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On transition probabilities of symmetric strong Markov processes

By

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Introduction

A Markov process $\boldsymbol{M}$ over a measure space $(X, \mathcal{B}, m)$ is said to be m-symmetric if the equality $\int_X R_\alpha f(x) \cdot g(x) m(dx) = \int_X f(x) \cdot R_\alpha g(x) m(dx)$ holds for any non-negative measurable functions f and g on X. Here R_α is the integral operator with the resolvent kernel $R_\alpha(x, dy)$ on $(X, \mathcal{B})$ associated with the process $\boldsymbol{M}$. The transition probability of $\boldsymbol{M}$ is denoted by $P(t, x, dy)$.

We are mainly concerned with the equivalence of the following two conditions: (0. 1) $P(t, x, \cdot)$ is absolutely continuous with respect to m for each $t>0$ and $x \in X$, (0. 2) $R_\alpha(x, \cdot)$ is absolutely continuous with respect to m for each $\alpha>0$ and $x \in X$.

The only thing one has to check is the implication (0. 2)$\Rightarrow$(0. 1). This is violated by some non-symmetric processes, for instance, by the uniform motion on a line with a positive constant velocity. We will show however that the equivalence in question is valid if the process $\boldsymbol{M}$ is symmetric and strong Markov. More specifically the following theorem will be established at the end of §3.

Theorem 6. *Let X be a Borel subset of a compact metric space, $\mathcal{B}$ be the field of all Borel subsets of X and m be a σ-finite measure on $\mathcal{B}$. Consider an m-symmetric right continuous strong*

Markov process $\boldsymbol{M}$ *over* $(X, \mathscr{B})$. *Conditions* (0.1) *and* (0.2) *for* $\boldsymbol{M}$ *are then equivalent.*

In §1 we derive an inequality for general symmetric contraction semigroups on L^2-spaces (Theorem 1).

In §2 (Theorem 3), by making use of this inequality, we show the equivalence of (0.1) and (0.2) especially for those processes which are *properly associated with regular Dirichlet spaces*, the term being specified in §2. Here we use the potential theory developed in [7].

Consider $(X, \mathscr{B}, m)$ and M in the statement of Theorem 6 and suppose that M satisfies the condition (0.2). We reduce in §3 (Theorem 5) the stituation to that of §2 by embedding X into the underlying space of an appropriate regular D-space. This amounts to replacing the topology of X by a new one which is however still coarser than the fine topology for the process $\boldsymbol{M}$. Theorem 3 and 5 added up lead us to Theorem 6.

Theorem 6 is clearly a generalization of A. D. Wentzell's results ([14] and [15]) that the transition probabilities of the one dimensional diffusion on a regular interval and of the multi-dimensional diffusion with Brownian hitting probabilities are always absolutely continuous with respect to the corresponding speed measures. Theorem 6 also generalizes a result in [4] where a transition density of a multi-dimensional reflecting barrier Brownian motion was constructed. All three strong Markov processes above are known to be symmetric and have the property (0.2).

In the one dimensional case, H. Mckean [10] had constructed smooth transition densities alerady by means of eigendifferential expansions. But Wentzell [15] extended his own method of [14] to the multi-dimensional case and indeed the core of his method was a usage of an inequality similar to (1.5). Here we utilize (1.5) differently and we derive from it the *quasi-everywhere fine continuity* of the function $P_t f$ for any non-negative universally measurable and square integrable function f (Theorem 2). In this sense the transition functions attain almost the same degree of regularity as the resolvents as far as symmetric strong Markov processes are concerned. This is the reason why Theorem 6 becomes possible.

It has been proved in [7] that, given any regular D-space, there exists a Hunt process which is properly associated with it. Theorem 5 of the present paper tells us to what extent and in what sense the converse statement is true.

In the final section (§4), we will look at the well-known class of one dimensional diflusions on a regular interval from oue viewpoint of regular Dirichlet spaces. One of the special characters of this class is that each point of the underlying space has a positive capacity, the capacity being evaluated by means of the associated Dirichlet form. This property makes the above mentioned regularity of the transition function into the usual continuity.

It seems hard to treat the whole class of the multi-dimensional diffusions with Brownian hitting probabilities directly within the framework of regular Dirichlet spaces, unless one appeals to some sort of a modification or a reduction like the argument of §3.

1. An inequality for symmetric semi-group

Let (X, m) be a σ-finite measure space. Denote by $(\ ,\ )_X$ the inner product for the real $L^2(X)=L^2(X; m)$. A linear operator T on $L^2(X)$ is said to be sub-Markov if $0\leqq Tu\leqq 1$ m-a.e. whenever $u\in L^2(X)$ and $0\leqq u\leqq 1$ m-a.e.

A symmetric sub-Markov resolvent (on $L^2(X)$) is a family of linear symmetric operators $\{G_\alpha, \alpha>0\}$ such that αG_α is sub-Markov and $G_\alpha-G_\beta+(\alpha-\beta)G_\alpha G_\beta=0$.

A pair $(\mathcal{F}, \mathcal{E})$ is said to be a Dirichlet space (relative to $L^2(X)$) if it satisfies the following conditions: $\mathcal{F}$ is a linear subspace of $L^2(X)$, $\mathcal{E}$ is a non-negative definite bilinear form on $\mathcal{F}$, $\mathcal{F}$ is a real Hilbert space with respect to the inner product

$$\mathcal{E}^{\alpha}(u, v) = \mathcal{E}(u, v) + \alpha(u, v)_X \tag{1.1}$$

for each $\alpha>0$, and finally $u\in\mathcal{F}$ implies $v=(0\vee u)\wedge 1\in\mathcal{F}$ and $\mathcal{E}(v, v)\leqq\mathcal{E}(u, u)$.

There is a one-to-one correspondence between all symmetric sub-Markov resolvents and all Dirichlet spaces [5].

