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On a canonical operation of sl(2m) on the exterior algebra of the vector space of complex m×n-matrices

By

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1. We mean by Z_{ip} $(1\leq i\leq m,\ 1\leq p\leq n)$ independent complex variables and denote by E the exterior algebra generated by dZ_{ip}, $d\overline{Z}_{jq}$ $(1\leq i, j\leq m;\ 1\leq p, q\leq n)$. For brevity we put

$$\theta_{ip}=dZ_{ip},\ \overline{\theta}_{jq}=d\overline{Z}_{jq}\ \ (1\leq i, j\leq m;\ 1\leq p, q\leq n).$$

Operators L_{ij}, Λ_{ij} acting on E are defined by

$$L_{ij}=\sqrt{-1}\sum_{p=1}^{n}e(\theta_{ip})e(\overline{\theta}_{jp}),$$

$$\Lambda_{ij}=-\sqrt{-1}\sum_{p=1}^{n}i(\theta_{ip})i(\overline{\theta}_{jp})$$

where $e(\xi)\eta=\xi\wedge\eta$ and $i(\xi)\eta$ is the inner product of ξ with η with respect to the metric

$$2\sum_{i=1}^{m}\sum_{p=1}^{n}(dZ_{ip},\ d\overline{Z}_{ip}).$$

In the present note we shall show that there exists a representation ρ of Lie algebra $sl(2m)$ on E such that

$$\rho\left(\begin{array}{c|c} 0 & (a_{ij}) \\ \hline 0 & 0 \end{array}\right)=\sum_{i,j=1}^{m}(-1)^{j+1}a_{ij}L_{ij},$$

$$\rho\left(\begin{array}{c|c} 0 & 0 \\ \hline (b_{ij}) & 0 \end{array}\right)=\sum_{i,j=1}^{m}(-1)^{j+1}b_{ji}\Lambda_{ij}.$$

$$
\rho\begin{array}{c}
\begin{array}{ccccccccc} & & & i & & & m+j & & \end{array}\\
\begin{pmatrix}
0 & & & & & & & & \\
 & \ddots & & & & & & & \\
 & & 0 & & & & & & \\
 & & & 1 & & & & & \\
 & & & & 0 & & & & \\
 & & & & & \ddots & & & \\
 & & & & & & 0 & & \\
 & & & & & & & -1 & \\
 & & & & & & & & 0 \\
 & & & & & & & & & \ddots \\
 & & & & & & & & & & 0
\end{pmatrix}
\end{array}
$$

$$
=(-1)\sum_{p,q}^{j+1}(p+q-n)\pi_{i,j}^{p,q}(1\leq i,\ j\leq m),
$$

where π_{ij}^{pq} is the projection of E onto the vector subspace spaned by monomials of degree p in $\{\theta_{il}\}1\leq l\leq n$, of degree q in $\{\bar{\theta}_{jl}\}1\leq l\leq n$ and of arbitrary degree in other variables.[1)]

2. The outer and inner product $e(\theta_{ip})$, $e(\bar{\theta}_{ip})$, $i(\theta_{ip})$, $i(\bar{\theta}_{ip})$ are anti-commutative except the following case:

(1) $$e(\theta_{ip})i(\theta_{ip})+i(\theta_{ip})e(\theta_{ip})=id_E\,,$$

(2) $$e(\bar{\theta}_{ip})i(\bar{\theta}_{ip})+i(\bar{\theta}_{ip})e(\bar{\theta}_{ip})=id_E\,.$$

Lemma 1.

(3) $$[A_{ij},L_{ik}]=\sum_p e(\bar{\theta}_{kp})i(\bar{\theta}_{jp})\qquad(j\neq k),$$

(4) $$[A_{ji},L_{ki}]=\sum_p e(\theta_{kp})i(\theta_{jp})\qquad(j\neq k),$$

(5) $$[A_{ij},L_{ij}]=\sum_p\{e(\theta_{ip})i(\theta_{ip})-i(\bar{\theta}_{jp})e(\bar{\theta}_{jp})\}.$$

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1) For a kählerian structure ω on a complex n-manifold there exists a representation ρ of Lie algebra on the cohomology such that

$$\rho\begin{pmatrix}0&1\\0&0\end{pmatrix}=e(\omega),\quad \rho\begin{pmatrix}0&0\\1&0\end{pmatrix}=i(\omega),$$

$$\rho\begin{pmatrix}1&0\\0&-1\end{pmatrix}=\sum_{p,q}(p+q-n)\pi^{p,q}.$$

Our result is an analogy of this representation.

