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Topological submanifolds and homology classes of a topological manifold

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This note is devoted to the problem of the realization of homology classes of a topological manifold by topological submanifolds. Firstly the C^∞-case of this problem was studied by R. Thom [6], and secondly the PL-case in [1], [2].

The present study is founded on the Kirby-Siebenmann's transversality theorem [3]. We shall apply R. Thom's method [6] to topological manifolds.

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1. Statement of the results

We shall obtain the following results.

i) *Homology classes* mod 2.

Theorem 1. *Let V^n be a closed topological manifold of dimension n, and $n\neq 4$. Then the following homology classes mod 2 are realizable by topological submanifolds which have normal vector bundles in V^n:*

(a) $H_{n-1}(V^n, \mathbf{Z}_2)$, *for* $n\neq 5$, $n\geqq 1$;

(b) $H_{n-2}(V^n, \mathbf{Z}_2)$, *for* $2\leqq n<6$;

(c) $H_{n-3}(V^n, \mathbf{Z}_2)$, *for* $3 \leqq n < 7$;

(d) $H_i(V^n, \mathbf{Z}_2)$, *for* $i \leqq n/2$, $i \neq 4$, *and all* $n \geqq 1$.

ii) *Integral homology classes.*

Theorem 2. *Let* V^n *be a closed orientable topological manifold of dimension n and* $n \neq 4$. *Then the following integral homology classes are realizable by oriented topological submanifolds which have normal vector bundles in* V^n:

(a) $H_{n-1}(V^n, \mathbf{Z})$, *for* $n \neq 5$, $n \geqq 1$;

(b) $H_{n-2}(V^n, \mathbf{Z})$, *for* $n \neq 6$, $n \geqq 2$;

(c) $H_i(V^n, \mathbf{Z})$, *for* $i \leqq 5$, $i \neq 4$ *and for all* $n \geqq 1$.

Remark 1. The normal bundle of an orientable submanifold in an orientable manifold is a priori orientable.

Remark 2. A topological submanifold which has a normal vector bundle is clearly a locally flatly embedded submanifold.

These results are quite in parallel to those of the C^∞-case in Thom [6].

2. Generalities

We shall work in the category of topological spaces and continuous maps.

Let V^n be a topological manifold of dimension n. Then we shall say that W^p is a topological *submanifold* of dimension p, if W^p is a closed topological manifold of dimension p and a topological subspace of V^n.

Let V^n be a closed topological manifold of dimension n. Let W^p be a topological submanifold of V^n of dimension p. The inclusion map $i: W^p \to V^n$ induces the homomorphism

$$i_*: H_p(W^p, \mathbf{Z}_2) \to H_p(V^n, \mathbf{Z}_2).$$

Let $z \in H(V^n, \mathbf{Z}_2)$ be the image by i_* of the fundamental class w of the topological manifold W^p. Then we say that the homology class z is *realized* by the topological submanifold W^p. Let V^n be orientable, and W^p be an oriented topological submanifold of dimension p. Then the inclusion map i_*: $W^p \to V^n$ induces the homomorphism

$$i_*: H_p(W^p, \mathbf{Z}) \to H_p(V^n, \mathbf{Z}).$$

Let $z \in H(V^n, \mathbf{Z})$ be the image by i_* of the fundamental class w of the oriented topological manifold W^p. Then we say that the homology class z is *realized* by the oriented topological submanifold W^p.

Here the following questions are considered: Let a homology class z mod 2 of a closed topological manifold V^n be given. Is it realizable by a topological submanifold? ; Let an integral homology class z of a closed orientable topological manifold V^n be given. Is it realizable by an oriented topological submanifold?

Following J. Kister [4], let $\mathscr{H}_0(k)$ be the space of all homeomorphisms of the Euclidean k-space $\mathbf{R}^k$ onto itself preserving the origin 0 with compact-open topology. Then $\mathscr{H}_0(k)$ is a topological group with respect to the composition of maps. Let $S\mathscr{H}_0(k)$ be the subgroup of $\mathscr{H}_0(k)$ of those elements that preserve orientation.

