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On the algebraic fundamental group of an algebraic group

By

Masayoshi MIYANISHI

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0. Introduction. Let k be an algebraically closed field of characteristic p, let G be a connected algebraic linear group defined over k and let a be a k-rational point of G. In [2], Exposé V, A. Grothendieck constructed the algebraic fundamental group $\pi_1(G, a)$ which is defined in the following fashion: Let $\mathrm{Et}(G)$ be the category of finite etale coverings of G and let F be a covariant functor from $\mathrm{Et}(G)$ to the category of finite sets which assigns to $X \in \mathrm{Et}(G)$ the set $F(X)$ of k-rational points over a. Then F is strictly prorepresentable. Namely there exists an inverse system $\{P_i, q_{ji}\}_{i,j \in \boldsymbol{Z}^+}$ in $\mathrm{Et}(G)$ with surjection $q_{ji} : P_j \rightarrow P_i$ for $j > i$ such that $F(X) = \varinjlim_{i \in \boldsymbol{Z}^+} \mathrm{Hom}_G(P_i, X)$. Then $\pi_1(G, a) = \varprojlim_{i \in \boldsymbol{Z}^+} \mathrm{Aut}_G(P_i)$. More precisely, the inverse system $\{P_i\}_{i \in \boldsymbol{Z}^+}$ has a cofinal subsystem consisting of Galois etale coverings, where by a Galois etale covering we mean a finite etale covering X over G with group of G-automorphisms $\mathrm{Aut}_G(X)$ of X such that the order of $\mathrm{Aut}_G(X)$ is equal to the rank of the covering.

If a' is another k-rational point of G, $\pi_1(G, a')$ is then isomorphic to $\pi_1(G, a)$. With the unit point e of G for a, we call $\pi_1(G, e)$ the algebraic fundamental group of G and denote it simply by $\pi_1(G)$.

When k is the complex number field $\boldsymbol{C}$ and G^h is the analytic space associated with G, $\pi_1(G)$ is then the profinite completion of the topological fundamental group of G^h (cf. M. Artin and B. Mazur, Etale homotopy, Springer Lecture Notes in Mathematics, No. **100**).

When the characteristic p of k is positive and when $\pi_1(G)=\varprojlim_{i\in \boldsymbol{Z}^+} G_i$ with finite groups G_i ($i\in \boldsymbol{Z}^+$), we denote by $\pi_{1,+p}(G)$ the completion $\varprojlim_{i\in \boldsymbol{Z}^+} G_{i,+p}$ of the maximal quotient $G_{i,+p}$ of G_i such that the order of $G_{i,+p}$ is prime to p.

Our first main result is

THEOREM 1. *Let k be an algebraically closed field of characteristic p and let G be a connected algebraic linear group defined over k. If $p=0$, $\pi_1(G)$ is abelian. If $p\neq 0$, $\pi_{1,+p}(G)$ is abelian.*

When k is the complex number field $\boldsymbol{C}$, Theorem 1 follows from P.A. Smith [5], where it is shown that the topological fundamental group of G^h is abelian. However Smith's method is too topological. Hence it cannot be extended to the positive characteristic case. So we present an algebraic proof of Theorem 1. Roughly speaking, our proof runs as follows: Let U be the unipotent radical of G, let P be the associated reductive group, let T be a maximal torus of P, let R be the root system of P with respect to T, let R_+(resp. R_-) be a positive (resp. negative) root system of R, once fixed in R and let $P_r (r\in R_+\cup R_-)$ be one parameter subgroup of P corresponding to a root r. Then there exists an affine open set Ω_{R_+} (a gross cell) in P such that Ω_{R_+} is isomorphic to $\prod_{r\in R_-} P_r\times T\times \prod_{r\in R_+} P_r$ where the product is taken following an appropriate order fixed in R_+ and R_-. On the other hand, T is considered a maximal torus of G and the pull back of Ω_{R_+} is an affine open dense set in G which is a direct product $\Omega_{R_+}\times_k U$. Since U is isomorphic to an affine space, the pull back of Ω_{R_+} is a product of T with an affine space.

The results to be shown are the following: (1) $\pi_1(T)$ is abelian. (2) The canonical homomorphism $\pi_1(T)\rightarrow\pi_1(G)$ induced from the restriction map $\mathrm{Et}(G)\rightarrow\mathrm{Et}(T)$ is surjective. These results provide a proof of Theorem 1. To prove (2), it suffices to show that if X is a connected

Galois etale covering of G, the restriction of X on T is connected too. This is shown as follows: Assume X_T is not connected. Then by induction on $2\,\mathrm{card}(R_+)+\dim(U)$, one shows that X restricted on a sub-product of $\Omega_{R_+}\times U$ is not connected. The final step of induction is that $X|_{\Omega_{R_+}\times U}$ is not connected. Taking the closure of each component of $X|_{\Omega_{R_+}\times U}$ in X, X is not connected. This is a contradiction.

For a finite abelian group F(resp. a finite abelian group of order prime to p if $p\neq 0$), the set of all Galois etale coverings of G with group F up to isomorphisms is an abelian group denoted by $\mathrm{H}^1(G, F)$. The set of all extensions of G by F up to isomorphisms, $\mathrm{Ext}^1(G, F)$, is an abelian group too. With these notations, we have

THEOREM 2. *Let G and F be as above. Then we have*

$$\mathrm{H}^1(G, F)\cong \mathrm{Ext}^1(G, F)\cong\begin{cases}\mathrm{Hom}_{gr}(\pi_1(G), F) & \text{if } p=0\\ \mathrm{Hom}_{gr}(\pi_{1,\dagger p}(G), F) & \text{if } p\neq 0.\end{cases}$$

This result leads us to a complete determination of $\pi_1(G)$ (or $\pi_{1,\dagger p}(G)$). Let $\bar{G}$ be the associated semi-simple group of G, i.e. $G/\mathrm{rad}(G)$ and let r be the dimension of the torus part of the radical of G. Then we have

THEOREM 3. $\pi_1(G)$ $(\pi_{1,\dagger p}(G)$ *if* $p\neq 0)\cong(\hat{\mathbf{Z}})^r\times \mathrm{Ext}^1(\bar{G}, G_m)$. *Here $\hat{\mathbf{Z}}$ is the profinite completion of $\mathbf{Z}$.*