Proof. From the definitions it follows

$$\begin{aligned}[][\Lambda_{ij}, L_{hk}] &= \sum_{p,q} [i(\theta_{ip}) i(\bar{\theta}_{jp}), e(\theta_{hq}) e(\bar{\theta}_{kq})] \\ &= -\sum_{p} \{i(\theta_{ip}) e(\theta_{hp}) i(\bar{\theta}_{jp}) e(\bar{\theta}_{kp}) - e(\theta_{hp}) i(\theta_{ip}) e(\bar{\theta}_{kp}) i(\bar{\theta}_{jp})\}.\end{aligned}$$

If $i=h$ and $j\neq k$

$$\begin{aligned}[][\Lambda_{ij}, L_{ik}] &= \sum_{p} \{i(\theta_{ip}) e(\theta_{ip}) + e(\theta_{ip}) i(\theta_{ip})\} e(\bar{\theta}_{kp}) i(\bar{\theta}_{jp}) \\ &= \sum_{p} e(\bar{\theta}_{kp}) i(\bar{\theta}_{jp}).\end{aligned}$$

Similarly we have (4). If $i=h$ and $j=k$, then

$$\begin{aligned}[][\Lambda_{ij}, L_{ij}] &= -\sum_{p} \{i(\theta_{ip}) e(\theta_{ip}) i(\bar{\theta}_{jp}) e(\bar{\theta}_{jp}) - e(\theta_{ip}) i(\theta_{ip}) e(\bar{\theta}_{jp}) i(\bar{\theta}_{jp})\} \\ &= -\sum_{p} \{i(\theta_{ip}) e(\theta_{ip}) + e(\theta_{ip}) i(\theta_{ip})\} i(\bar{\theta}_{jp}) e(\bar{\theta}_{jp}) \\ &\quad + \sum_{p} e(\theta_{ip}) i(\theta_{ip}) \{i(\bar{\theta}_{jp}) e(\bar{\theta}_{jp}) + e(\bar{\theta}_{jp}) i(\bar{\theta}_{jp})\} \\ &= \sum_{p} \{e(\theta_{ip}) i(\theta_{ip}) - e(\bar{\theta}_{jp}) i(\bar{\theta}_{jp})\}.\end{aligned}$$

Lemma 1 shows that $[\Lambda_{ij}, L_{ik}]$ and $[\Lambda_{ji}, L_{ki}]$ are independent on the choice of i. We use the following notations:

$$\begin{aligned}
&K_{jk} = [\Lambda_{ij}, L_{ik}], \quad M_{jk} = [\Lambda_{ji}, L_{ki}], \\
&X_1 = M_{21}, \quad X_2 = M_{23}, \cdots, X_{m-1} = M_{m,m-1}, \\
&X_m = L_{m1}, \quad X_{m+1} = K_{12}, \quad X_{m+2} = K_{23}, \cdots, X_{2m-1} = K_{m-1,m}, \\
&Y_1 = M_{12}, \quad Y_2 = M_{23}, \cdots, Y_{m-1} = M_{m-1,m}, \\
&Y_m = -\Lambda_{m1}, \quad Y_{m+1} = K_{21}, \cdots, Y_{2m-1} = K_{m,m-1}, \\
&H_1 = [X_1, Y_1], \cdots, H_{2m-1} = [X_{2n-1}, Y_{2m-1}].
\end{aligned}$$

We mean by $\mathfrak{g}$ the Lie algebra generated by L_{ij}, Λ_{ij} $(1\leq i, j\leq m)$.

Lemma 2. *If $j\neq k$, then we have*

(6) $$[K_{jk}, L_{hj}] = L_{hk},$$

(7) $$[K_{jk}, \Lambda_{hk}] = \Lambda_{hj},$$

(8) $$[M_{jk}, L_{jh}] = L_{kh},$$

(9) $$[M_{jk}, \Lambda_{kh}] = \Lambda_{jh}.$$

Proof. From (3) the condition $j \neq k$ implies

$$[K_{jk}, L_{hj}] = \sqrt{-1} \sum_{p,q} [e(\bar{\theta}_{kp}) i(\bar{\theta}_{jp}), e(\theta_{hq}) e(\bar{\theta}_{jq})]$$