Now let us consider a symmetric sub-Markov resolvent $\{G_\alpha, \alpha>0\}$ and the corresponding Dirichlet space $(\mathcal{F}, \mathcal{E})$. These are related to each other by

$$\mathcal{F} = \{u\in L^2(X)\,;\, \mathcal{E}(u, u) = \lim_{\beta\to+\infty}\beta(u-\beta G_\beta u, u)_X<+\infty\}, \tag{1.2}$$

$$\mathcal{E}^{\alpha}(G_\alpha v, v) = (u, v)_X, \qquad v\in\mathcal{F}, \tag{1.3}$$

u in (1.3) being an arbitrarily fixed element of $L^2(X)$. We put

$$L_0^2(X) = \{u\in L^2(X)\,;\, \alpha G_\alpha u \underset{\alpha\to+\infty}{\longrightarrow} u \text{ in } L^2(X)\}.$$

According to Hille-Yosida theorem, $\{G_\alpha, \alpha>0\}$ is the resolvent of the infinitisimal generator A of a strongly continuous contraction semi-group $\{T_t, t>0\}$ on $L_0^2(X)$. $-A$ is a non-negative definite self-adjoint operator and T_t is symmetric and sub-Markov. Relation (1.2) means that $\mathcal{F}$ is just the domain of $\sqrt{-A}$ and $\mathcal{E}(u, u) = (\sqrt{-A}u, \sqrt{-A}u)_X$, $u\in\mathcal{F}$. Therefore it is easy to see the following.

$$\begin{aligned} &\mathcal{F} = \{u\in L_0^2(x)\,;\, \lim_{s\downarrow 0}\mathcal{E}_s(u, u)<+\infty\} \\ &\mathcal{E}(u, u) = \lim_{s\downarrow 0}\mathcal{E}_s(u, u) \qquad u\in\mathcal{F}, \end{aligned} \tag{1.4}$$

where $\mathcal{E}_s(u, v) = \frac{1}{s}(u - T_s u, v)$, $u, v\in L_0^2(X)$.

Notice that the function $f_t(u) = (T_t u, u)_X$ is a non-negative non-increasing convex function of $t\geqq 0$ for each $u\in L_0^2(X)$:

$$\begin{aligned} f_{t+s}(u) &= (T_t u, T_s u)_X \leqq \sqrt{f_{2t}(u)\cdot f_{2s}(u)} \\ &\leqq \frac{1}{2}\{f_{2t}(u) + f_{2s}(u)\}, \qquad s, t\geqq 0. \end{aligned}$$

Thus we have an inequality

$$\frac{1}{2t}\{f_{2t}(u) - f_0(u)\}\leqq f_{2t}^{+}(u)\leqq 0, \qquad t>0,$$

where f^+ denotes the right derivative in t.

In view of (1.4) and the identity $f^+_{2t}(u) = -\lim_{s\downarrow 0} \mathcal{E}_s(T_t u, T_t u)$, we have obtained the next theorem.

Theorem 1. *T_t, $t>0$, transforms $L^2_0(X)$ into $\mathcal{F}$ and $T_t u$, $u \in L^2_0(X)$, satisfies*

$$(1.5) \qquad \mathcal{E}(T_t u, T_t u) \leqq \frac{1}{2t} \{(u, u)_X - (T_t u, T_t u)_X\}.$$

Inequality (1.5) is a generalization of A. D. Wentzell's ([15]; Theorem 3)

$$\int_G |\operatorname{grad} T_t u|^2(x) dx \leqq \frac{1}{\pi t} \int_G u(x)^2 m(dx)$$

which holds for the diffusion with Brownian hitting probabilities under certain restrictions on the domain G and the speed measure m.

Starting with the inequality (1.5), we can go along the line analogous to [15] to get the equivalence of (0.1) and (0.2). However we will take another course by deriving from (1.5) a sort of continuity of the transition probability $P(t, x, E)$ in x.

2. Proof of the equivalence for a process properly associated with a regular Dirichlet space

We say a Dirichlet space $(X, m, \mathcal{F}, \mathcal{E})$ to be *regular* if following conditions are satisfied.

(2.1) X is a locally compact Hausdorff and separable space,

(2.2) m is an everywhere dense Radon measure on X,

(2.3) $\mathcal{F} \cap C(X)$ is dense in $\mathcal{F}$ with metric $\mathcal{E}^{\alpha_0}$ and in $C(X)$ with uniform norm, α_0 being a fixed positive constant and $C(X)$ being the space of all continuous functions on X vanishing at infinity.

Given a regular Dirichlet space, we define the (α_0)-*capacity* of an open set $E \subset X$ by

$$\text{(2.4)} \qquad \mathrm{Cap}(E) = \begin{cases} \inf_{u \in \mathcal{L}_E} \mathcal{E}^{\alpha_0}(u, u) & \text{if } \mathcal{L}_E \neq \phi \\ +\infty & \text{otherwise} \end{cases}$$

where

$$\text{(2.5)} \qquad \mathcal{L}_E = \{u \in \mathcal{F};\ u \geqq 1 \ m\text{-a.e. on } E\}.$$

The capacity of an arbitrary set F is defined by:

$$\mathrm{Cap}(F) = \inf_{F \subset E,\, E;\text{open}} \mathrm{Cap}(E).$$

$F \subset X$ is called *polar* if $\mathrm{Cap}(F) = 0$. The expression "quasi-everywhere" or "q.e." means "except for a polar set". A function u defined q.e. on X is called *quasi-continuous* if for any $\varepsilon > 0$ there is an open set ω with $\mathrm{Cap}(\omega) < \varepsilon$ such that the restriction of u to $X - \omega$ is continuous there. It is easy to see that each element u of $\mathcal{F}$ admits a quasi-continuous modification $u^*: u = u^*$ m-a.e. $\mathcal{F}^*$ will stand for the collection of all quasi-continuous modifications of elements of $\mathcal{F}$.

Consider a universally measurable subset B of X and denote by $\mathcal{B}_0$ the field of all universally measurable subsets of $X_0 = X - B$. Let $\boldsymbol{M} = (X_t, \zeta, P_x)$ be a standard Markov process over $(X_0, \mathcal{B}_0)$, namely, $\boldsymbol{M}$ is a normal right continuous strong Markov process with the quasi-left continuity (up to ζ). We assume that almost all sample paths have left limits in X_0 up to ζ.