1. The proof of Theorem 1. We shall begin with

LEMMA 1. *Let k be an algebraically closed field of characteristic* 0, *let $\boldsymbol{A}^1$ be the affine line and let $G_{m,k}=\boldsymbol{A}^1-(0)$. Then we have*: (1) *Let X be a connected Galois covering of $G_{m,k}$ with group F. X is then isomorphic to $G_{m,k}$ and after an appropriate change of parameters in X and $G_{m,k}$ (the base scheme), the base map $q: X\to G_{m,k}$ is the multiplication by some positive integer n. Hence $F\cong \mathbf{Z}/n\mathbf{Z}$.*

(2) *Let X be a connected Galois etale covering of $\boldsymbol{A}^1$ with group F. Then $F \cong (1)$. Therefore there is no non-trivial etale covering of $\boldsymbol{A}^1$.*

PROOF. We prove (1). (2) is proved in a similar fashion. Embed $\boldsymbol{A}^1$ into $\boldsymbol{P}^1$ canonically. Since X is affine and of dimension 1, we can embed X into a non-singular complete curve C of genus g. The base map $q: X \to G_{m,k}$ is extended to a morphism $\pi: C \to \boldsymbol{P}^1$. Moreover since F acts on C as birational $\boldsymbol{P}^1$-automorphisms, F acts on C as biregular $\boldsymbol{P}^1$-automorphisms. Let $S_0 = \pi^{-1}((0))$ and $S_\infty = \pi^{-1}((\infty))$. Then F acts transitively on the sets S_0 and S_∞. Therefore each point of S_0 (resp. S_∞) has the same ramification index e (resp. e'). Let $m = \operatorname{card}(S_0)$ and let $m' = \operatorname{card}(S_\infty)$. Noting that C has no ramified points other than points of S_0 and S_∞ over $\boldsymbol{P}^1$, one applies the Riemann-Hurvitz formula to $\pi: C \to \boldsymbol{P}^1$;

$$2(g-1) = 2n(0-1) + m(e-1) + m'(e'-1)$$

where n is the order of F, $me = m'e' = n$ and $m, m' \geq 1$. Therefore $g = 0$, $m = m' = 1$ and $e = e' = n$. Then $C - (S_0 \cup S_\infty) \cong G_{m,k}$. Let $G_{m,k} = \operatorname{Spec}(k[t, t^{-1}])$ with an indeterminate t. The base map $q: X \to G_{m,k}$ is given by an invertible element of $k[t, t^{-1}]$, say at^n. Now by a parameter change $t \rightsquigarrow a^{1/n}t$ in X, q is the multiplication by n. q.e.d.

REMARK 1. In case of the positive characteristic, the Riemann-Hurwitz formula holds true if a function field K is a finite separable extension of a rational field of dimension 1 and if all ramification indices are prime to the characteristic p. Therefore Lemma 1 is valid if the order of F is prime to p since e and e' are prime to p. (Note that $me = m'e' = n$.)

LEMMA 2. *Let k be a field of characteristic p and let X be a Galois etale covering of $\boldsymbol{A}^1$ with group F. Assume the order of F is prime to p if p is positive. Then there exists a finite etale k-algebra*

R such that $X \cong \boldsymbol{A}^1 \times_k \mathrm{Spec}(R)$. F acts on R and the action of F on X is given from an action of F on $\mathrm{Spec}(R)$.

PROOF. (I) The case where $p=0$. Let $\bar{k}$ be an algebraic closure of k, let $\boldsymbol{A}^1 = \mathrm{Spec}(k[t])$ and let $X = \mathrm{Spec}(A)$. Then $X_{\bar{k}} = X \otimes_K \bar{k}$ is a Galois etale covering of $\boldsymbol{A}^1_{\bar{k}}$ with group F. In virtue of Lemma 1, (2), $X_{\bar{k}}$ is split, i.e. $X_{\bar{k}} \cong \underbrace{\boldsymbol{A}^1_{\bar{k}} \amalg \boldsymbol{A}^1_{\bar{k}} \amalg \dots \boldsymbol{A}^1_{\bar{k}}}_{\mathrm{card}\,(F)}$ and F permutes the components. In other words, $A_{\bar{k}} = A \otimes_k \bar{k} \cong \underbrace{\bar{k}[t] \oplus \dots \oplus \bar{k}[t]}_{n} = \oplus^n \bar{k}[t]$ (a direct sum of $\bar{k}$-algebras), where $n = \mathrm{card}(F)$. Look at the following commutative diagram,

$$\begin{array}{ccc} A & \hookrightarrow & A_{\bar{k}} \cong \oplus^n \bar{k}[t] \\ \cup\!\!\uparrow & & \cup\!\!\uparrow \\ k[t] & \hookrightarrow & \bar{k}[t] \end{array}$$

$k[t]$ has a k-rational derivation D with $D(t) = 1$. This derivation is locally nilpotent, i.e. for every element a of $k[t]$, there exists an integer l such that $D^m(a) = 0$ for all $m > l$. Since A is etale over $k[t]$, D is extended uniquely to a derivation $\tilde{D}$ on A. (Let F act on a derivation D' on A by $({}^f D')(a) = {}^{f^{-1}}D'({}^f a)$ for $a \in A$ and for $f \in F$. $\tilde{D}$ is then F-invariant with this F-action.) On the other hand, D is extended uniquely to a $\bar{k}$-trivial derivation $\bar{D}$ on $\bar{k}[t]$. The same arguments apply to $\tilde{D}$ and $\bar{D}$ to extend them onto $A_{\bar{k}}$. Both extensions coincide on $A_{\bar{k}}$. Denote it by D'. D' acts on each component $\bar{k}[t]$ in the same way as $\bar{D}$ does on $\bar{k}[t]$.*) Hence D' is locally nilpotent. Then $\tilde{D}$ is locally nilpotent on A, since A is canonically embedded in $A_{\bar{k}}$ and D' is the extension of $\tilde{D}$. Moreover A has the element t which satisfy $\tilde{D}(t) = 1$. Then we know that $A = R[t]$ with $R = \{a \in A \mid \tilde{D}(a) = 0\}$ (cf. [3], Lemma 2). The remaining assertions are easy to prove.