$$= -\sqrt{-1} \sum_{p} e(\bar{\theta}_{kp}) \{i(\bar{\theta}_{jp}) e(\bar{\theta}_{jp}) + e(\bar{\theta}_{jp}) i(\bar{\theta}_{jp})\} e(\theta_{hp})$$

$$= -\sqrt{-1} \sum_{p} e(\bar{\theta}_{kp}) e(\theta_{hp}) = L_{hk},$$

$$[K_{jk}, \Lambda_{hk}] = -\sqrt{-1} \sum_{p,q} [e(\bar{\theta}_{kp}) i(\bar{\theta}_{jp}), i(\theta_{hq}) i(\bar{\theta}_{jq})]$$

$$= \sqrt{-1} \sum_{p} \{e(\bar{\theta}_{kp}) i(\bar{\theta}_{kp}) + i(\bar{\theta}_{kp}) e(\bar{\theta}_{kp})\} i(\bar{\theta}_{jp}) i(\theta_{hp})\}$$

$$= -\sqrt{-1} \sum_{p} i(\theta_{hp}) i(\bar{\theta}_{jp}) = \Lambda_{hj}.$$

Similarly we can prove (8), (9).

Lemma 3.

$$(10) \quad adX_{m-1+j}\, adX_{m-1+j-1} \cdots adX_{m+1}\, adX_i\, adX_{i+1} \cdots adX_{m-1} X_m = L_{ij},$$

$$(11) \quad adY_i\, adY_{i+1} \cdots adY_{m-1}\, adY_{m-1+j-1} \cdots adY_{m+1} Y_m = \Lambda_{ij}.$$

Proof. These are immediate consequences of the definitions of X_i, Y_j $(1 \leq i, j \leq m)$ and (6), (7), (8), (9).

Lemma 4.

$$(12) \quad [K_{i,i+1}, K_{i+1,i}] = \sum_{p} \{e(\bar{\theta}_{i+1,p}) i(\bar{\theta}_{i+1,p}) - e(\bar{\theta}_{ip}) i(\bar{\theta}_{ip})\},$$

$$(13) \quad [M_{i+1,i}, M_{i,i+1}] = \sum_{p} \{-e(\theta_{i+1,p}) i(\theta_{i+1,p}) + e(\theta_{p}) i(\theta_{ip})\},$$

Proof. From (3) we have

$$[K_{i,i+1}, K_{i+1,i}] = \sum_{p,q} [e(\bar{\theta}_{i+1,p}) i(\bar{\theta}_{ip}), e(\bar{\theta}_{iq}) i(\bar{\theta}_{i+1,q})]$$

$$= \sum_{p} e(\bar{\theta}_{i+1,p}) \{i(\bar{\theta}_{ip}) e(\bar{\theta}_{ip}) + e(\bar{\theta}_{ip}) i(\bar{\theta}_{ip})\} i(\bar{\theta}_{i+1,p})$$

$$- \sum_{p} \{e(\bar{\theta}_{i+1,p}) i(\bar{\theta}_{i+1,p}) + i(\bar{\theta}_{i+1,p}) e(\bar{\theta}_{i+1,p})\} e(\bar{\theta}_{ip}) i(\theta_{ip})$$

$$= \sum_{p} \{e(\bar{\theta}_{i+1}) i(\bar{\theta}_{i+1,p}) - e(\bar{\theta}_{ip}) i(\bar{\theta}_{ip})\}.$$

Similarly we have (13).

Lemma 5.

$$(14) \quad [H_i, H_j] = 0.$$

Proof. From (12) the inequality $i+1\leq j$ implies

$$\begin{aligned}&[[K_{i,i+1}, K_{i+1,i}], [K_{j,j+1}, K_{j+1,j}]]\\&=\sum_{p,q}[e(\bar\theta_{i+1,p})i(\bar\theta_{i+1,p})-e(\bar\theta_{ip})i(\bar\theta_{ip}), e(\bar\theta_{j+1,q})i(\bar\theta_{j+1,q})-e(\bar\theta_{jq})i(\bar\theta_{jq})]\\&=-\sum_{p}[e(\bar\theta_{i+1,p})i(\bar\theta_{i+1,p}), e(\bar\theta_{jp})i(\bar\theta_{jp})]=0.\end{aligned}$$