We call $\boldsymbol{M}$ a *Markov process properly associated with the regular D-space* $(X, m, \mathcal{F}, \mathcal{E})$ if further B is polar and if, for any $u \in L^2(X; m) \cap C(X)$, the function $R_\alpha u$ belongs to $\mathcal{F}^*$ and satisfies the equation (1.3). Here R_α is the resolvent of $\boldsymbol{M}$:

$$R_\alpha u(x) = E_x\left(\int_0^\zeta e^{-\alpha t} u(X_t) dt\right), \qquad x \in X_0.$$

Let us assume throughout this section that we are given B and $\boldsymbol{M}$ as above, $\boldsymbol{M}$ being properly associated with $(X, m, \mathcal{F}, \mathcal{E})$. Their existence has been proved in [7] where more specifically a Borel set B and a Hunt process $\boldsymbol{M}$ are constructed.

The equivalence of (0.1) and (0.2) is now to be established for the present process $\boldsymbol{M}$. Here are several related facts whose proof can be found in [7].

1°. If u is quasi-continuous and $u=0$ m-a.e., then $u=0$ q.e.

2°. If $u_n \in \mathcal{T}^*$, $n=1, 2, \cdots$, form a Cauchy sequence with metric $\mathcal{E}^{\alpha_0}$ and converge q.e. on X to a function u, then $u \in \mathcal{T}^*$ and $\mathcal{E}^{\alpha_0}(u_n-u, u_n-u) \to 0$ as $n \to +\infty$.

3°. If u is a non-negative universally measurable function belonging to $L^2(X; m)$, then $R_\alpha u$ belongs to $\mathcal{T}^*$ and satisfies (1.3).

4°. Any quasi-continuous function is finely continuous q.e., the fine topology being defind in terms of the process $\boldsymbol{M}$. Conversely if a function $u \in \mathcal{T}$ is finely continuous q.e. and equal q.e. to a universally measurable function, then $u \in \mathcal{T}^*$.

5°. A nearly Borel set $A \subset X$ is polar if and only if

$$\mathrm{P}_x(X_t \in A \text{ for some } t>0)=0 \tag{2.6}$$

for m-a.e. $x \in X_0$.

A natural question arises: when is it possible to replace "m-a.e. $x \in X_0$" by "all $x \in X_0$" in the statement 5°? Obviously our condition (0.2) is sufficient. Furthermore it is proved in [7; Theorem 4.6] that (0.2) is also a necessary condition for this.

Denote by P_t the integral operator with the transition kernel

$$P(t, x, E)=P_x(X_t \in E), \qquad x \in X_0 .$$

Theorem 2. *For any non-negative universally measurable function u of $L^2(X; m)$ and for any $t>0$, the function $P_t u$ belongs to the space $\mathcal{T}^*$ and satisfies*

$$\mathcal{E}^{\alpha_0}(P_t u, P_t u) \leqq \left(\frac{1}{2t}+\alpha_0\right)(u, u)_X . \tag{2.7}$$

Proof. Let T_t and G_α be the semigroup and the resolvent on $L_0^2(X)$ associated with the present D-space $(\mathcal{T}, \mathcal{E})$ (see §1). Since $(\mathcal{T}, \mathcal{E})$ is assumed to be regular, $L_0^2(X)$ is the entire space $L^2(X)$.

First we show that $P_t u$ is a version of $T_t u$ for any non-negative Borel measurable function $u \in L^2(X)$. 3° means that $R_\alpha u$ is a version of $G_\alpha u$. Hence we have $\int_0^\infty e^{-\alpha t}(P_t u, v)_X\, dt = (R_\alpha u, v)_X = (G_\alpha u, v)_X = \int_0^\infty e^{-\alpha t}(T_t u, v)_X\, dt$ for any $\alpha > 0$ and any non-negative continuous function v with compact support. But $(P_t u, v)_X = (u, P_t v)_X$ is right continuous in $t > 0$. Therefore $P_t u = T_t u$ m-a.e. Moreover, by virtue of Theorem 1, the function $P_t u$ belongs to the space $\mathcal{F}$ and satisfies inequality (2.7).

Denote by $\mathcal{H}$ the collection of all non-negative Borel measurable functions u of $L^2(X)$ for which $P_t u \in \mathcal{F}^*$. If $u_1, u_2 \in \mathcal{H}$ and if $c_1 u_1 + c_2 u_2 \geqq 0$ for some constant c_1, c_2, then $c_1 u_1 + c_2 u_2 \in \mathcal{H}$. If $u_n \in \mathcal{H}$ increases to $u \in L^2(X)$, then $u \in \mathcal{H}$ because of inequality (2.7) and property 2°. Finally take a function $u \in L^2(X) \cap C(X)^+$. Then $\alpha R_\alpha P_t u(x) = \alpha P_t R_\alpha u(x)$ converges to $P_t u(x)$ as α tends to infinity for each $x \in X_0$. According to [6, Lemma 2.1], $\alpha R_\alpha P_t u$ converges to $P_t u (\in \mathcal{F})$ in $\mathcal{E}^{\alpha_0}$-norm as well. Since $R_\alpha P_t u \in \mathcal{F}^*$ by 3°, we can use 2° again to get $P_t u \in \mathcal{F}^*$, namely, $u \in \mathcal{H}$. Thus we have proved Theorem 2 for every non-negative Borel measurable function $u \in L^2(X)$ ([7; §3, Proposition]). The proof for the case that u is universally measurable can be carried out in the same way as in [7; Lemma 3.1].

Theorem 3. *Conditions* (0.1) *and* (0.2) *are equivalent for the process* $\boldsymbol{M}$ *properly associated with the regular Dirichlet space.*

Proof. Fix $t > 0$ and suppose that E is an m-negligible Borel set. Then, by virtue of Theorem 2 and property 1°, $P(t/2, x, E) = P_{t/2} I_E(x) = 0$ except on a polar set $C \subset X_0$. By taking a polar G_δ-set containing C if necessary, we can assume that C is Borel. If we assume the condition (0.2), then (2.6) is valid for every $x \in X_0$ and we obtain

$$P(t, x, E) = E_x(P_{t/2} I_E(X_{t/2}); X_{t/2} \notin C) = 0,$$

$x \in X_0$, getting the condition (0.1).

