2.14 to the second integral of (2.28) to obtain

$$F\left(r, \frac{5}{2}-n, \frac{7}{4}-n, m_1\right)=$$

$$=e^{2m_1\sqrt{r}}r^{\frac{5}{2}-n}\int_{S_r}\left\{J_1(x, v, Dv)-\frac{1}{2}|x|^{-\frac{3}{4}}<A\tilde{x}, \tilde{x}>|v|^2\right\}dS+$$

$$+\frac{m_1}{2}e^{2m_1\sqrt{r}}r\frac{d}{dr}r^{1-n}\int_{S_r}<A\tilde{x}, \tilde{x}>^2|v|^2dS+ \tag{2.29}$$

$$+e^{2m_1\sqrt{r}}r^{\frac{5}{2}-n}\int_{S_r}K_8(x, m_1)|v|^2dS,$$

where $K_8(x, m)$ is given by

$$K_8(x, m)=\frac{m^2}{4|x|}<A\tilde{x}, \tilde{x}>^2-$$

$$-\frac{m}{4|x|^{3/2}}\Big\{(Sp\,A-n<A\tilde{x}, \tilde{x}>+x_i\partial_j a_{ij})<A\tilde{x}, \tilde{x}>+$$

$$+x_k\partial_l a_{ij}\cdot\tilde{x}_i\tilde{x}_j a_{kl}+2|A\tilde{x}|^2-2<A\tilde{x}, \tilde{x}>^2+$$

$$+\frac{2n+1}{8}<A\tilde{x}, \tilde{x}>^2-\frac{Sp A}{4}<A\tilde{x}, \tilde{x}>\Big\}.$$

We can take an R_5 so large that we have for $|x|\geq R_5$

$$K_8(x, m_1)\leq\frac{1}{2}|x|^{-3/4}<A\tilde{x}, \tilde{x}>^{5)} \tag{2.30}$$

(note that m_1 is fixed). Since by assumption $r^{1-n}\int_{S_r}<A\tilde{x}, \tilde{x}>^2|v|^2dS$ is not monotone increasing for $r\geq R_5$, we must have some $R_4\geq R_5$ such that

$$\frac{d}{dr}r^{1-n}\int_{S_r}<A\tilde{x}, \tilde{x}>^2|v|^2dS\Big|_{r=R_4}\leq 0. \tag{2.31}$$

Combining (2.29), (2.30), (2.31) and Definition 2.4, we see that

$$F\Big(R_4, \frac{5}{2}-n, \frac{7}{4}-n, m_1\Big)\leq e^{2m_1\sqrt{R_4}}F\Big(R_4, \frac{5}{2}-n\Big). \tag{2.32}$$

Now by Lemma 2.13 the left member of (2.32) is positive, whence readily follows what we intended to show. Q.E.D.

Having prepared all the necessary tools for proving Theorem 1.1, we now proceed to the

Proof of Theorem 1.1. Let $u(x)$ be an eigenfunction of the eigenvalue problem (1.1) satisfying the assumption of Theorem 1.1.

First let us assume that $\int_{S_r}<A\tilde{x}, \tilde{x}>^2|u|^2dS$ is monotone increasing in $r\in(R, \infty)$ for some $R\geq R_0$. Then the assertion of the-theorem is almost trivial if we note that

$$<A\tilde{x}, \tilde{x}>|u|^2\geq\frac{1}{2}<A\tilde{x}, \tilde{x}>^2|u|^2$$

for all sufficiently large $|x|$. The above inequality is seen to hold in

5) Here we can see the reason why we introduced the $|x|^{\beta-\alpha}$ term in Definition 2.8 (see $K_3(x, \alpha, \beta, m)$) that is the only part depending on the parameter β. We can make use of the freedom in the choice of β so that we can finally reach the inequality (2.32) connecting the two functionals $F(r,\alpha)$ and $F(r, \alpha, \beta, m)$.

virtue of (A4).

Next we assume that $\int_{S_r} \langle A\tilde{x}, \tilde{x}\rangle^2 |u|^2 dS$ is *not* monotone increasing in $r \in (R, \infty)$ for any $R \geq R_0$. Let us put

$$M(r) = \lambda \int_{S_r} \langle A\tilde{x}, \tilde{x}\rangle |u|^2 dS,$$

$$N(r) = \int_{S_r} \langle A\tilde{x}, \tilde{x}\rangle \langle ADu, Du\rangle dS.$$

Then what we have to show is that

$$\lim_{r\to\infty} r^{\varepsilon}(M(r)+N(r)) = \infty \tag{2.33}$$

for any $\varepsilon > 0$.