Similarly the inequality $i+1\leq j$ implies

$$[[M_{i+1,i}, M_{i,i+1}], M_{j+1,j}, M_{j,j+1}]]=0.$$

If $i+2\leq j$, then M_i and K_j contain no common $e(\theta)$ and $i(\theta)$ and thus

$$[[M_{i+1,i}, M_{i,i+1}], [K_{j,j+1}, K_{j+1,j}]]=0.$$

From (5) and (12) we see that

$$\begin{aligned}&[[K_{12}, K_{21}], [L_{m1}, -\Lambda_{m1}]]\\&=\sum_{p,q}[e(\bar\theta_{2p})i(\bar\theta_{2p})-e(\bar\theta_{1p})i(\bar\theta_{1p}), e(\theta_{mq})i(\theta_{mq})-i(\bar\theta_{1q})e(\bar\theta_{1q})]\\&=-\sum_{p}[e(\bar\theta_{1p})i(\bar\theta_{1p}), i(\bar\theta_{1p})e(\bar\theta_{1p})]=0.\end{aligned}$$

Similarly we have

$$[[M_{m,m-1}, M_{m-1,m}], [L_{m1}, -\Lambda_{m1}]]=0.$$

This completes the proof of $[H_i, H_j]=0$.

Lemma 6.

$$[H_i, X_i]=2X_i\,, \tag{15}$$

$$[H_i, Y_i]=-2Y_i\,. \tag{16}$$

Proof. From (3) and (12) it follows that

$$\begin{aligned}&[[K_{i,i+1}, K_{i+1,i}], K_{i,i+1}]\\&=\sum_{p,q}[e(\bar\theta_{i+1,p})i(\bar\theta_{i+1,p})-e(\bar\theta_{ip})i(\bar\theta_{ip}), e(\bar\theta_{i+1,p})i(\bar\theta_{ip})]\\&=\sum_{p}e(\bar\theta_{i+1,p})\{i(\bar\theta_{i+1,p})e(\bar\theta_{i+1,p})-e(\bar\theta_{i+1,p})i(\bar\theta_{i+1,p})\}i(\bar\theta_{ip})\\&\qquad+\sum_{p}e(\bar\theta_{i+1,p})\{i(\bar\theta_{ip})e(\bar\theta_{ip})-e(\bar\theta_{ip})i(\bar\theta_{ip})\}i(\bar\theta_{ip})\\&=2\sum_{p}e(\bar\theta_{i+1,p})i(\bar\theta_{ip})=2K_{i,i+1}\,.\end{aligned}$$

Similarly we have

$$[[M_{i+1,i}, M_{i,i+1}], M_{i+1,i}] = 2M_{i+1,i}.$$

From (5) we have

$$\begin{aligned}
[H_m, X_m] &= [[L_{m1}, -\Lambda_{m1}], L_{m1}] \\
&= \sqrt{-1}\sum_{p,q}[e(\theta_{mp})i(\theta_{mp}) - e(\bar\theta_{1p})i(\bar\theta_{1p}), e(\theta_{mq})e(\bar\theta_{1q})] \\
&= \sqrt{-1}\sum_p e(\theta_{mp})\{i(\theta_{mp})i(\theta_{mp}) - e(\theta_{mp})i(\theta_{mp})\}e(\bar\theta_{1p}) \\
&\quad - \sqrt{-1}\sum_p e(\theta_{mp})\{e(\bar\theta_{1p})i(\bar\theta_{1p}) - i(\bar\theta_{1p})e(\bar\theta_{1p})\}e(\bar\theta_{1p}) \\
&= 2\sqrt{-1}\sum_p e(\theta_{mp})e(\bar\theta_{1p}) = 2L_{m1}.
\end{aligned}$$

Similarly we can prove (16).

Lemma 7.

(17) $\qquad [H_i, X_j] = 0 \qquad (j \neq i-1, i, i+1),$

(18) $\qquad [H_i, X_{i-1}] = -X_{i-1},$

(19) $\qquad [H_i, X_{i+1}] = -X_{i+1}.$

Proof. If $i+2 \leq j$, then H_i, X_j contain no common $e(\theta)$, $i(\theta)$, and thus $[H_i, X_j] = 0$. From (3) and (12) it follows that

$$\begin{aligned}
&[[K_{i,i+1}, K_{i+1,i}], K_{i-1,i}] \\
&\quad = \sum_{p,q}[e(\bar\theta_{i+1,p})i(\bar\theta_{i+1,p}) - e(\bar\theta_{ip})i(\bar\theta_{ip}), e(\bar\theta_{iq})i(\bar\theta_{i-1,q})] \\
&\quad = -\sum_p[e(\bar\theta_{ip})i(\bar\theta_{ip}), e(\bar\theta_{ip})i(\bar\theta_{i-1,p})] \\
&\quad = -\sum_p e(\bar\theta_{ip})\{i(\bar\theta_{ip})e(\bar\theta_{ip}) + e(\bar\theta_{ip})i(\bar\theta_{ip})\}i(\bar\theta_{i-1,p}) \\
&\quad = -K_{i-1,i}.
\end{aligned}$$