*) Let D be a derivation on a k-algebra C and let e be an idempotent of C. Then $D(e) = 0$. Indeed, $D(e^2) = 2eD(e) = D(e)$. Hence $eD(e) = (1-e)D(e)$, where $eD(e) = (1-e)D(e) = 0$. Then $De = eD(e) + (1-e)D(e) = 0$.

(II) The case where $p\neq 0$. We use the same notations as in (I). $k[t]$ has a k-trivial iterative infinite higher derivation $D=\{D_i\}_{i\geq 0}$ with $D_1(x)=1$. D is locally nilpotent in $k[t]$, i.e. for any element a of $k[t]$, $D_l(a)=0$ for all $l\gg 0$. D is extended uniquely to a $\bar{k}$-trivial iterative infinite higher derivation $\tilde{D}=\{\tilde{D}_i\}_{i\geq 0}$ on $\bar{k}[t]$ and $\tilde{D}$ is in turn uniquely extended to a $\bar{k}$-trivial iterative infinite higher derivation $D'=\{D'_i\}_{i\geq 0}$ on $A_{\bar{k}}$. Note that any iterative infinite higher derivation kills idempotent elements. D' is locally nilpotent.

We can assume $\bar{k}$ is a quasi-Galois extension of k with Galois group $\mathfrak{G}$. An element γ of $\mathfrak{G}$ acts on D' by ${}^{\gamma}D'_i(\bar{a})={}^{\gamma^{-1}}D'_i({}^{\gamma}\bar{a})$ for $\bar{a}\in A_{\bar{k}}$. Then D' is $\mathfrak{G}$-invariant. Note that $\mathfrak{G}$ acts on $A_{\bar{k}}=A\underset{k}{\otimes}\bar{k}$ via the canonical $\mathfrak{G}$-action on $\bar{k}$. Hence the ring of $\mathfrak{G}$-invariants in $A_{\bar{k}}$ is $A_{k'}=A\underset{k}{\otimes}k'$ with $k'=(\bar{k})^{\mathfrak{G}}$. k' is a purely inseparable algebraic extension of k.

Let a' be an element of $A_{k'}$. Then ${}^{\gamma}D'_i(a')={}^{\gamma^{-1}}D'_i({}^{\gamma}a')={}^{\gamma^{-1}}D'_i(a')=D'_i(a')$ for all $\gamma\in\mathfrak{G}$. Namely $D'_i(a')$ is $\mathfrak{G}$-invariant. Therefore $D'_i(A_{k'})\subseteq A_{k'}$ for all $i\geqslant 0$. Since $D'=\{D'_i\}_{i\geq 0}$ is locally nilpotent on $A_{k'}$, D' defines an action of $G_{a,k'}$ on Spec$(A_{k'})$. Now we apply Lemma 2 of [3]. Since $H^1(G_{a,k'},A_{k'})\underset{k'}{\otimes}\bar{k}=H^1(G_{a,\bar{k}},A_{\bar{k}})=0$, the action of $G_{a,k'}$ on Spec$(A_{k'})$ is free and there exists a quotient of Spec$(A_{k'})$ by $G_{a,k'}$. Let $\mathfrak{v}'$ be the subring of $A_{k'}$ consisting of elements killed by $D'|_{A_{k'}}$. Then Spec$(\mathfrak{v}')$ is the quotient of Spec$(A_{k'})$ by $G_{a,k'}$ and $A_{k'}=\mathfrak{v}'[t]$.

Let $\mathfrak{v}=A/tA$. Then $\mathfrak{v}$ is separable over k. On the other hand, $\mathfrak{v}'=\mathfrak{v}\underset{k}{\otimes}k'$ and $\mathfrak{v}'$ is separable over k'. This means that if $\mathfrak{v}_0$ is the subring of all separable elements in $\mathfrak{v}'$ over k, then $\mathfrak{v}_0\cong\mathfrak{v}$ and $\mathfrak{v}'=\mathfrak{v}_0\underset{k}{\otimes}k'$. Since A is the set of all separable elements of $A_{k'}$ over $k[t]$, one has $\mathfrak{v}_0\subset A$. It is now easy to see that $A=\mathfrak{v}_0[t]$. q.e.d.

The algebraic fundamental group of a torus is shown to be abelian in the following

PROPOSITION 1. *Let k be an algebraically closed field of characteristic p, let T be a torus defined over k and let X be a connected Galois etale* *covering of T with group F. Then F is abelian, X is a torus and after an appropriate change of parameters in X and T, the base map $q: X \to T$ is a homomorphism of tori.*

PROOF. (I) The case where $p=0$. An induction argument on the order n of F reduces the proof of Proposition 1 to prove the following:

(1) When $F \cong \boldsymbol{Z}/a\boldsymbol{Z}$ with a prime number a, Proposition 1 is true.

(2) If F is a non-commutative simple group, $F \cong (1)$.

Since k is algebraically closed, T is a split torus. Hence T has a decomposition $T = G_{m,k} \times T'$, where T' is a torus with $\dim(T') = \dim(T) - 1$. Let K be the function field of T'. Then $X_K = X \underset{T'}{\times} \mathrm{Spec}(\bar{K})$ is a connected Galois etale covering of $G_{m,K}$ with group F. Let $\bar{K}$ be an algebraic closure of K and let $\mathfrak{G}$ be the Galois group of $\bar{K}$ over K. In virtue of Lemma 1, (1), $X_{\bar{K}} = G_{m,\bar{K}} \coprod \cdots \coprod G_{m,\bar{K}}$. Let X_0 be the first component and let $\mathfrak{G}_0$ be the stabilizer group*) of X_0, i.e. $\mathfrak{G}_0 = \{\gamma \in \mathfrak{G} \mid \gamma X_0 \subseteq X_0\}$. Look at the following commutative diagram,

$$\begin{array}{ccc} X_K & \longrightarrow & X_{\bar{K}} = \underset{\substack{\| \\ X_0}}{G_{m,\bar{K}}} \coprod \cdots \coprod G_{m,\bar{K}} \\ \downarrow q & & \bar{q} \downarrow \\ G_{m,K} & \longleftarrow & G_{m,\bar{K}} \end{array}$$

After an appropriate change of parameters in X_0 and $G_{m,\bar{K}}$ below, one can assume the restriction of the base map q on X_0 is the multiplication by some positive integer l. Let $X_0 = \mathrm{Spec}(\bar{K}[t, t^{-1}])$ and let $G_{m,\bar{K}} = \mathrm{Spec}(\bar{K}[t^l, t^{-l}])$. Furthermore, once obtained the above splitting of $X_{\bar{K}}$, one can assume $\bar{K}$ is a finite normal extension of K. Hence $\mathfrak{G}$ is a finite group. Then there exists a quotient scheme $X_0/\mathfrak{G}_0$ whose structure we shall give more explicitly.