3. Embedding theorem for symmetric strong Markov processes

Let $(X, \mathcal{B}, m)$ be a σ-finite topological measure space in the statement of Theorem 6 mentioned in the introduction. Let $\boldsymbol{M}=(X_t, \zeta, P_x, x \in X)$ be an m-symmetric right continuous strong Markov process over $(X, \mathcal{B})$. $\boldsymbol{M}$ is assumed to be normal in the sense that $P_x(X_0=x)=1$ for every $x \in X$.

We assume that $\boldsymbol{M}$ satisfies condition (0. 2). Because of this, $R_\alpha u(x)=\int_X R_\alpha(x, dy)u(y)$ defines a Borel measurable function on X for any $u \in L^\infty(X; m)$. Being a difference of α-excessive functions, $R_\alpha u$ is finely continuous. On the other hand, condition (0. 2) implies the following:

(3. 1) m is finely positive: $m(A)>0$ for every non-empty nearly Borel finely open set $A \subset X$.

Actually (0. 2) and (3. 1) are equivalent on account of the symmetry of R_α. Thus we can regard the space of bounded Borel measurable finely continuous functions on X with uniform norm as a closed subalgebra of $L^\infty(X; m)$.

As was stated in [6; §2], the resolvent kernel $\{R_\alpha, \alpha>0\}$ gives rise to a symmetric resolvent on $L^2(X; m)$, to which corresponds a D-space $(X, m, \mathcal{F}, \mathcal{E})$. However the latter may fail to be a regular one. We will rather take its strongly regular representation $(\widetilde{X}, \widetilde{m}, \widetilde{\mathcal{F}}, \widetilde{\mathcal{E}})$ to work with in this section.

There is a sequence $E_i \in \mathcal{B}$ with $m(E_i)<+\infty$, such that $\{E_i\}$ generates $\mathcal{B}$. Put $S_0=\{R_{\alpha_0} I_{E_i}, i=1, 2, \cdots\}$, I_{E_i} being the indicator function of E_i. Starting with S_0, we can exactly in the same manner as in [5; §6] construct a closed subalgebra L of $L^\infty(X; m)$ satisfying the following. Denote by L_+^∞ the set of all non-negative elements of $L^\infty(X; m)$.

(3. 2) $\qquad R_\alpha(L) \subset L, \qquad \alpha>0,$

(3. 3) L is generated by a countable subfamily $L_0 \subset L$ such as

$$S_0 \subset L_0 \subset R_\alpha(L_+^\infty \cap L^1).$$

In view of the preceding remark, all functions in L_0 are finely continuous and Borel measurable, and so are all functions in L.

Denote by $\tilde{X}$ the character space of L, namely, $\tilde{X}$ is the space of all non zero real linear multiplicative functionals on L endowed with the weak* topology. $\tilde{X}$ is a locally compact Hausdorff and separable space. For each $u\in L$, defines a function u on $\tilde{X}$ by

$$(3.4) \qquad \tilde{u}(\chi)=\tilde{X}(u), \qquad \chi\in\tilde{X},$$

then the correspondence $u\rightarrow\tilde{u}$ becomes an isometircal isomorph from L onto the space $C(\tilde{X})$ (c.f. [6; §4]).

Theorem 4. (i) *There is a Borel measurable one to one transform Π from X onto a dense subset of $\tilde{X}$. $\Pi(A)$ is an analytic set in $\tilde{X}$ for any $A\in\mathcal{B}$. In particular $\Pi(X)$ is analytic in $\tilde{X}$.*

(ii) *The induced measure $\tilde{m}$ on $\tilde{X}$ is an everywhere dense Radon measure on $\tilde{X}$, where by definition $\tilde{m}(\tilde{A})=m(\Pi^{-1}(\tilde{A}))$, $\tilde{A}\in\tilde{\mathcal{B}}$.*

(iii) *The induced process $\Pi\boldsymbol{M}=(\Pi X_t, \zeta, P_{\Pi^{-1}\tilde{x}}, \tilde{x}\in\Pi(X))$ is a right continuous strong Markov process on $\Pi(X)$. The resolvent of $\Pi\boldsymbol{M}$ is absolutely continuous with respect to $\tilde{m}$.*

Proof. (i) Each $x\in X$ defines a linear multiplicative functional on L by $l_x(u)=u(x)$, $u\in L$. In view of the inclusion $L\supset S_0$, we see that $l_x\neq 0$ for any $x\in X$ and that $l_x=l_y$ only if $x=y$. Since each element of L is a Borel function on X, it is easy to see that the map $\Pi: x\rightarrow l_x$ is a Borel measurable one to one map from X onto a dense subset of $\tilde{X}$ (c.f. [6; Lemma 4.2]). Now the second statement of (i) follows from P. A. Meyer [12; III, T13], because the space X, being a Souslin set in the terminology of Bourbaki, is a continuous image of a Polish space.

(ii) For $u\in L\cap L^1(X; m)$, we have

$$\int_{\tilde{X}}|u(\tilde{x})|\,\tilde{m}(dx)=\int_X|\tilde{u}(\Pi x)|\,m(dx)=\int_X|u(x)|\,m(dx)<+\infty.$$

This particularly means that $\widetilde{m}$ is finite on each compact set of $\widetilde{X}$, because, $L\cap L^1(\supset L_0)$ being dense in L, the space $\Phi(L\cap L^1)$ is dense in $C(\widetilde{X})$. Here Φ denotes the isometry $u\to\tilde{u}$ from L onto $C(\widetilde{X})$. Let us next prove that $\widetilde{m}$ is everywhere dense on $\widetilde{X}$. Suppose that $\tilde{u}=0$ $\widetilde{m}$-a.e. for some $u\in L\cap L^1$. Then, by the above identity, $u=0$ m-a.e. and hence $\sup_{\tilde{x}\in\widetilde{X}}|\tilde{u}(\tilde{x})|=\|u\|_\infty=0$ by the isometry of Φ, yielding that u is identically zero on $\widetilde{X}$. Now we have, for any $u\in\Phi(L\cap L^1)$, the equality $\sup_{\tilde{x}\in\widetilde{X}}|\tilde{u}(\tilde{x})|=\widetilde{m}-\operatorname{ess}-\sup_{\tilde{x}\in X}|\tilde{u}(\tilde{x})|$, because we can see that $\tilde{u}\wedge a\in\Phi(L\cap L^1)$ whenever $\tilde{u}\in\Phi(L\cap L^1)$ and $a>0$ and that $\tilde{u}-\tilde{u}\wedge a=0$ $\widetilde{m}$-a.e. if $|\tilde{u}(\tilde{x})|\leqq a$ $\widetilde{m}$-a.e. This equality is extended to every $\tilde{u}\in C(\widetilde{X})$, proving that $\widetilde{m}$ is everywhere dense.