$M(r)$ and $N(r)$ can be rewritten in terms of v by use of (2.2) as follows:

$$M(r) = \lambda r^{1-n} \int_{S_r} \langle A\tilde{x}, \tilde{x}\rangle |v|^2 dS,$$

$$N(r) = r^{1-n} \int_{S_r} \Big\{ \langle A\tilde{x}, \tilde{x}\rangle \langle ADv, Dv\rangle -$$

$$- \frac{n-1}{r} \langle A\tilde{x}, \tilde{x}\rangle \mathrm{Re}\, v \langle \tilde{x}, ADv\rangle +$$

$$+ \frac{(n-1)^2}{4r^2} \langle A\tilde{x}, \tilde{x}\rangle^2 |v|^2 \Big\} dS.$$

Making use of the inequalities

$$|ab| \leq \frac{1}{2}(|a|^2 + |b|^2),$$

$$|\langle \tilde{x}, ADv\rangle|^2 \leq \langle A\tilde{x}, \tilde{x}\rangle \langle ADv, Dv\rangle$$

and Definition 2.4, therefore, we have

$$M(r)+N(r) \geq r^{1-n} \int_{S_r} \Big\{ \lambda \langle A\tilde{x}, \tilde{x}\rangle |v|^2 + |\langle \tilde{x}, ADv\rangle|^2 -$$

$$- \frac{1}{2}\Big[\frac{(n-1)^2}{r^2} \langle A\tilde{x}, \tilde{x}\rangle^2 |v|^2 + |\langle \tilde{x}, ADv\rangle|^2 \Big] +$$

$$+\frac{(n-1)^2}{4r^2}<A\tilde{x},\ \tilde{x}>^2|v|^2\Big\}dS$$

$$\geq r^{1-n}\int_{S_r}\Big\{|<\tilde{x},\ ADv>|^2-\frac{1}{2}<A\tilde{x},\ \tilde{x}><ADv,\ Dv>-$$

$$-\frac{(n-1)^2}{4r^2}<A\tilde{x},\ \tilde{x}>^2|v|^2+\lambda<A\tilde{x},\ \tilde{x}>|v|^2\Big\}dS$$

$$=F(r,\ 1-n)+r^{1-n}\int_{S_r}\Big\{\frac{\lambda}{2}<A\tilde{x},\ \tilde{x}>^2|v|^2-$$

$$-\frac{(n-1)(3n+1)}{8r^2}<A\tilde{x},\ \tilde{x}>^2|v|^2+$$

$$+\frac{n-1}{4r^2}SpA<A\tilde{x},\ \tilde{x}>|v|^2\Big\}dS.$$

The last surface integral turns out to be non-negative if we take r large enough, say $r\geq R_6$. (R_6 can be regarded not to be less than any of the R_j's that have so far appeared.) Thus

$$M(r)+N(r)\geq F(r,\ 1-n)\qquad \text{for } r\geq R_6. \tag{2.34}$$

Now let $\varepsilon>0$ be as in Theorem 1.1, and let η be such that $0<\eta<\min(\varepsilon,\ 2)$. The present hypothesis on $\int_{S_r}<A\tilde{x},\ \tilde{x}>|u|^2dS$ tells us that we can apply Lemma 2.15. Hence, Lemma 2.15 together with Lemma 2.6 implies

$$F(r,\ 1-n+\eta)\geq F(R_7,\ 1-n+\eta)>0\qquad \text{for } r\geq R_7 \tag{2.35}$$

with some R_7 fulfilling $R_7\geq R_6$. Let us multiply (2.34) by r^{ε} and (2.35) by $r^{\varepsilon-\eta}$, and compare the results in view of the relation $F(r,\alpha)=r^{\alpha-\alpha'}F(r,\ \alpha')$. (2.33) then follows immediately. This completes the proof of Theorem 1.1.

§3. Proof of Theorem 1.2

In this section the proof of Theorem 1.2 will be given on the basis of Theorem 1.1, and for this purpose we start with proving a lemma which concerns itself with an asymptotic behavior of an L_2 function.

Lemma 3.1. *Let* $u \in L_2(E_{R_8})$ *for some* $R_8 \geq R_0$. *Assume, further, that* $\int_{S_r} <A\tilde{x}, \tilde{x}> |u|^2 dS$ *is differentiable in* r *for* $r \geq R_8$. *Then we have*

$$\liminf_{r \to \infty} \frac{d}{dr} \int_{S_r} <A\tilde{x}, \tilde{x}> |u|^2 dS \leq 0.$$

Proof. Assume the contrary, i.e., for all sufficiently large $r(\geq R_8)$ let there exist a positive number ε such that

$$\frac{d}{dr} \int_{S_r} <A\tilde{x}, \tilde{x}> |u|^2 dS \geq \varepsilon > 0.$$

Then for s and $r(s<r)$ large enough we have

$$\left[\int_{S_r} - \int_{S_s}\right] <A\tilde{x}, \tilde{x}> |u|^2 dS =$$

$$= \int_s^r dt \frac{d}{dt} \int_{S_t} <A\tilde{x}, \tilde{x}> |u|^2 dS \geq \varepsilon(r-s).$$

The left member goes to ∞ as r tends to ∞. But this is a contradiction, since $u \in L_2(E_{R_8})$ implies

$$\liminf_{r \to \infty} \int_{S_r} <A\tilde{x}, \tilde{x}> |u|^2 dS = 0.$$

This completes the proof of the lemma. Q.E.D.