$\mathfrak{G}$(resp. $\mathfrak{G}_0$) acts on $G_{m,\bar{K}} = \mathrm{Spec}(\bar{K}[t^l, t^{-l}])$, trivially on t^l and via the canonical $\mathfrak{G}$-action on $\bar{K}$. For any element γ of $\mathfrak{G}_0$, ${}^{\gamma}t$ is of the form at^m with $a \in \bar{K}$ and $m \in \boldsymbol{Z}$. However ${}^{\gamma}(t^l) = t^l$. Hence

*) $\mathfrak{G}$ acts on $X_{\bar{K}}$ via the canonical action of $\mathfrak{G}$ on $\bar{K}$.

$a^l t^{lm} = t^l$, whence $m=1$ and a is a l-th root of the unity. This implies an existence of a multiplicative character $\varphi: \mathfrak{G}_0 \to \{$the l-th roots of the unity$\}$ such that ${}^{\gamma}t = \varphi(\gamma)t$ for all $\gamma \in \mathfrak{G}_0$.

Let $a_{-r}t^{-r} + \dots + a_{-1}t^{-1} + a_0 + a_1 t + \dots + a_s t^s$ be any element of $\bar{K}[t, t^{-1}]$ with $a_{-r}, \dots, a_0, \dots, a_s \in \bar{K}$. If this element is $\mathfrak{G}_0$-invariant, we know from the $\mathfrak{G}_0$-action given explicitly that each term should be $\mathfrak{G}_0$-invariant. Let $at^{l'}$ be a $\mathfrak{G}_0$-invariant element of $\bar{K}[t, t^{-1}]$ with the least positive degree with respect to t. Then the set of $\mathfrak{G}_0$-invariants in $\bar{K}[t, t^{-1}]$ is $L[(at^{l'}), (at^{l'})^{-1}]$ with $L = (\bar{K})^{\mathfrak{G}_0}$ and it is the affine algebra of $X_0/\mathfrak{G}_0$.

$\bar{K}[t, t^{-1}]$ has rank $l' \times \mathrm{card}(\mathfrak{G}/\mathfrak{G}_0)$ as a finite $L[(at^{l'}), (at^{l'})^{-1}]$ algebra, while, on the other hand, it should be $\leq \mathrm{card}(\mathfrak{G}/\mathfrak{G}_0)$ since $\mathfrak{G}_0$ acts freely on $X_{\bar{K}}$. This shows that $l'=1$. Therefore $X_0/\mathfrak{G}_0 = \mathrm{Spec}(L[(at), (at)^{-1}])$. After a parameter change $t \rightsquigarrow at$ in X_0, we may assume $X_0/\mathfrak{G}_0 = \mathrm{Spec}(L[t, t^{-1}])$.

Let $(e_i)_{1\leqslant i\leqslant r}$ be all idempotent elements in the affine algebra of $X_{\bar{K}}$ which correspond to connected components of $X_{\bar{K}}$. $\mathfrak{G}$ acts transitively on $(e_i)_{1\leqslant i\leqslant r}$ since X_K is connected. Let $a_1e_1 + \dots + a_re_r$ be a $\mathfrak{G}$-invariant element in the affine algebra of $X_{\bar{K}}$. If ${}^{\gamma}e_1 = e_i$ for some $\gamma \in \mathfrak{G}$, then ${}^{\gamma}a_1 = a_i$. Hence the above element is of the form $\sum_{\bar{\gamma} \in \mathfrak{G}/\mathfrak{G}_0} {}^{\bar{\gamma}}(a)^{\bar{\gamma}}(e_1)$ with $a \in L[t, t^{-1}]$. Conversely any element of this type is $\mathfrak{G}$-invariant. Let $L' = \{\sum_{\bar{\gamma} \in \mathfrak{G}/\mathfrak{G}_0} ({}^{\bar{\gamma}}\lambda)({}^{\bar{\gamma}}e_1);\ \lambda \in L\}$ and let $t' = \sum_{\bar{\gamma} \in \mathfrak{G}/\mathfrak{G}_0} ({}^{\bar{\gamma}}t)({}^{\bar{\gamma}}e_1)$. Then $L'[t', t'^{-1}]$ is K-isomorphic to $L[t, t^{-1}]$ and $X_{\bar{K}}/\mathfrak{G} = X_K \cong \mathrm{Spec}(L'[t', t'^{-1}])$. The base map $q: X_K \to G_{m,K} = \mathrm{Spec}(K[t, t^{-1}])$ is given by $t \rightsquigarrow t'^l$ and a natural injection $K \hookrightarrow L'$. Hereafter t is a parameter of $G_{m,K}$ below.

Now F acts on X_K as follows: there exists a multiplicative character $\psi: F \to \{$the l-th roots of the unity$\}$ such that ${}^{f}t' = \psi(f)t'$ for all $f \in F$. F acts on L' as K-automorphisms. With a similar argument, one knows that $L'^F = K$ and that there exists an element f_0 of F such that $\psi(f_0)$ is a primitive l-th root of the unity. Namely ψ is surjective.

First of all, consider the second case. If F is non-commutative and simple, l must be 1 since ψ is surjective. Then X_K is isomorphic to $\mathrm{Spec}(L'[t, t^{-1}])$. Let Y be the closure in X of a L'-rational point $t'=t=1$ of X_K. The affine algebra of Y is the integral closure of the affine algebra of T' in L'. Then $Y \subset X|_{(t=1)\times T'}$ and in fact, Y coincides with $X|_{(t=1)\times T'}$ since $X|_{(t=1)\times T'}$ is connected. Then by induction on $\dim(T)$, $F \cong (1)$.