Similarly we have

$$[[K_{i,i+1}, K_{i+1,i}], K_{i+1,i+2}] = -K_{i+1,i+2}.$$

From (4) and (13) we have

$$\begin{aligned}
&[[M_{j+1,j}, M_{j,j+1}], M_{j,j-1}] \\
&\quad = \sum_{p,q}[-e(\theta_{j+1,p})i(\theta_{j+1,p}) + e(\theta_{jp})i(\theta_{jp}), e(\theta_{j-1,q})i(\theta_{j,q})] \\
&\quad = \sum_p[e(\theta_{jp})i(\theta_{jp}), e(\theta_{j-1,p})i(\theta_{jp})]
\end{aligned}$$

$$= -\sum_p e(\theta_{j-1,p})\{i(\theta_{jp})e(\theta_{jp}) - e(\theta_{jp})i(\theta_{jp})\}i(\theta_{jp})$$
$$= -\sum_p e(\theta_{j-1,p})i(\theta_{jp}) = -M_{j,j-1}\,.$$

Similarly we have

$$[[M_{j+1,j}, M_{j,j+1}], M_{j+2,j+1}] = -M_{j+2,j+1}\,.$$

From (5) and (12) it follows that

$$\begin{aligned}[H_{m1}, X_{m+1}] &= [[\Lambda_{m1}, L_{m1}], K_{12}] \\ &= \sum_{p,q}[e(\theta_{mp})i(\theta_{mp}) - i(\bar\theta_{1p})e(\bar\theta_{1p}), e(\bar\theta_{2q})i(\bar\theta_{1q})] \\ &= -\sum_p [i(\bar\theta_{1p})e(\bar\theta_{1p}), e(\bar\theta_{2p})i(\bar\theta_{1p})] \\ &= \sum_p \{i(\bar\theta_{1p})e(\bar\theta_{1p})i(\bar\theta_{1p}) + i(\bar\theta_{1p})i(\bar\theta_{1p})e(\bar\theta_{1p})\}e(\bar\theta_{2p}) \\ &= \sum_p i(\bar\theta_{1p})e(\bar\theta_{2p}) = -\sum_p e(\bar\theta_{2p})i(\bar\theta_{1p}) \\ &= -K_{12} = -X_{m+1}\,.\end{aligned}$$

Similarly we have

$$[H_m, X_{m-1}] = [[\Lambda_{m1}, L_{m1}], M_{m,m-1}] = -M_{m,m-1} = -X_{m-1}\,.$$

From (12) we have

$$\begin{aligned}[H_{m+1}, X_m] &= [[K_{12}, K_{21}], L_{m1}] \\ &= \sqrt{-1}\sum_{p,q}[e(\bar\theta_{2p})i(\bar\theta_{2p}) - e(\bar\theta_{1p})i(\bar\theta_{1p}), e(\theta_{mq})e(\bar\theta_{1q})] \\ &= -\sqrt{-1}\sum_p [e(\bar\theta_{1p})i(\bar\theta_{1p}), e(\theta_{mp})e(\bar\theta_{1p})] \\ &= \sqrt{-1}\sum_p e(\bar\theta_{1p})\{i(\bar\theta_{1p})e(\bar\theta_{1p}) + e(\bar\theta_{1p})i(\bar\theta_{1p})\}e(\theta_{mp}) \\ &= -\sqrt{-1}\sum_p e(\theta_{mp})e(\bar\theta_{1p}) = -L_{m1} = -X_m\,.\end{aligned}$$

Similarly we have

$$[H_{m-1}, X_m] = [[M_{m,m-1}, M_{m-1,m}], L_{m1}] = -L_{m1} = -X_m\,.$$

By virtue of (10) and (11) we can conclude that the Lie algebra $\mathfrak{g}$ is generated by $X_1, \cdots, X_{2m-1}$, $Y_1, \cdots, Y_{2m-1}$. By virtue of Lemma 5, 6, 7 we see that $\{H_1, \cdots, H_{2m-1}\}$ is the Weyl base of the Cartan subalgebra of g and its Cartan matrix[2] is given by