By an $\mathbf{R}^k$*-bundle* we shall mean a fibre bundle whose fibre is the Euclidean k-space $\mathbf{R}^k$ and structure group is $\mathscr{H}_0(k)$. Let $\xi = \{E(\xi), \pi_\xi, B(\xi), \mathbf{R}^k, \mathscr{H}_0(k)\}$ be an $\mathbf{R}^k$-bundle. Then ξ has the 0-cross-section

$$i_\xi: B(\xi) \to E(\xi).$$

By an *orientable* $\mathbf{R}^k$*-bundle* we shall mean a fibre bundle whose fibre is the Euclidean k-space $\mathbf{R}^k$ and structure group is $S\mathscr{H}_0(k)$. The orthogonal group $O(k)$ can be canonically considered as a topological subgroup of $\mathscr{H}_0(k)$. Therefore, we can canonically consider a vector bundle of dimension k to be an $\mathbf{R}^k$-bundle.

Let V^n be a closed topological manifold and W^k be a topological submanifold of V^n. Then by a *normal* $\mathbf{R}^{n-k}$*-bundle* of W^k in V^n, we shall mean an R^{n-k}-bundle $\nu = \{E(\nu), \pi_\nu, B(\nu), \mathbf{R}^{n-k}, \mathscr{H}_0(n-k)\}$ whose base space $B(\nu)$ is W^k and total space $E(\nu)$ is a neighborhood of W^k in V^n.

Let V^n be a closed topological manifold and W^k a topological submanifold of V^n. Then by a *normal vector bundle* of W^k in V^n, we shall mean a vector bundle $\nu=\{E(\nu), \pi_\nu, B(\nu), \mathbf{R}^{n-k}, O(n-k)\}$ whose base space $B(\nu)$ is W^k and total space $E(\nu)$ is a neighborhood of W^k in V^n.

3. Transversality theorem of Kirby-Siebenmann

Let $\xi=\{E(\xi), \pi_\xi, B(\xi), \mathbf{R}^n, \mathscr{H}_0(n)\}$ be an $\mathbf{R}^n$-bundle and M^m be a topological m-manifold. By considering the zero-cross-section, we can consider $B(\xi)\subset E(\xi)$. A continuous map $f\colon M^m\to E(\xi)$ is called to be *transverse* to $B(\xi)$, if $P=f^{-1}(B(\xi))$ is an $(m-n)$-dimensional topological submanifold with normal $\mathbf{R}^n$-bundle ν in M^m and ν is isomorphic to the induced bundle $(f|P)^*\xi$.

Kirby-Siebenmann [3] have proved the following transversality theorem.

Theorem 3. *Let $\xi=\{E(\xi), \pi_\xi, B(\xi), \mathbf{R}^n, \mathscr{H}_0(n)\}$ be an $\mathbf{R}^n$-bundle and M^m be a topological m-manifold. Let $f\colon M^m\to E(\xi)$ be a continuous map. Then, if $m\neq 4$, $m-n\neq 4$, f is homotopic to a map f_1 which is transverse to $B(\xi)$. If f is transverse to $B(\xi)$ near a closed set $C\subset M^m$, then the homotopy equals to f near C.*

4. Fundamental theorem.

Definition. We say that a cohomology class $u\in H^k(A, \mathbf{Z}_2)$ of a space A is *$O(k)$-realizable*, if there exists a mapping $f\colon A\to MO(k)$ such that u is the image, for the homomorphism f^* induced by f, of the fundamental class $U_{o(k)}$ of the Thom complex $MO(k)$. We say that an integral cohomology class $u\in H^k(A, \mathbf{Z})$ of a space A is *$SO(k)$-realizable*, if there exists a mapping $f\colon A\to MSO(k)$ such that u is the image, for the homomorphism f^* induced by f, of the fundamental class $U_{SO(k)}$ of the Thom complex $MSO(k)$.

Then we have the following fundamental theorem.