Consider the first case. Let Y be the closure in X of a L'-rational point $t'=1$ of X_K. Y is a connected component of $X|_{(t=1)\times T'}$ which has just l connected components. Y is a connected Galois etale covering of T' with group $F_0=\mathrm{Im}(F \to \mathrm{Gal}(L'/K))$. l might be 1. If so, $F_0=F$. If $l\neq 1$, $l=a=$ the order of F. By induction on $\dim(T)$, Y is a torus and after an appropriate change of parameters, the base map $Y \to (t=1)\times T' \cong T'$ is a homomorphism of tori.

If $l=1$, then $X \cong G_{m,k} \underset{k}{\times} Y$. Hence X is a torus. If $F_0=(1)$, then $X \cong \mathrm{Spec}(k[t', t'^{-1}]) \underset{k}{\times} T'$. Again X is a torus. Finally assume that $l\neq 1$ and $F_0=F$. One can assume that $T'=\mathrm{Spec}(k[t_2, t_2^{-1}, \ldots, t_n, t_n^{-1}])$ and $Y=\mathrm{Spec}(k[t'_2, t_2'^{-1}, t_3, t_3^{-1}, \ldots, t_n, t_n^{-1}])$ with $t_2'^{a}=t_2$ and $n=\dim(T)$. $F \cong \boldsymbol{Z}/a\boldsymbol{Z}$ acts on $t'_2, t_3, \ldots, t_n$ by ${}^f t'_2=\psi'(f)t'_2$, ${}^f t_3=t_3, \ldots, {}^f t_n=t_n$ for all $f \in F$, where ψ' is a multiplicative character of F onto $\{l\text{-th roots of the unity}\}$. Now make the following change of parameters in $X_K=\mathrm{Spec}(L'[t', t'^{-1}])$ with $L'=k(t'_2, t_3, \ldots, t_n)$: $(t', t'_2, t_3, \ldots, t_n) \rightsquigarrow (t''=t't_2'^{-r}, t'_2, t_3, \ldots, t_n)$ if $\psi=\psi'^{r}$. Then t'' is F-invariant, $X_K=\mathrm{Spec}(L'[t'', t''^{-1}])$ and $G_{m,K}=\mathrm{Spec}(K[t'', t''^{-1}])$. Thus this case is reduced to the case where $l=1$.

(II) The case where $p\neq 0$. With the same notation as in (I), there exist an integer r and a finite separable extension L of $K^{p^{-r}}$ such that $X_{K^{p^{-r}}} \cong G_{m,K^{p^{-r}}} \underset{K^{p^{-r}}}{\otimes} L$. If $T'=\mathrm{Spec}(k[t_2, t_2^{-1}, \ldots, t_n, t_n^{-1}])$, let $T'^{(p^{-r})} = \mathrm{Spec}(k[t_2^{p^{-r}}, t_2^{-p^{-r}}, \ldots, t_n^{p^{-r}}, t_n^{-p^{-r}}])$. Then $K^{p^{-r}}$ is the function field of $T'^{(p^{-r})}$.

If F is a non-commutative simple group, $l=1$ and F acts regularly on L, hence on the integral closure of $k[t_2^{p^{-r}}, t_2^{-p^{-r}}, \ldots, t_n^{p^{-r}}, t_n^{-p^{-r}}]$ in

L whose spectrum is $X \underset{T'}{\times} T'^{(p^{-r})}|_{(t'=1)\times T'^{p^{-r}}}$. Induction on $\dim(T)$ shows that $F \cong (1)$.

Consider the case where $F \cong \boldsymbol{Z}/a\boldsymbol{Z}$ with a prime number a. If $l=1$, we finish the proof by the same argument as above. If $F'_0 = \mathrm{Im}(F \rightarrow \mathrm{Gal}(L/K^{p^{-r}}))$ is not trivial, one can assume that $L \cong k((t_2^{p^{-r}})^{a^{-1}}, t_3^{p^{-r}}, \ldots, t_n^{p^{-r}}) = k(t_2^{a^{-1}}, t_3, \ldots, t_n)^{p^{-r}}$ after an appropriate change of parameters in T'. Since the function field of X_K is the set of all separable elements in the function field of $X_{K^{p^{-r}}}$ over $K(t)$ and since t', $t_2^{l^{-1}}$, $t_3, \ldots, t_n$ are all separable over $K(t)$, the function field of X_K is $k(t', t_2^{l^{-1}}, t_3, \ldots, t_n)$, in which the integral closure of $K[t, t^{-1}]$ is $k[t', t'^{-1}] \underset{K}{\otimes} K(t_2^{l^{-1}})$. Then $X_K = \mathrm{Spec}(k[t', t'^{-1}] \underset{K}{\otimes} K(t_2^{l^{-1}}))$. This case is reduced to the case $l=1$. If $F'_0 = (0)$, the proof is a little modification of the case where $F'_0 \neq 0$. q.e.d.

As stated in Introduction, we shall prove

LEMMA 3. *Let k be an algebraically closed field of characteristic p. If $p \neq 0$, we assume the order of F is prime to p. Let X be a connected Galois etale covering of G with group F. Then the restriction of X on a maximal torus T is connected too.*

PROOF. Assuming the contrary, we shall derive a contradiction. Assume the restriction X_T is not connected. Take the root system R, a positive (resp. negative) root system R_+ (resp. R_-) and the gross cell $\Omega_{R_+} = \prod_{r \in R_-} P_r \times T \times \prod_{r \in R_+} P_r$ as indicated in Introduction. Write $\Omega_{R_+} \times U$ simply in the form $\Omega_{R_+} = T \times \underbrace{\boldsymbol{A}^1 \times \ldots \times \boldsymbol{A}^1}_{2\,\mathrm{card}(R_+) + \dim(U)}$. Let $V = T \times \underbrace{\boldsymbol{A}^1 \times \ldots \times \boldsymbol{A}^1}_{r}$ be a sub-product of $\Omega_{R_+} \times U$, i.e. $r < 2\,\mathrm{card}(R_+) + \dim(U)$. Assuming that X_V is not connected, we shall show that $X_{V'}$ with $V' = V \times \boldsymbol{A}^1$ is not connected either.