2) See Ch. IV 5, 6 in [1].

$$\begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & \ddots & & \\ & & & \ddots & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}.$$

Therefore $\mathfrak{g}$ must be isomorphic to $sl(2m)$ and the isomorphism ρ is given by

$$\rho\begin{pmatrix} 0 & 1 & & \\ 0 & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} = X_1, \qquad \rho\begin{pmatrix} 0 & 0 & & \\ 1 & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} = Y_1$$

$$(20) \qquad \rho\begin{pmatrix} 0 & & & & \\ & 0 & 1 & & \\ & 0 & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix} = X_2, \qquad \rho\begin{pmatrix} 0 & & & & \\ & 0 & 0 & & \\ & 1 & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix} = Y_2$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$\rho\begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & 1 \\ & & 0 & 0 \end{pmatrix} = X_{2m-1}, \qquad \rho\begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & 0 \\ & & 1 & 0 \end{pmatrix} = Y_{2m-1}.$$

Finally we have proved the following theorem:

Theorem 1. *Let Z_{ip} $(1\leq i\leq m;\ 1\leq p\leq n)$ be independent complex variables and let E be the exterior algebra generated by dZ_{ip}, $d\overline{Z}_{jq}$ $(1\leq i, j\leq m;\ 1\leq p, q\leq n)$. Let L_{ij}, Λ_{ij} be the operators acting on E given by*

$$L_{ij} = \sqrt{-1}\sum_{p=1}^{n} e(dZ_{ip})e(d\overline{Z}_{jp}),$$

$$\Lambda_{ij} = -\sqrt{-1}\sum_{p=1}^{n} i(dZ_{ip})i(d\overline{Z}_{jp})$$

where $i(\ )$ means the inner product with respect to the metric $2\sum_{i=1}^{m}\sum_{p=1}^{n}(dZ_{ip}, d\overline{Z}_{ip})$. Denote by $\{h_1, \cdots, h_{2m-1}, e_1, \cdots, e_{2m-1}, f_1, \cdots, f_{2m-1}\}$ the system of canonical generators of Lie algebra $sl(2m)$ such that

$$[h_i, h_j]=0,\ [h_i, e_i]=-2e_i,\ [h_i, f_i]=-2f_i,$$
$$[h_i, e_{i-1}]=-e_{i-1},\ [h_i, e_{i+1}]=-e_{i+1}\quad (1\leq i\leq 2m-1).$$

Then there exists a representation ρ *of* $\mathrm{sl}(2m)$ *on* E *such that*

$$L_{ij}=\rho(ade_{m-1+j}\, ade_{m-1+j-1}\cdots ade_{m+1}\, ade_i\, ade_{i+1}\cdots ade_{m-1}\, e_m), \tag{21}$$

$$\Lambda_{ij}=\rho(adf_i\, adf_{i+1}\cdots adf_{m-1}\, adf_{m-1+j}\, adf_{m-1+j-1}\cdots adf_{m+1}\, f_m), \tag{22}$$

$$\begin{aligned}&\rho(e_1)=[\Lambda_{2i}, L_{1i}],\ \cdots,\ \rho(e_{m-1})=[\Lambda_{m,i}, L_{m-1,i}],\\ &\rho(e_m)=L_{m1},\ \rho(e_{m+1})=[\Lambda_{i1}, L_{i2}],\ \cdots,\ \rho(e_{2m-1})=[e_{i,m-1}, L_{i,m}]\end{aligned} \tag{23}$$

$$\begin{aligned}&\rho(f_1)=[\Lambda_{1i}, L_{2i}],\ \cdots,\ \rho(f_{m-1})=[\Lambda_{m-1,i}, L_{m,i}]\\ &\rho(f_m)=-\Lambda_{m1},\ \rho(f_{m+1})=[\Lambda_{i2}, L_{i1}],\ \cdots,\ \rho(f_{2m-1})=[\Lambda_{im}, L_{i,m-1}].\end{aligned} \tag{24}$$

Theorem 2.

