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Cohomology mod 3 of the classifying space BF_4 of the exceptional group F_4

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1. Introduction and the statement of the results

Let $\boldsymbol{F}_4$ be the compact simply connected exceptional Lie group of rank 4. The mod p cohomology rings are known [3]:

$$H^*(\boldsymbol{F}_4;\ \boldsymbol{Z}_2)=\boldsymbol{Z}_2[x_3]/(x_3^4)\otimes\Lambda(Sq^2x_3,\ x_{15},\ Sq^8x_{15}),$$

$$(1.1)\qquad H^*(\boldsymbol{F}_4;\ \boldsymbol{Z}_3)=\boldsymbol{Z}_3[\delta\mathcal{P}^1x_3]/((\delta\mathcal{P}^1x_3)^3)\otimes\Lambda(x_3,\ \mathcal{P}^1x_3,\ x_{11},\ \mathcal{P}^1x_{11}),$$

$$H^*(\boldsymbol{F}_4;\ \boldsymbol{Z}_p)=\Lambda(x_3,\ x_{11},\ x_{15},\ x_{23})\quad \textit{for } p\geq 5,$$

where $x_i\in H^i$.

For the classifying space $B\boldsymbol{F}_4$ of $\boldsymbol{F}_4$, its mod p cohomology ring is known except the case $p=3$:

$$H^*(B\boldsymbol{F}_4;\ \boldsymbol{Z}_2)=\boldsymbol{Z}_2[x_4,\ Sq^2x_4,\ Sq^3x_4,\ x_{16},\ Sq^8x_{16}],$$

$$H^*(B\boldsymbol{F}_4;\ \boldsymbol{Z}_p)=\boldsymbol{Z}_p[x_4,\ x_{12},\ x_{16},\ x_{24}]\quad \textit{for } p\geq 5.$$

These results are consequences of (1.1) by applying Borel's transgression theorems [2] to the universal $\boldsymbol{F}_4$-bundle over $B\boldsymbol{F}_4$. For the case $p=3$, however, it seems very difficult to determine $H^*(B\boldsymbol{F}_4;\ \boldsymbol{Z}_3)$ directly from (1.1) because the element $x_{11}\in H^{11}(B\boldsymbol{F}_4;\ \boldsymbol{Z}_3)$ is not transgressive and there is a relation [1] of Araki

$$x_4(\delta\mathcal{P}^1x_4)=0$$

for the transgression image $x_4\in H^4(B\boldsymbol{F}_4;\ \boldsymbol{Z}_3)$ of x_3.

The purpose of the present paper is to determine the structure of $H^*(BF_4; Z_3)$ by use of the bundle

$$(1.2)\qquad \Pi \longrightarrow B\mathbf{Spin}(9) \xrightarrow{p} BF_4$$

where $\Pi = F_4/\mathbf{Spin}(9)$ is the Cayley plane.

Let T be a maximal torus of $\mathbf{Spin}(9) \subset F_4$ and let $\Phi(G)$ be the Weyl group of G for $G = \mathbf{Spin}(9), = F_4$. As is well-known [2] the natural map $\rho : BT \to BG$ induces a homomorphism

$$(1.3)\qquad \rho^* : H^*(BG; Z_3) \longrightarrow H^*(BT; Z_3)$$

such that the image of ρ^* is contained in the subalgebra $H^*(BT; Z_3)^{\Phi(G)}$ which consists of the elements invariant under the action of $\Phi(G)$.

For $G = \mathbf{Spin}(9)$, ρ^* is injective and the image coincides with the invariant subalgebra which is a polynomial algebra on the Pontrjagin classes $p_i \in H^{4i}$. Thus we may identify as follows.

$$H^*(B\mathbf{Spin}(9); Z_3) = H^*(BT; Z_3)^{\Phi(\mathbf{Spin}(9))} = Z_3[p_1, p_2, p_3, p_4].$$

First we shall determine $H^*(BT; Z_3)^{\Phi(F_4)}$ which is a subalgebra of $Z_3[p_1, p_2, p_3, p_4]$, and the result (Lemma 2.1) is

$$H^*(BT; Z_3)^{\Phi(F_4)} = Z_3[p_1, \bar{p}_2, \bar{p}_5, \bar{p}_9, \bar{p}_{12}]/(r_{15})$$

where

$$\bar{p}_2 = p_2 - p_1^2, \qquad \bar{p}_5 = p_4 p_1 + p_3 \bar{p}_2,$$
$$\bar{p}_9 = p_3^3 - p_4 p_3 p_1^2 + p_3^2 \bar{p}_2 p_1 - p_4 \bar{p}_2 p_1^3,$$
$$\bar{p}_{12} = p_4^3 + p_4^2 \bar{p}_2^2 + p_4 \bar{p}_2^4$$

and $$r_{15} = \bar{p}_5^3 + \bar{p}_5^2 \bar{p}_2^2 p_1 - \bar{p}_{12} p_1^3 - \bar{p}_9 \bar{p}_2^3.$$