Let K be the function field of V and let $X' = X|_{V'} \underset{V'}{\times} \boldsymbol{A}^1_K$. Then X' is a Galois etale covering of $\boldsymbol{A}^1_K$ with group F. Due to Lemma 2, there exists a finite etale K-algebra R such that $X' = \boldsymbol{A}^1 \underset{K}{\times} \mathrm{Spec}(R)$

and that the action of F on X' comes from an action of F on R. Let B be the integral closure of the affine algebra of V in R. Then $\mathrm{Spec}(B)=X|_V$ and $X|_{V'}=A_k^1 \times_k X_V$. Since $X|_V$ is not connected, $X|_{V'}$ is not connected either. Therefore by induction on $2\,\mathrm{card}(R_+)+\dim(U)$, $X|_{\Omega_{R_+}\times U}$ is not connected. Then X is not connected. This is a contradiction. q.e.d.

Thus we have finished the proof of Theorem 1.

2. The proof of Theorems 2 and 3. Our next aim is to give a complete determination of the algebraic fundamental group $\pi_1(G)$ (or $\pi_{1,+p}(G)$) of a connected algebraic linear group G. We need the following Lemma 4 in order to prove Theorem 2.

LEMMA 4. *Let k be an algebraically closed field of characteristic p, let G be a connected algebraic linear group defined over k, let $G(k)$ be the group of k-rational points of G and let F be a finite abelian group. One assumes the order of F is prime to p if $p\neq 0$. Denote by* $\mathrm{Ext}^1(G(k), F)$ *the ordinary extension group of $G(k)$ (considered as an abstract group) by F. Then we have injective homomorphisms of abelian groups,*

$$\mathrm{Ext}^1(G, F) \overset{\theta}{\hookrightarrow} \mathrm{H}^1(G, F) \overset{\rho}{\hookrightarrow} \mathrm{Ext}^1(G(k), F).$$

PROOF. θ is a canonical homomorphism which regards an extension as an etale covering. Its injectivity is easy to see. ρ is a homomorphism defined as follows: First of all, $\mathrm{H}^1(G, F)$ is a finite abelian group in virtue of Lemma 6 in [4]. The group of k-rational points $G(k)$ acts on $\mathrm{H}^1(G, F)$ in the following way; for $g\in G(k)$ and for a class $[X]$ of $\mathrm{H}^1(G, F)$, ${}^g[X]$ is the class obtained from a base change $r_g: G\rightarrow G$, r_g being the right translation by g. This action of $G(k)$ on $\mathrm{H}^1(G, F)$ is algebraic and therefore trivial since $G(k)$ is connected (with its k-Zariski topology) and $\mathrm{H}^1(G, F)$ is a finite set. Let σ_g be a morphism defined from a commutative diagram,

$$\begin{array}{ccccc} F\underset{k}{\times}X & \rightrightarrows & X & \longrightarrow & G \\ \uparrow{\scriptstyle 1_F\times\sigma_g} & & \uparrow{\scriptstyle \sigma_g} & & \uparrow{\scriptstyle r_g} \\ F\underset{k}{\times}X & \rightrightarrows & X & \longrightarrow & G \end{array}$$

Then for any $g, g' \in G(k)$, $\sigma_{gg'}\sigma_{g'}^{-1}\sigma_g^{-1} \in \mathrm{Aut}_G^F(X)$; the group of automorphisms of the Galois etale covering X. $\mathrm{Aut}_G^F(X)$ is indeed isomorphic to F (cf. [4]). Let $\tau(g, g')=\sigma_{gg'}\sigma_{g'}^{-1}\sigma_g^{-1}$. Then we have

$$\tau(g, g'g'')(\sigma_g\tau(g', g'')\sigma_g^{-1}) = \tau(gg', g'')\tau(g, g')$$

where $\sigma_g\tau(g', g'')\sigma_g^{-1}=\tau(g', g'')$, for the action of $G(k)$ on F defined by ${}^gf=\sigma_g f\sigma_g^{-1}$ is trivial since this action is algebraic and $G(k)$ is connected. Thus $\tau(g, g')$ defines a 2-cocycle $Z^2(G(k), F)$. Hence it defines an element $\rho([X])$ of $\mathrm{Ext}^1(G(k), F)$. The homomorphism ρ is an application which assigns $\rho([X])$ to $[X]$.

Suppose $\rho([X])$ be trivial. Then there exists an application μ: $G(k)\to F$ such that $\tau(g, g')=\mu(gg')\mu(g)^{-1}\mu(g')^{-1}$. Let $\sigma'_g=\mu(g)^{-1}\sigma_g$. Then $\sigma'_{gg'}=\sigma'_g\sigma'_{g'}$, for $g, g'\in G(k)$. The same argument with k replaced by a universal domain Ω over k shows that σ'_g defines an algebraic action of G on X. Take any k-rational point x of X. The orbit Gx of x defines a section from G to X. Therefore X is split. Thus ρ is injective. q.e.d.

We can now prove

THEOREM 2. *Let k be an algebraically closed field of characteristic p, let G be a connected algebraic linear group defined over k and let F be a finite abelian group. If $p\neq 0$, assume the order of F is prime to p. Then we have,*

$$\mathrm{Ext}^1(G, F)\cong \mathrm{H}^1(G, F)=\begin{cases} \mathrm{Hom}_{gr}(\pi_1(G), F) & \textit{if} \quad p=0. \\ \mathrm{Hom}_{gr}(\pi_{1,\nmid p}(G), F) & \textit{if} \quad p\neq 0. \end{cases}$$

PROOF. Let e be the unit element of G and let l_e and r_e be morphisms defined by

$$l_e:\ g\in G \longrightarrow (e\times g)\in G\times G$$

$$r_e:\ g\in G \longrightarrow (g\times e)\in G\times G.$$

Then we have an isomorphism,

$$\mathrm{Ext}^1(G(k)\times G(k),\ F) \xrightarrow[\sim]{l_e^*\times r_e^*} \mathrm{Ext}^1(G(k),\ F)\times\mathrm{Ext}^1(G(k),\ F).$$