$$\rho\left(\begin{array}{c|c} 0 & (a_{ij}) \\ \hline 0 & 0\end{array}\right)=\sum_{i,j=1}^{m}(-1)^{j+1}a_{ij}L_{ij}, \tag{25}$$

$$\rho\left(\begin{array}{c|c} 0 & 0 \\ \hline (b_{ij}) & 0\end{array}\right)=\sum_{i,j=1}^{m}(-1)^{j+1}b_{ji}\Lambda_{ji}, \tag{26}$$

$$\rho\begin{array}{c} \begin{array}{ccccccc} & & i & & & m+1 & \end{array} \\ \begin{pmatrix} 0 & & & & & & & \\ & \ddots & & & & & & \\ & & 0 & & & & & \\ & & & 1 & & & & \\ & & & & 0 & & & \\ & & & & & \ddots & & \\ & & & & & & 0 & \\ & & & & & & & -1 \\ & & & & & & & & 0 \\ & & & & & & & & & \ddots \\ & & & & & & & & & & 0 \end{pmatrix}\end{array} \tag{26}$$

$$=(-1)^{j+1}\sum_{p,q}(p+q-n)\pi_{i,j}^{p,q}\qquad (1\leq i,\ j\leq m),$$

where $\pi_{i,j}^{p,q}$ *is the projection of* E *onto the vector subspace spaned by monomials of degree* p *in* $\{dZ_{il}\}$ $1\leq l\leq n$, *of degree* q *in* $\{d\overline{Z}_{jl}\}$ $1\leq l\leq n$ *and of arbitrary degree in other variables.*

Proof. We mean by ε_{ij} the $2m\times 2m$-matrix whose non-zero entries are only the (i,j)-component 1. Then, if $1\leq i,\ j\leq m$, we have

$$[e_i,\ \varepsilon_{j,m+k}]=\begin{cases}\varepsilon_{i,m+k} & \text{for } i=j-1,\\ 0 & \text{otherwise,}\end{cases}$$

$$[e_{m+i},\ \varepsilon_{j,m+k}]=\begin{cases}-\varepsilon_{j,m+i} & \text{for } i=k+1,\\ 0 & \text{otherwise,}\end{cases}$$

where

$$e_i=\begin{array}{cc} & \begin{array}{ccccc} & & i+1 & & \end{array} \\ & \left(\begin{array}{ccccc} 0 & 0 & & & \\ & \ddots & \ddots & & \\ & & & 1 & \\ & & & \ddots & 0 \\ & & & & 0 \end{array}\right) \begin{array}{c} \\ \\ -i \\ \\ \\ \end{array}\end{array}$$

By virtue of Theorem 1

$$\rho(e_i)=X_i \qquad (1\leq i\leq 2m-1),$$

hence conclude that

$$\begin{aligned}\rho(\varepsilon_{i,m+j})&=(-1)^{j+1}\rho(ade_{m-1+j}\,ade_{m-j+1}\cdots ade_{m+1}\,ade_i\,ade_{i+1}\cdots ade_{m-1}e_m)\\ &=(-1)^{j+1}L_{ij}\,.\end{aligned}$$

Similarly we have

$$\rho(\varepsilon_{j+m,i})=(-1)^{j+1}\Lambda_{i,j}\,.$$

For the proof of (27) it is enough to prove for a monomial of the following type

$$\xi=\theta_{il_1}\wedge\cdots\wedge\theta_{il_s}\wedge\bar\theta_{jh_1}\wedge\cdots\wedge\bar\theta_{jh_t}\wedge(\theta_{ir_1}\wedge\bar\theta_{jr_1})\wedge\cdots\wedge(\theta_{ri_u}\wedge\bar\theta_{jr_u})$$

$$(-1)^{j+1}\rho\begin{array}{c}\begin{array}{ccccccc} & & i & & m+j & & \end{array}\\ \left(\begin{array}{ccccccc} 0 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & & & \\ & & & 1 & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \\ & & & & & & & -1 \\ & & & & & & & & 0 \\ & & & & & & & & & \ddots \\ & & & & & & & & & & 0 \end{array}\right)\end{array}\xi=[L_{ij},\ \Lambda_{ij}]\xi$$

$$=\sum_{p,q}\{e(\theta_{ip})e(\bar{\theta}_{jp})i(\theta_{iq})i(\bar{\theta}_{jq})-i(\theta_{iq})i(\bar{\theta}_{jq})e(\theta_{ip})e(\bar{\theta}_{jp})\}\xi$$
$$=(u-(n-s-t-u))\xi=(s+t+2u-n)\xi.$$

This completes the proof of Theorem 2.

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