(iii) It suffices to show that $Y_t=\Pi X_t$ is right continuous on $\Pi X\subset\widetilde{X}$, P_x-a.s., for any $x\in X$. For each $u\in L$, the process $\tilde{u}(Y_t)=u(X_t)$, $0\leqq t<\zeta$, is right continuous P_x-a.s., because u is finely continuous and Borel measurable on X. Since $\tilde{u}$ exhausts the space $C(\widetilde{X})$, the proof is finished.

Let us define the family of operators $\{\widetilde{R}_\alpha,\ \alpha>0\}$ on $C(\widetilde{X})$ by

$$\widetilde{R}_\alpha\tilde{u}=\widetilde{R_\alpha u},\qquad u\in L. \tag{3.5}$$

Owing to (3.2) and (3.3), $\{\widetilde{R}_\alpha,\ \alpha>0\}$ becomes a symmetric Ray resolvent over $\widetilde{X}$. Let $(\widetilde{\mathcal{F}},\widetilde{\mathcal{E}})$ be a D-space generated by this resolvent. Then $(\widetilde{X},\widetilde{m},\widetilde{\mathcal{F}},\widetilde{\mathcal{E}})$ turns out to be strongly regular. It is in fact a strongly regular representation of the original D-space $(X,m,\mathcal{F},\mathcal{E})$. This has been proved in [6; §6] in a more general context.

(3.5) implies that the kernel $\widetilde{R}_\alpha(\tilde{x},dy)$ over $\widetilde{X}$ is an extension of the resolvent of $\Pi\boldsymbol{M}$:

$$\widetilde{R}_\alpha(\tilde{x},\widetilde{A})=R_\alpha(\Pi^{-1}\tilde{x},\Pi^{-1}\widetilde{A}),\quad \tilde{x}\in\Pi(x),\ \widetilde{A}\subset\widetilde{X},\ \widetilde{A}\in\widetilde{\mathcal{B}}, \tag{3.6}$$

$$\widetilde{R}_\alpha(\tilde{x},\widetilde{X}-\Pi(X))=0,\qquad \tilde{x}\in\Pi(X). \tag{3.7}$$

Notice that $\widetilde{X}-\Pi(X)$ is universally measurable and $\widetilde{m}(\widetilde{X}-\Pi(X))=0$. We have now a regular D-space $(\widetilde{X},\widetilde{m},\widetilde{\mathcal{F}},\widetilde{\mathcal{E}})$ with which the transformed process $\Pi\boldsymbol{M}$ on $\Pi(X)$ is associated. Unfortunately we do not know whether this association is proper in the sense of §2 because we are not able to prove that $\tilde{X}-\Pi(X)$ is polar. Hence we are forced to consider the extended process $\tilde{\boldsymbol{M}}=(\tilde{X}_t, \tilde{\zeta}, \tilde{P}_{\tilde{x}}, \tilde{x}\in\tilde{X})$ the Ray process on $\tilde{X}$ with the resolvent $\{\tilde{R}_\alpha, \alpha>0\}$. If $\tilde{x}\in\Pi(X)$, then the process $(\tilde{X}_t, \tilde{P}_{\tilde{x}})$ is equivalent to $(\Pi X_t, P_{\Pi^{-1}\tilde{x}})$ because both of them are right continuous and have the same resolvent. But we still have trouble: the entire space $\tilde{X}$ might be too large to assure the absolute continuity of $\tilde{R}_\alpha(\tilde{x}, \cdot)$ with respect to $\tilde{m}$ for $\tilde{x}\in\tilde{X}-\Pi(X)$.

In order to get rid of this difficulty, we consider the set[1)]

(3.8) $$\tilde{X}_R=\{x\in\tilde{X};\ \tilde{R}_{\alpha_0}(\tilde{x}, \tilde{X}-\Pi(x))=0\}.$$

The set $\tilde{X}_R$ contains $\Pi(X)$ according to (3.7). Notice that all the potential theoretical criterions $1^\circ\sim5^\circ$ work for the present Ray process $\tilde{\boldsymbol{M}}$ ([7; §3]). Since $\tilde{X}-\Pi(x)$ is $\tilde{m}$-negligible and $\tilde{R}_{\alpha_0}$ is $\tilde{m}$-symmetric, 1° and 3° imply that

(3.9) $$\tilde{X}-\tilde{X}_R \quad \text{is polar.}$$

By virtue of Lemma 3.1 (iii), it is further clear that $\tilde{R}_\alpha(\tilde{x}, \cdot)$ is absolutely continuous with respect to $\tilde{m}$ whenever $x\in\tilde{X}_R$. Combining this with (3.9) and 3°, we arrive at

(3.10) $$\tilde{P}_{\tilde{x}}(\tilde{X}_t\in\tilde{X}-\tilde{X}_R \text{ for some } t\geqq0)=0, \qquad x\in\tilde{X}_R.$$

Here we used the fact that $\tilde{X}-\tilde{X}_R$ is nearly Borel. We can see this from (3.8) because $\tilde{R}_{\alpha_0}(\cdot, \tilde{X}-\Pi(X))$ is α_0-excessive.