Then by use of the cohomology spectral sequence associated with the bundle (1.2) we have the following

Theorem I. *There exist elements $x_i \in H^i(BF_4; Z_3)$ for $i = 4, 8, 9, 20, 21, 25, 26, 36, 48$ such that*

$$\rho^*(x_4) = p_1,\ \rho^*(x_8) = \bar{p}_2,\ \rho^*(x_{20}) = \bar{p}_5,\ \rho^*(x_{36}) = \bar{p}_9,\ \rho^*(x_{48}) = \bar{p}_{12}$$

and that by means of cup-product we have an additive isomorphism

$$Z_3[x_{36}, x_{48}]\otimes C\cong H^*(BF_4;\, Z_3)$$

for

$$C=Z_3[x_4, x_8]\otimes\{1, x_{20}, x_{20}^2\}+\Lambda(x_9)\otimes Z_3[x_{26}]\otimes\{1, x_{20}, x_{21}, x_{25}\}$$

where two terms of C *has the intersection* $\{1, x_{20}\}$. *Thus the kernel of* ρ^* *is the ideal generated by* x_9, x_{21}, x_{25} *and* x_{26}.

In order to determine the ring structure of $H^*(BF_4;\, Z_3)$ we shall prove the non-triviality of $\delta\mathcal{P}^4\delta\mathcal{P}^1x_4$ (Lemma 4.1). Then the ring structure is determined by the following

Theorem II. *We can choose the generators* x_i *in Theorem I such that* $x_9=\delta x_8$, $x_{21}=\delta x_{20}$, $x_{25}=\mathcal{P}^1x_{21}$ *and* $x_{26}=\delta x_{25}$. *Then the relations in* $H^*(BF_4;\, Z_3)$ *are generated by the following ones*:

$$x_9x_4=x_9x_8=x_9^2=x_{21}x_4=x_{25}x_8=x_{21}x_{20}=x_{21}^2=x_{25}x_{20}=x_{25}^2=0,$$
$$x_{21}x_8=x_{25}x_4=-x_{20}x_9, \qquad x_{26}x_4=-x_{21}x_9,$$
$$x_{26}x_8=x_{25}x_9, \qquad x_{25}x_{21}=x_{26}x_{20}$$

and $$x_{20}^3=x_{48}x_4^3+x_{36}x_8^3-x_{20}^2x_8^2x_4.$$

Thus the homomorphism ρ^* *maps the subalgebra* $Z_3[x_4, x_8, x_{36}, x_{48}]\otimes\{1, x_{20}, x_{20}^2\}$ *generated by* x_4, x_8, x_{20}, x_{36}, x_{48} *isomorphically onto the invariant subalgebra* $H^*(BT;\, Z_3)^{\Phi(F_4)}$.

Finally we shall determine the reduced power operations. By means of Cartan formula and Adem relations and by dimensional reasons, it is sufficient to determine the values of $\mathcal{P}^1$, $\mathcal{P}^3$ and $\mathcal{P}^9$ for the generators, and the results are stated as follows.

Theorem III.

(i) $$\mathcal{P}^1(x_9)=\mathcal{P}^1(x_{20})=\mathcal{P}^1(x_{25})=\mathcal{P}^1(x_{26})=0,$$
$$\mathcal{P}^1(x_4)=-x_8+x_4^2, \qquad \mathcal{P}^1(x_8)=x_8x_4,$$
$$\mathcal{P}^1(x_{21})=x_{25}, \qquad \mathcal{P}^1(x_{36})=-x_{20}^2$$

and $$\mathcal{P}^1(x_{48})=x_{26}^2.$$

(ii) $$\mathcal{P}^3(x_4)=\mathcal{P}^3(x_{21})=\mathcal{P}^3(x_{25})=\mathcal{P}^3(x_{26})=0,$$
$$\mathcal{P}^3(x_8)=x_{20}-x_8^2x_4, \qquad \mathcal{P}^3(x_9)=x_{21},$$

$$\mathcal{P}^3(x_{20})=x_{20}(-x_8+x_4^2)x_4,$$

$$\mathcal{P}^3(x_{36})=x_{48}-x_{36}(x_8+x_4^2)x_4+x_{20}^2(x_8+x_4^2)$$

and $$\mathcal{P}^3(x_{48})=-x_{48}(x_8+x_4^2)x_4.$$

(iii) $$\mathcal{P}^9(x_4)=\mathcal{P}^9(x_8)=\mathcal{P}^9(x_9)=0,$$

$$\mathcal{P}^9(x_{20})=(x_{48}+x_{20}^2x_8)(-x_8+x_4^2)+x_{36}(x_{20}+x_8^2x_4)+x_{26}x_{21}x_9,$$