Moreover we have a commutative diagram,

$$\begin{array}{ccc}
\mathrm{H}^1(G\times G,\ F) & \xrightarrow{l_e^*\times r_e^*} & \mathrm{H}^1(G,\ F)\times\mathrm{H}^1(G,\ F) \\
\big\downarrow\cap & & \big\downarrow\cap \\
\mathrm{Ext}^1(G(k)\times G(k),\ F) & \xrightarrow{l_e^*\times r_e^*} & \mathrm{Ext}^1(G(k),\ F)\times\mathrm{Ext}^1(G(k),\ F).
\end{array}$$

Therefore $\mathrm{H}^1(G\times G,\ F) \xrightarrow{l_e^*\times r_e^*} \mathrm{H}^1(G,\ F)\times\mathrm{H}^1(G,\ F)$ is injective. Since $l_e^*\times r_e^*$ is surjective, $l_e^*\times r_e^*$ is an isomorphism. Namely two Galois etale coverings of $G\times G$ with group F are isomorphic to each other if and only if they are so on $(e)\times G$ and $G\times(e)$.

Let m (resp. μ) be the multiplication of G (resp. F). Given a Galois etale covering X of G with group F, we have two Galois etale coverings $m^*(X)$ and $\mu_*(X\times X)$ of $G\times G$ with group F. They are easily shown to be isomorphic on $(e)\times G$ and $G\times(e)$. Hence they are isomorphic to each other on $G\times G$. Thus we have a morphism γ: $X\times X\rightarrow\mu_*(X\times X)\xrightarrow{\sim} m^*(X)\rightarrow X$ which commutes a diagram,

$$\begin{array}{ccccc}
F\times X & \rightrightarrows & X & \longrightarrow & G \\
\uparrow & & \uparrow & & \uparrow m \\
F\times m^*(X) & \rightrightarrows & m^*(X) & \longrightarrow & G\times G \\
\uparrow & & \uparrow & & \| \\
F\times\mu_*(X\times X) & \rightrightarrows & \mu_*(X\times X) & \longrightarrow & G\times G \\
\uparrow \mu\times\delta & & \uparrow \delta & & \| \\
(F\times F)\times(X\times X) & \rightrightarrows & X\times X & \longrightarrow & G\times G.
\end{array}$$

Let x_0 be an element of X over the unit element e of G. We can assume ν satisfies $\nu(x_0\times x_0)=x_0$. Then ν is associative, i. e. $\nu\times 1_X=1_X\times\nu$ and $\nu(x, x_0)=\nu(x_0, x)=x$ for all point x of X. Under these conditions, X is an algebraic group with the multiplication ν and with the unit element x_0. The base map $q: X\rightarrow G$ is a homomorphism of algebraic groups. Thus the homomorphism θ of Lemma 4 is surjective.

The second isomorphism in Theorem 2 is almost trivial from the definition of $\pi_1(G)$ and $\pi_{1,+p}(G)$. q.e.d.

REMARK 2. (Due to M. Maruyama) If the order of F is a power of the characteristic p, Theorem 2 ceases to be valid. Let $p=2$ and let E be an elliptic curve of Hasse invariant 0. Consider E as an abelian variety of dimension 1, taking a point e as the point of unity. Then E has no 2-division point. Consider an automorphism of E; $a \rightsquigarrow -a$. Let X be the quotient variety of E by this action of $\boldsymbol{Z}/2\boldsymbol{Z}$. Then X-(the image of e) is the affine line $\boldsymbol{A}^1$ and E-(e) is a Galois etale covering of $\boldsymbol{A}^1$ with group $\boldsymbol{Z}/2\boldsymbol{Z}$. However, since the function field of an algebraic linear group should be rational, E-(e) is never isomorphic to an algebraic linear group. Thus $\mathrm{Ext}^1(A^1, \boldsymbol{Z}/2\boldsymbol{Z})\subsetneqq \mathrm{H}^1(A^1, \boldsymbol{Z}/2\boldsymbol{Z})$.

Due to Theorem 2, we can give a complete determination of $\pi_1(G)$ (or $\pi_{1,+p}(G)$). This is shown in

THEOREM 3. *Let k be an algebraically closed field of characteristic p and let G be a connected algebraic linear group defined over k. Let r be the dimension of the torus part of the radical of G and let $\bar{G}$ be the associated semi-simple group of G. Then $\pi_1(G)$ (or $\pi_{1,+p}(G)$ if $p\neq 0$)$\cong(\hat{\boldsymbol{Z}})^r\times\mathrm{Ext}^1(\bar{G}, G_m)$ (or $(\hat{\boldsymbol{Z}}/\hat{\boldsymbol{Z}}_p)^r\times\mathrm{Ext}^1$ (G, G_m) if $p\neq 0$), where $\hat{\boldsymbol{Z}}$ is the profinite completion of $\boldsymbol{Z}$ and $\hat{\boldsymbol{Z}}_p$ is the p-adic completion of $\boldsymbol{Z}$.*

PROOF. Let U be the unipotent radical of G, let $T=\mathrm{rad}(G)/U$, let $P=G/U$ and let $\bar{G}=G/\mathrm{rad}(G)=P/T$. Let $\bar{G}$ be a simply connected semi-simple algebraic linear group over $\bar{G}$ which is an isogeny with kernel F_0.

Let $0 \to F \to G' \xrightarrow{\pi} G \to 0$ be an extension of G by a finite abelian group F, where the order of F is prime to p if $p \neq 0$. One can assume G' is connected. Let U' be the unipotent radical of G'. Then the restriction of π on U' is an isomorphism, since $\mathrm{Ker}(\pi|_{U'}) = U' \cap F = (0)$. Hence π induces an isogeny $\bar{\pi}$ from the associated reductive group $P' = G'/U'$ of G' to P with $\mathrm{Ker}\,\bar{\pi} = F$. This implies $\pi_1(G)$ (or $\pi_{1,\nmid p}(G)$ if $p \neq 0$) is isomorphic to $\pi_1(P)$ (or $\pi_{1,\nmid p}(P)$ if $p \neq 0$). Thus one can assume G is reductive.