We finally put

(3.11) $$\tilde{X}_0=\tilde{X}_R-\tilde{X}_b,$$

where $\tilde{X}_b$ denotes the branch set of $\tilde{\boldsymbol{M}}$. The properties (3.9) and (3.10) are not violated by passing from $\tilde{X}_R$ to $\tilde{X}_0$ (c.f. Theorem 3.2

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1) This subset has been considered already by D. Ray and H. Kunita-T. Watanabe. See [6] for the bibliography.

of [7]). Denote by $\widetilde{\boldsymbol{M}}_0$ the process $\widetilde{\boldsymbol{M}}$ restricted to $\widetilde{X}_0$. Almost all sample paths of $\widetilde{\boldsymbol{M}}_0$ have left limits $\widetilde{X}_0$ up to their lifetimes. This can be verified by making use of the method of time reversion [7; 3.3] and the absolute continuity of the resolvent.

Summarizing what has been mentioned, we conclude as follows.

Theorem 5. *Let the measure space* $(X, \mathcal{B}, m)$ *and the process* $\boldsymbol{M}=(X_t, \zeta, P_x;\ x\in X)$ *be those in the statement of Theorem 6 (see Introduction). We assume the condition* (0.2) *for* $\boldsymbol{M}$. *Then there exist a locally compact Hausdorff and separable space* $\widetilde{X}$ *and a Borel measurable one to one map* Π *from* X *into* $\widetilde{X}$ *satisfying the following*: (i) $(\widetilde{X}, \Pi)$ *has those properties of Theorem 4. In particular the induced measure* $\widetilde{m}=m\Pi^{-1}$ *is an everywhere dense Radon measure on* $\widetilde{X}$. (ii) *There are a universally measurable subset* $\widetilde{X}_0$ *of* $\widetilde{X}$ *and a normal standard process* $\widetilde{\boldsymbol{M}}_0=(\widetilde{X}_t, \widetilde{\zeta}, \widetilde{P}_{\tilde{x}};\ \tilde{x}\in\widetilde{X}_0)$ *on* $\widetilde{X}_0$ *with properties* (a), (b) *and* (c) *below.*

(a) $\Pi(X)\subset\widetilde{X}_0$. *For each* $\tilde{x}\in\Pi(X)$, *the process* $(\widetilde{X}_t, \widetilde{P}_{\tilde{x}})$ *is equivalent to* $(\Pi X_t, P_{\Pi^{-1}\tilde{x}})$.

(b) $\widetilde{\boldsymbol{M}}_0$ *is properly associated with a regular D-space* $(\widetilde{\mathcal{F}}, \widetilde{\mathcal{E}})$ *over* $(\widetilde{X}, \widetilde{m})$.

(c) $\widetilde{\boldsymbol{M}}_0$ *satisfies condition* (0.2): *the resolvent* $\widetilde{R}_\alpha(\tilde{x}, \cdot)$ *of* $\widetilde{\boldsymbol{M}}_0$ *is absolutely continuous with respect to* $\widetilde{m}$ *for each* $\tilde{x}\in\widetilde{X}_0$ *and* $\alpha>0$.

Theorem 5 combined with Theorem 3 prove Theorem 6 stated in the Introduction. In fact under the assumption of Theorem 5 we see that $\widetilde{\boldsymbol{M}}_0$ also satisfies condition (0.1): its transition probability is absolute continuous with respect to $\widetilde{m}$. This in turn implies the property (0.1) for $\boldsymbol{M}$ because of Theorem 4 (i) and Theorem 5 (ii) (a).

4. One dimensional diffusions

Consider an open interval $X=(r_1, r_2)\subset R^1$. Let $s(x)$ be a strictly increasing continuous function on X and m be a Borel measure on X finite on each compactum and positive on each non-empty open interval.

Let us put

$$\mathcal{E}(u, v) = \int_{r_1}^{r_2} \frac{du}{ds} \frac{dv}{ds} ds \tag{4.1}$$

$$\mathcal{F}^R = \{u \in L^2(X;\, m);\ u \text{ is absolutely continuous with respect to } s \text{ and } \mathcal{E}(u, u) < +\infty\}. \tag{4.2}$$

The space $\mathcal{F}^R$ is complete with metric $\mathcal{E}^\alpha$ $(\alpha > 0)$ and in fact the convergence in this metric implies the uniform convergence on each compactum. If $u \in \mathcal{F}^R$, then it is easy to see that $v = (o \vee u) \wedge 1 \in \mathcal{F}^R$ and $\mathcal{E}(v, v) \leqq \mathcal{E}(u, u)$. Hence $(X, m, \mathcal{F}^R \mathcal{E})$ is a D-space.

Any solution u of the differential equation

$$d \cdot \frac{du}{ds} = \alpha u \,\mathrm{dm} \tag{4.3}$$

on a subinterval $(a, b) \subset X$ is called α-harmonic on (a, b). α-harmonic function on $X = (r_1, r_2)$ is simply said to be α-harmonic. The following observation is useful: $u \in \mathcal{F}^R$ is α-harmonic on $(a, b) \subset X$ if and only if

$$\mathcal{E}^\alpha(u, v) = 0 \tag{4.4}$$

for all $v \in K_{(a,b)}$, where $K_{(a,b)}$ denotes the space of those functions on X which are continuously differentiable with respect to s and have compact supports inside (a, b). This is proved by means of the "formal" integration by parts.[2]

Denote by $\mathcal{H}_\alpha$ the space of all α-harmonic functions of $\mathcal{F}^R$. Let $\mathcal{F}^0$ be the closure of $K_{(r_1, r_2)}$ in $(\mathcal{F}^R, \mathcal{E}^{\alpha_0})$, $\alpha_0 > 0$ being fixed. The above observation implies that these two are orthogonally complementary spaces of $(\mathcal{F}^R, \mathcal{E}^\alpha)$: $\mathcal{F}^R = \mathcal{F}^0 \oplus \mathcal{H}_\alpha$. In order to get a more explicit expression of these spaces, let us call the boundary point r_1

2) For the justification of this procedure, see §§54 of F. Riesz-B. St. Nagy, Functional analysis, Ungar, New York, 1955.

regular if both $s(r_1)=\lim_{x\downarrow r_1} s(r)$ and $m(r_1, c)$ $(r_1<c<r_2)$ are finite. The regularity of r_2 is defined similarly. If r_1 is regular, then any function $u\in\mathcal{F}^R$ has a finite limit $u(r_i)=\lim_{x\to r_i} u(x)$. If further $u\in\mathcal{F}^0$, then $u(r_i)=0$.