Lemma 3.2. *Let* u *satisfy the assumption of Theorem* 1.2. *Then we have* $|Du| \in L_2(E_{R'})$, $R' > R_0$.

Proof. It follows by partial integration applied to equation (1.1) multiplied with $\overline{u(x)}$ that for any $r > R'$

$$\text{(3.1)} \quad Re \int_{B_{R',r}} (\lambda - \bar{c}) |u|^2 dx = -Re \int_{B_{R',r}} u \overline{D_i a_{ij} D_j u}\, dx$$

$$= -Re\left[\int_{S_r} - \int_{S_{R'}}\right] u <\tilde{x}, ADu> dS + \int_{B_{R',r}} <AD, Du> dx.$$

In order to rewrite the surface integral $\int_{S_r} u<\tilde{x}, ADu>dS$ in (3.1), let us apply Proposition 2.3 with $f=g\equiv 1$, $\alpha=0$ and $v=u$. We then obtain

$$Re\int_{B_{sr}} u<\tilde{x}, ADu>dx=\frac{1}{2}\left[\int_{S_r}-\int_{S_s}\right]<A\tilde{x}, \tilde{x}>|u|^2dS-$$

$$-\frac{1}{2}\int_{B_{sr}}(-<A\tilde{x}, \tilde{x}>+Sp\,A+x_i\partial_j a_{ij})|x|^{-1}|u|^2dx,$$

which in turn yields through differentiation in r that for $r>R_0$

$$Re\int_{S_r} u<\tilde{x}, ADu>dS=$$

$$=\frac{1}{2}\frac{d}{dr}\int_{S_r}<A\tilde{x}, \tilde{x}>|u|^2dS- \tag{3.2}$$

$$-\frac{1}{2r}\int_{S_r}(-<A\tilde{x}, \tilde{x}>+Sp\,A+x_i\partial_j a_{ij})|u|^2dS.$$

If we substitute (3.2) in (3.1), we have

$$Re\int_{B_{R'r}}(\lambda-\bar{c})|u|^2dx=$$

$$=-\frac{1}{2}\frac{d}{dr}\int_{S_r}<A\tilde{x}, \tilde{x}>|u|^2dS+ \tag{3.3}$$

$$+\frac{1}{2r}\int_{S_r}(Sp\,A+x_i\partial_j a_{ij}-<A\tilde{x}, \tilde{x}>)|u|^2dS+$$

$$+Re\int_{S_{R'}}u<\tilde{x}, ADu>dS+\int_{B_{R'r}}<ADu, Du>dx.$$

Since by (A2) and (A4) we have the second term on the right-hand side of (3.3) non-negative for sufficiently large r, and since we can choose by Lemma 3.1 a sequence (r_n) tending to ∞ for $n\to\infty$ such that the first term on the right-hand side of (3.3) tends to a non-negative number along this sequence, the following inequality obtains:

$$\infty>Re\int_{E_{R'}}(\lambda-\bar{c})|u|^2dx-Re\int_{S_{R'}}u<\tilde{x}, ADu>dS\geq$$

$$\geq \int_{E_{R'}} < ADu,\ Du > dx.$$

Together with (A1) and the fact $u \in L_2(E_{R_0})$ the above inequality implies $|Du| \in L_2(E_{R'})$. Q.E.D.

Proof of Theorem 1.2. Suppose that $u(x) \not\equiv 0$ in E_R for any $R \geq R_0$. Then by (A4), Lemma 3.2 and the fact $u \in L_2(E_{R_0})$ we have

$$\liminf_{r \to \infty} r \int_{S_r} < A\tilde{x},\ \tilde{x} > \{\lambda |u|^2 + < ADu,\ Du >\} dS = 0,$$

which obviously contradicts Theorem 1.1 with $\varepsilon = 1$. Therefore, we must have $u(x) \equiv 0$ in E_R for some $R \geq R_0$. The unique continuation property (UC) then implies that $u(x) \equiv 0$ in E_{R_0}. We have thus completed the proof of Theorem 1.2.

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Added in proof: Theorem 1.1 admits of a slight improvement which is sometimes more convenient for application:

Theorem 1.1′. *Under the same assumption as Theorem* 1.1 *we have for any* $\varepsilon>0$

$$\lim_{r\to\infty} r^{\varepsilon}\int_{S_r} <A\tilde{x},\ \tilde{x}>\{\lambda|u|^2+|<ADu,\ \tilde{x}>|^2\}\,dS=\infty.$$

The proof is not essentially different from that of Theorem 1.1 given on pp. 442-444. We have only to replace $N(r)$ by $N'(r)=\int_{S_r}<A\tilde{x},\ \tilde{x}>|<ADu,\ \tilde{x}>|^2 dS$.