Let $\tilde{G}' = G \underset{\bar{G}}{\times} \tilde{G}$ and look at a commutative diagram,

$$\begin{array}{ccccc} T & \longrightarrow & G & \longrightarrow & \bar{G} \\ \| & & \uparrow & & \uparrow \\ T & \longrightarrow & \tilde{G}' & \longrightarrow & \tilde{G} \\ & & \uparrow & & \uparrow \\ & & F_0 & = & F_0 \,. \end{array}$$

Consider the derived group $\mathrm{der}(\tilde{G}'^0)$ of the neutral component $\tilde{G}'^0$ of $\tilde{G}'$ which is a simi-simple algebraic group and which is normal in $\tilde{G}'$. The projection $\tilde{G}' \to \tilde{G}$ restricted on $\mathrm{der}(\tilde{G}'^0)$ is a separable isogeny, hence it is an isomorphism since $\tilde{G}$ is simply connected. Thus $\tilde{G}' = T \times \tilde{G}$. $\tilde{G}$ has no connected Galois etale covering (cf. [4]). Therefore a Galois etale covering of $\tilde{G}'$ comes from a Galois etale covering of T. This implies an exact sequence

$$0 \longrightarrow \pi_1(T) \longrightarrow \pi_1(G) \longrightarrow F_0 \longrightarrow 0$$

where one replaces $\pi_1(T)$ and $\pi_1(G)$ by $\pi_{1,\nmid p}(T)$ and $\pi_{1,\nmid p}(G)$ if $p \neq 0$. In virtue of Proposition 1, $\pi_1(T) = \hat{\boldsymbol{Z}}^r$ (or $\pi_{1,\nmid p}(T) \cong (\hat{\boldsymbol{Z}}/\hat{\boldsymbol{Z}}_p)^r$ if $p \neq 0$).

The above sequence of fundamental groups is split exact. In fact, let T' be a maximal torus of G such that $T' \supset T$. Then $T'' = T'/T$ is a maximal torus of $\bar{G}$ and $T' \cong T \times T''$. Then the argument which proved that $\pi_1(T)\to\pi_1(G)$ is surjective applies to the present situation in order to obtain the following exact commutative diagram,

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & K \arrow[r, equal] \arrow[d] & K \arrow[d] & \\
0 \arrow[r] & \pi_1(T) \arrow[r, shift left, "j"] \arrow[d, equal] & \pi_1(T') \arrow[l, shift left, "\xi'"] \arrow[r] \arrow[d] & \pi_1(T'') \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \pi_1(T) \arrow[r, shift left, "i"] & \pi_1(G) \arrow[l, shift left, "\xi"] \arrow[r] \arrow[d] & \pi_1(\bar{G})=F_0 \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

where K is the kernel of the surjection $\pi_1(T'')\to F_0$, i and j are the canonical injections and where $\pi_1(\ \ )$ should be replaced by $\pi_{1,\nmid p}(\ \ )$ if $p\neq 0$. Since $\pi_1(T)\cong\hat{\boldsymbol{Z}}^d$ (or $\pi_{1,\nmid p}(T)\cong(\hat{\boldsymbol{Z}}/\hat{\boldsymbol{Z}}_p)^d$ if $p\neq 0$) for any torus group T with $d=\dim T$, there exists a homomorphism ξ': $\pi_1(T')\to\pi_1(T)$ such that $\xi'\cdot j=1_{\pi_1(T)}$ and $\mathrm{Ker}\ \xi'\supset K$. Therefore ξ' gives rise to a homomorphism ξ: $\pi_1(G)\to\pi_1(T)$ such that $\xi\cdot i=1_{\pi_1(T)}$. Therefore we get $\pi_1(G)\cong\pi_1(T)\times F_0$ (or $\pi_{1,\nmid p}(G)\cong\pi_{1,\nmid p}(T)\times F_0$ if $p\neq 0$), where $F_0\cong\mathrm{Ext}^1(G, G_m)$ (cf. [4]). q.e.d.

As a corollary of Theorem 2, we can prove

PROPOSITION 2. *Let k be an algebraically closed field of characteristic p and let G be a connected algebraic linear group defined over k. Then we have isomorphisms,*

$$\mathrm{Ext}^1(G, G_m)\cong \mathrm{H}^1(G, G_m)\cong \mathrm{Pic}(G).$$

When $p\neq 0$, the above isomorphisms are still valid except on the p-th components.

PROOF. For a positive integer n, consider an exact sequence,

$$0 \longrightarrow \boldsymbol{Z}/n\boldsymbol{Z} \longrightarrow G_m \xrightarrow{\times n} G_m \longrightarrow 0.$$

From this, we have an exact commutative diagram,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_n(G) & \longrightarrow & \mathrm{H}^1(G, \boldsymbol{Z}/n\boldsymbol{Z}) & \longrightarrow & {}_n\mathrm{Pic}(G) & \longrightarrow & 0 \\
 & & \wr\uparrow & & \wr\uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathrm{Hom}_{gr}(G, G_m)_n & \longrightarrow & \mathrm{Ext}^1(G, \boldsymbol{Z}/n\boldsymbol{Z}) & \longrightarrow & {}_n\mathrm{Ext}^1(G, G_m) & \longrightarrow & 0
\end{array}$$

(See [4] for the notation.) $C_n(G)$ is actually isomorphic to $\mathrm{Hom}_{gr}(G, G_m)_n$ for any invertible regular function on a connected algebraic group G is a rational character of G up to a non-zero constant. (See, for example, H. Sumihiro, J. Math. Kyoto Univ., **11** (1971), p. 542.) The assertion of Proposition 2 follows immediately from this diagram. q.e.d.

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Added in Proof: The proof of Theorem 3 is incomplete. Theorem should be read as follows: Let T be the torus part of the radical of G. Then we get an exact sequence

$$0 \longrightarrow \pi_1(T) \longrightarrow \pi_1(G) \longrightarrow \mathrm{Ext}(\bar{G}, G_m) \longrightarrow 0$$

where $\pi_1(\ \ )$ is replaced by $\pi_{1,\nmid p}(\ \ )$ if $p\neq 0$ and where $\pi_1(T)\cong\hat{\boldsymbol{Z}}^r$ (or $\pi_{1,\nmid p}(T)\cong(\hat{\boldsymbol{Z}}/\hat{\boldsymbol{Z}}_p)^r$).

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