Theorem 4. *Let V^n be a closed topological manifold of dimension n. Suppose that $n \neq 4$, $n-k \neq 4$.*

(a) *In order that homology class $z \in H_{n-k}(V^n, \mathbf{Z}_2)$, $k>0$ can be realized by a topological submanifold W^{n-k} which has a normal vector bundle in V^n, it is necessary and sufficient that the cohomology class $u \in H^k(V^n, \mathbf{Z}_2)$, corresponding to z be the Poincaré duality, is $O(k)$-realizable.*

(b) *Let V^n be orientable. In order that an integral homology class $z \in H_{n-k}(V^n, \mathbf{Z})$, $k>0$, can be realized by an oriented topological submanifold W^{n-k} which has a normal vector bundle in V^n, it is necessary and sufficient that the cohomology class $u \in H^k(V^n, \mathbf{Z})$, corresponding to z by the Poincaré duality, is $SO(k)$ -realizable.*

Proof. We shall prove the case (a) of the theorem. The case (b) can be proved quite in parallel with the case (a).

(i) *The condition is necessary.* Suppose that there exists a topological submanifold W^{n-k} in V^n which realizes the homology class z, and has a normal vector bundle ν in V^n:

$$\nu = \{E(\nu),\ \pi_\nu,\ B(\nu),\ \mathbf{R}^k,\ O(k)\},\ B(\nu) = W^{n-k}.$$

Let N be the total space of the associated k-disk bundle ν_D of ν, and T be the total space of the associated S^{k-1}-bundle ν_S of ν; namely T is the boundary of N. Then we can consider

$$W^{n-k} = B(\nu) \subset N \subset E(\nu) \subset V^n.$$

Let

$$\nu_D = \{N,\ \pi_D,\ W^{n-k},\ D^k,\ 0(k)\}.$$

Then ν_D is induced from the universal D^k-bundle $\nu_D^k = \{A_{0(k)}, \pi_k, B_{0(k)}, D^k, O(k)\}$ by a continuous mapping $g\colon W^{n-k} \to B_{0(k)}$. Therefore, there exists a bundle map $\tilde{g}$ which induces g:

$$\begin{array}{ccc} N & \xrightarrow{\tilde{g}} & A_{0(k)} \\ \pi_D \downarrow & & \downarrow \pi_k \\ W^{n-k} & \xrightarrow{g} & B_{0(k)}. \end{array}$$

Let $\gamma_s^k = \{E_{0(k)}, \pi_k, B_{0(k)}, S^{k-1}, O(k)\}$ be the universal S^{k-1}-bundle. Then the restriction of $\tilde{g}$ on the boundary T of N maps T in the boundary $E_{0(k)}$ of $A_{0(k)}$. Therefore, we have the following commutative diagram:

$$\begin{array}{ccc} H^k(N, T; \mathbf{Z}_2) & \xleftarrow{\tilde{g}^*} & H^k(A_{0(k)}, E_{0(k)}; \mathbf{Z}_2) \\ \varphi_\nu^* \uparrow \wr\| & & \wr\| \uparrow \varphi_{0(k)}^* \\ H^0(W^{n-k}, \mathbf{Z}_2) & \xleftarrow{g^*} & H^0(B_{0(k)}, \mathbf{Z}_2), \end{array} \tag{1}$$

where φ_ν^* and $\varphi_{0(k)}^*$ are Thom isomorphisms.

On the other hand, we have the following canonical homomorphism $j_* = j^* \circ \alpha$:

$$j_*: H^k(N, T; \mathbf{Z}_2) \underset{\cong}{\xrightarrow{\alpha}} H^k(V^n, V^n\text{-int}\, N; \mathbf{Z}_2) \xrightarrow{j^*} H^k(V^n, \mathbf{Z}_2),$$

where α is the excision isomorohism and j^* is the relativization. We know that in the open manifold $N' = N - T = \text{int}\, N$, the class $\varphi_\nu^*(\omega)$ corresponds, by the Poincaré duality, to the fundamental homology class w of the base W^{n-k}, where is ω the unit of the cohomology ring $H^*(W^{n-k}, \mathbf{Z}_2)$(cf. Thom [5], Théorème I.8). Consequently, the class $j_* \circ \varphi_\nu^*(\omega) \in H^k(V^n, \mathbf{Z}_2)$ is the class u corresponding to z.