Let us show the following.

(4. 5) $\mathcal{F}^0\{u\in\mathcal{F}^R;\ u(r_i)=0$ if r_i is regular$\}$.

(4. 6) $\mathcal{H}_\alpha=\{c_1 h_1^\alpha+c_2 h_2^\alpha;\ c_i=0$ if r_i is non-regular$\}$,

where h_1^α (resp. h_2^α) denotes a positive α-harmonic function strictly decreasing (increasing) on (r_1, r_2). (4. 5) follows from (4. 6). (4. 6) is the same as the statement

(4. 7) $\mathcal{E}^\alpha(h_i^\alpha, h_i^\alpha)$ is finite if and only if r_i is regular.

But this in turn follows from the computation

$$\int_a^b\left(\frac{dh_i^\alpha}{ds}\right)^2 \mathrm{ds}+\alpha\int_a^b(h_i^\alpha)^2\,\mathrm{dm}=h_i^\alpha\frac{dh_i^\alpha}{ds}\bigg|_a^b$$

for $r_1<a<b<r_2$ and from the well known boundary behaviours of h_i^α (K. Ito [8; §60, §61]).

In the following we only discuss the absorbing barrier and the reflecting barrier processes. As for the formulation of the most general symmetric bondary conditions within the present framework, see the papers by J. Elliott and M. R. Silverstein [2] and by the present author [5; §9].[3)]

(I) The absorbing barrier process

$(X, m, \mathcal{F}^0, \mathcal{E})$ is clearly a regular D-space. Before discussing the associated process, let us notice two special properties of this regular

3) W. Feller [3; §10, 11] already determined the most general symmetric linear operator G_α which, for a fixed $\alpha>0$, associates with each $f\in L^2(m)$ an element $u\in L^2(m)$ such that $\alpha u-\frac{d}{dm}\frac{d}{ds}u=f$ m-a.e. Our formulation imposes an additional requirement of the submarkovity of G_α. The role of the regular boundary in our argument corresponds to that of the active boundary in Feller's [3].

D-space. First of all the Hilbert space $(\mathcal{F}^0, \mathcal{E}^\alpha)$ has a reproducing kernel, say $g_\alpha^0(x, y)$, since the map $u \in \mathcal{F}^0 \to u(y) \in R^1$ is continuous for each fixed $y \in X$:

$$(4.7) \qquad \mathcal{E}^\alpha(g_\alpha^0(\cdot, y), v) = v(y), \qquad v \in \mathcal{F}^0.$$

Second our D-space admits no non-empty polar set and indeed each point $y \ni X$ has a positive capacity:

$$(4.8) \qquad \overset{\circ}{\mathrm{Cap}}(\{y\}) = \frac{1}{g_{\alpha_0}^0(y, y)} > 0.$$

Recall the definition (2.4) of the capacity of open sets and observe $\overset{\circ}{\mathrm{Cap}}(E_n) = \mathcal{E}^\alpha(p_n, p_n)$ for open $E_n = \left(y - \frac{1}{n}, y + \frac{1}{n}\right)$, where p_n is characterized as the element of $\mathcal{F}^0$ such that $p_n = 1$ on E_n and $\mathcal{E}^{\alpha_0}(p_n, v) \geqq 0$ for every $v \in \mathcal{F}^0$ non-negative on E_n. We can show that p_n converges to a function $p \in \mathcal{F}^0$ in $\mathcal{E}^{\alpha_0}$-norm as well as in pointwise sense and that the limit function p (α_0-equilibrium potential for $\{y\}$) is characterized as the unique element of $\mathcal{F}^0$ such that $p(y) = 1$ and $\mathcal{E}^{\alpha_0}(p, v) \geqq 0$ for every $v \in \mathcal{F}^0$ with $v(y) \geqq 0$.[4] Comparing this with (4.7), we get

$$(4.9) \qquad p(x) = g_{\alpha_0}^0(x, y) / g_{\alpha_0}^0(y, y), \qquad x \in X.$$

Since ours is a Choquet capacity ([6; 1]),

$$\overset{\circ}{\mathrm{Cap}}(\{y\}) = \lim_n \overset{\circ}{\mathrm{Cap}}\left(\left[y - \frac{1}{n}, y + \frac{1}{n}\right]\right) = \lim_n \overset{\circ}{\mathrm{Cap}}(E_n) = \mathcal{E}^{\alpha_0}(p, p),$$

which combined with (4.7) and (4.9), implies (4.8).

Incidentally the above criterion for p implies that $0 \leqq p \leqq 1$ on X and that p is α_0-harmonic on (r_1, y) and (y, r_2) in accordance with (4.4). Hence we arrive at

$$p(x) = \frac{u_1(x)}{u_1(y)} (\text{if } x \leqq y), \quad = \frac{u_2(x)}{u_2(y)} (\text{if } x \geqq y).$$

Here u_1 (resp. u_2) is a positive strictly increasing (resp. decreasing)

4) See the proof of (1.28) of [7].

α_0-harmonic function on X obeying the condition $u_1(r_1)=0$ (resp. $u_2(r_2)=0$) if r_1 (resp. r_2) is regular. This condition uniquely determines u_i up to a constant factor ([8; Th. 61.2]). In particular p is strictly positive everywhere. Furthermore (4.9) and the symmetry of $g^0_{\alpha_0}$ lead us to

$$g^0_{\alpha_0}(x, y) = \begin{cases} cu_1(x)u_2(y), & x \leqq y \\ cu_2(x)u_1(y), & x \geqq y \end{cases} \tag{4.10}$$

with a constant $c>0$.