Let us denote by h: $A_{0(k)} \to MO(k)$ the mapping obtained by identifying to a point a the boundary $E_{0(k)}$ of $A_{0(k)}$.

The composite mapping $h \circ \tilde{g}$ maps the boundary T of N on the point a. Consequently, the mapping $h \circ \tilde{g}$ can be extended on the whole manifold V^n; it suffices to map the complement $V^n - N$ to the point a. Thus we have defined a mapping f of V^n into $MO(k)$, for which we have

$$f^*(U_{0(k)}) = f^* \circ \varphi_{0(k)}^*(\omega_{0(k)}) = j_* \circ \varphi_\nu^*(\omega)$$

by the commutative diagram (1), where $\omega_{0(k)}$ is the unit of the cohomology ring $H^*(B_{0(k)}, \mathbf{Z}_2)$, and

$$j_* \circ \varphi_\nu^*(\omega) = D_V \circ i_* \circ D_W(\omega) = u,$$

where D_V, D_W are the Poincaré duality of V^n, W^{n-k}, respectively, and i: $W^{n-k} \to V^n$ is the inclusion.

(ii) *The condition is sufficient.* Suppose that there exists a mapping f of V^n into $MO(k)$, such that $f^*(U_{0(k)}) = u$. The space $MO(k)$, with the exceptional point a deprived, can be considered as an $\mathbf{R}^k$-bundle $\gamma_R^k = \{A'_{0(k)}, \pi', B_{0(k)}, R^k, O(k)\}$ over $B_{0(k)}$. The restriction of f on the complement $V^n - f^{-1}(a)$ is a mapping of the topological n-manifold $V^n - f^{-1}(a)$ into an $\mathbf{R}^k$-bundle $A'_{0(k)}$. Let $\gamma_D^k = \{A_{0(K)}, \pi_D, B_{0(k)}, D^k, 0(k)\}$ be D^k-bundle associated to γ_R^k. Then we can consider $B_{0(k)} \subset A_{0(k)} \subset A'_{0(k)}$. Let $C = f^{-1}(A'_{0(k)} - \mathrm{int} A_{0(k)})$. Then C is a closed set in $V^n - f^{-1}(a)$ and f restricted on a neighborhood U of C is t-regular on $B_{0(k)}$. By Kirby-Siebenmann's transversality theorem (Theorem 3), we obtain a new mapping f_1 of $V^n - \mathrm{int} f^{-1}(a)$ into the $\mathbf{R}^k$-bundle $A'_{0(k)}$, which is t-regular on $B_{0(k)}$, and homotopic to $f|(V^n - f^{-1}(a))$. Moreover, we can take f_1 on the neighborhood U of C to be $f|(V^n - f^{-1}(a))$ on U. Therefore, we can extend the mapping f_1 on V^n:

$$f_1 \colon V^n \to MO(k),$$

and f_1 is homotopic to the given mapping f. Consequently, we have $u = f^*(U_{0(k)}) = f_1^*(U_{0(k)})$. On the other hand, by the definition of t-regularity (cf. §4), we obtain that $f_1^{-1}(B_{0(k)})$ is a topological submanifold W^{n-k} and this has a normal $\mathbf{R}^k$-bundle ν of W^{n-k} in V^n which is induced from γ_R^k by $f_1 | W^{n-k}$. However, the structure group of γ_R^k is $O(k)$, therefore, the structure group of the normal $\mathbf{R}^k$-bundle ν can be reduced to $O(k)$. Thus we can consider that the submanifold W^{n-k} has a normal vector bundle ν in V^n.

Let us denote by φ_ν^* the Thom isomorphism of the normal vector bundle ν. Then the class $u = f^*(U_{0(k)}) = f_1^*(U_{0(k)})$ is $j_* \circ \varphi_\nu^*(\omega)$. As we have seen in (i), this proves that u corresponds by the Poincaré duality to the fundamental class w of W^{n-k}.

5. Proof of Theorems.

We know that *R*. Thom studied on the homotopy types of Thom complexes $MO(k)$ and $MSO(k)$ (cf. Thom [6], Chapitre II). By these results and the fundamental theorem (Theorem 4), we have Theorem 1 and Theorem 2.

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