Now consider the Hunt process $\boldsymbol{M}^0=(X_t^0, \zeta^0, P_x^0, x\in X)$ properly associated with $(X, m, \mathcal{F}^0, \mathcal{E})$. The existence of $\boldsymbol{M}^0$ is assured in [7]. We call this the absorbing barrier process. Let us see how the behaviour of $\boldsymbol{M}^0$ reflects the analytic feature (4.8) of our D-space. First the state space of $\boldsymbol{M}^0$ should be the entire space $X=(r_1, r_2)$ because any polar set B is now empty. Each element of $\mathcal{F}$ has a quasi-continuous (and hence continuous) modification, but we already started with the continuous version exclusively. Properties $1^\circ\sim5^\circ$ listed in §2 now become trivial except that every function of the class $R^0_\alpha(L^2\cap\mathcal{U}^+)$ is continuous on X.[5] The last property implies the absolute continuity of the resolvent. More important property comes from Theorem 2: every function of $P_t(L^2\cap\mathcal{U}^+)$ is continuous, from which immediately follows the absolute continuity of the transition probability without appealing to Theorem 3.

It has been proved in [7; Theorem 3.8] that

$$p(x)=E^0_x(e^{-\alpha_0\sigma'_{\{y\}}};\ \sigma'_{\{y\}}<\zeta^0)$$

for quasi-every $x\in X$, where $\sigma'_{\{y\}}$ is the hitting time for y. This equality now holds everywhere. Thus we see that the hitting probability for one point is positive starting at any point and that each point $y\in X$ is regular with respect to itself.

The reproducing kernel g^0_α is just a density function of the resolvent of the process $\boldsymbol{M}^0$:

5) $\mathcal{U}^+$ denotes the space of non-negative universally measurable functions on X.

$$(4.11) \qquad R_\alpha^0 f(x) = \int_X g_\alpha^0(x, y) f(y) m(dy), \qquad x \in X,$$

for $f \in K_{(r_1, r_2)}$. By definition, $R_\alpha^0 f$ is indeed the unique element of $\mathcal{F}^0$ satisfying (1.3) for $(\mathcal{F}^0, \mathcal{E}^\alpha)$. Noting the expression (4.10), we can conclude from (4.7) that the right hand side of (4.11) has the same property. Hence we have (4.11).

Finally we make the following remark: if either r_1 or r_2 is a (pure) entrance boundary [8], then $(X, m, \mathcal{F}^0, \mathcal{E})$ is not strongly regular. For instance suppose that r_1 is entrance. By (4.11) and [8; Th. 62.1], $R_\alpha^0 f$ does not necessarily vanish at r_1 for $f \in C(X)$, namely, R_α^0 is not a Ray resolvent on X. If every entrance boundary point is added to the state space $X = (r_1, r_2)$, then the continuously extended resolvent R_α^0 becomes a Ray's one. But this amounts to taking a strongly regular representation of $(X, m, \mathcal{F}^0, \mathcal{E})$.

(II) The reflecting barrier process

Let $\bar{X}$ be the space obtained from $X = (r_1, r_2)$ by adjoining the boundary point r_i if r_i is regular $(i = 1, 2)$. The measure m is extended to $\bar{X}$ by setting simply $m(r_i) = 0$, r_i being the adjoined (regular) boundary point.

$(\bar{X}, m, \mathcal{F}^R, \mathcal{E})$ is then a regular D-space. This is clear when no (pure) entrance boundary point is present because $\mathcal{F}^R \cap C(\bar{X})$ then contains the algebra generated by $\mathcal{H}_{\alpha_0} \cup K_{(r_1, r_2)}$ which is dense not only in $(\mathcal{F}^R, \mathcal{E}^{\alpha_0})$ but also in $C(\bar{X})$ owing to Stone-Weirstrass theorem. Consider next the case when r_1 is entrance and r_2 is regular. Each element of $\mathcal{H}_{\alpha_0}$ is then a constant multiple of $h_2^{\alpha_0}$. Put $\varphi(x) = h_2^{\alpha_0}(x) - h_2^{\alpha_0}(r_1 +)$, $x \in (r_1, r_2]$, then $\varphi \in \mathcal{F}^R \cap C(r_1, r_2]$ because $\mathcal{F}^R$ contains all constant functions. Now $\mathcal{F}^R \cap C(r_1, r_2]$, containing the algebra generated by φ, is obviously dense in $C(r_1, r_2]$. If $u \in \mathcal{F}^R$ is orthogonal to $\mathcal{F}^R \cap C(r_1, r_2]$, then $u \in \mathcal{H}_{\alpha_0}$, namely, $u = c h_2^{\alpha_0}$ for some constant c. But then $0 = \mathcal{E}^{\alpha_0}(c h_2^{\alpha_0}, \varphi) = c \dfrac{d h_2^{\alpha_0}}{ds}(r_2) \varphi(r_2)$ from which we get $c = 0$.

Let $\boldsymbol{M}^R=(X_t^R, \zeta^R, P_x^R)$ be the Hunt process properly associated with $(\overline{X}, m, \mathcal{F}^R, \mathcal{E})$. We call this the reflecting barrier process. We can make the arguments exactly parallel to the preceding case of the absorbing barrier process. Each point of $\overline{X}$ has a positive capacity, the state space of $\boldsymbol{M}^R$ must be the entire space $\overline{X}$ and the function $P_t^R f$, $f \in L^2 \cap \mathcal{U}^+$, is continuous on X, P_t^R being the transition probability of $\boldsymbol{M}^R$. P_t^R is therefore absolutely continuous with respect to m. The reproducing kernel of $(\mathcal{F}^R, \mathcal{E}^\alpha)$ is just a density function of the resolvent of $\boldsymbol{M}^R$.

Let σ_i be the hitting time (of $\boldsymbol{M}^R$) to the point r_i if r_i is regular $(i=1,2)$. Otherwise we set $\sigma_i=\zeta^R$. Put $\sigma=\sigma_1 \wedge \sigma_2$. According to (4.5) and [7; Theorem 4.3], the absorbing barrier process $\boldsymbol{M}^0$ is equivalent to the process $\boldsymbol{M}^R$ killed upon the time σ.

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Added of Proof: Quite recently M. Takano introduced a new notion of capacity (On a fine capacity related to a symmetric Markov process, to appear in Proc. Japan Acad.). This will make it possible to show Theorem 6 in a similar manner as in the proof of Theorem 3.