$$\mathcal{P}^9(x_{21})=-x_{48}x_9+x_{36}x_{21},$$

$$\mathcal{P}^9(x_{25})=x_{36}x_{25}-x_{26}^2x_9,$$

$$\mathcal{P}^9(x_{26})=x_{36}x_{26},$$

$$\mathcal{P}^9(x_{36})=-x_{48}x_{20}x_4+x_{48}(x_8^2+x_4^4)x_4^2-x_{36}^2+x_{36}x_{20}(x_8+x_4^2)x_4^2$$
$$-x_{36}(x_8^2+x_4^4)^2x_4+x_{20}^2x_8(x_8^3+(x_8+x_4^2)^2x_4^2)$$

and $$\mathcal{P}^9(x_{48})=-x_{48}x_{36}+x_{48}x_{20}(-x_8^2-x_8x_4^2+x_4^4)-x_{48}(x_8^2+x_4^4)^2x_4.$$

Recently, N. Shimada has shown that E_2-term $\mathrm{Cotor}^{H^*(F_4;\, Z_3)}(\boldsymbol{Z}_3, \boldsymbol{Z}_3)$ of Eilenberg-Moore spectral sequence converging to $H^*(B\boldsymbol{F}_4; \boldsymbol{Z}_3)$ is additively isomorphic to $H^*(B\boldsymbol{F}_4; \boldsymbol{Z}_3)$. Thus the spectral sequence collapses.

Theorem I will be proved in section 3 after determining the invariant subalgebra $H^*(B\boldsymbol{T}; \boldsymbol{Z}_3)^{\Phi(\mathbf{F}_4)}$ in section 2. Theorems II and III will be proved in section 5 by auxiliary computations of cohomology operations in section 4.

2. Mod 3 invariant forms

Let $\boldsymbol{T}'$ be the usual maximal torus of $\boldsymbol{SO}(9)$, then $H^*(B\boldsymbol{T}')=\boldsymbol{Z}[t_1, t_2, t_3, t_4]$ for canonical generators $t_i \in H^2$ and the Weyl group $\Phi(\boldsymbol{SO}(9))$ of $\boldsymbol{SO}(9)$ acts on $H^*(B\boldsymbol{T}')$ as the permutations of t_i and the changements of the signs of t_i. Take a maximal torus $\boldsymbol{T}$ of $\mathbf{Spin}(9)$ as the inverse image of $\boldsymbol{T}'$ under the universal covering $\mathbf{Spin}(9)\rightarrow \boldsymbol{SO}(9)$. Denote by the same symbol $t_i \in H^2(B\boldsymbol{T})$ the image of t_i under the natural homomorphism $H^*(B\boldsymbol{T}')\rightarrow H^*(B\boldsymbol{T})$. Then $H^*(B\boldsymbol{T})=\boldsymbol{Z}[t_1, t_2, t_3, t_4](c_1/2)=\boldsymbol{Z}[t_1, t_2, t_3, c_1/2]$ and the action of $\Phi(\mathbf{Spin}(9))$ is same as $\Phi(\boldsymbol{SO}(9))$, where $c_1=t_1+t_2+t_3+t_4$.

Let p be an odd prime, then $H^*(B\boldsymbol{T}; \boldsymbol{Z}_p)=\boldsymbol{Z}_p[t_1, t_2, t_3, t_4]$ and

(2.1) $$H^*(BT;\ Z_p)^{\Phi(\mathbf{Spin}(9))}=Z_p[p_1, p_2, p_3, p_4]$$

where $p_i \in H^{4i}$ stands for the i-th elementary symmetric function on t_i^2, that is,

$$\sum_{i=0}^{4} p_i x^{2i}=\prod_{j=1}^{4}(1+t_j^2 x^2), \qquad p_0=1.$$

According to the section 19 of [4] we choose $\mathbf{Spin}(9)$ as a subgroup of F_4 such that $F_4/\mathbf{Spin}(9)$ is the Cayley plane Π. Then the Weyl group $\Phi(F_4)$ of F_4 is generated by $\Phi(\mathbf{Spin}(9))$ and an element R which acts as the reflection to the plane $t_1+t_2+t_3+t_4=0$, that is,

$$R(t_i)=t_i-(c_1/2), \qquad i=1, 2, 3, 4.$$

Now we discuss in Z_3-coefficient. Then

(2.2) $$H^*[BT;\ Z_3]^{\Phi(\mathrm{F}_4)}=Z_3[p_1, p_2, p_3, p_4]\cap Z_3[t_1, t_2, t_3, t_4]^R$$

and $$R(t_i)=t_i+c_1.$$

Let c_i be the i-th elementary symmetric function on t_i, that is,

$$\sum c_i x^i=\Pi(1+t_j x), \qquad c_0=1,$$

then we have easily

(2.3) $$R(c_i)=\sum_{j+k=i}\binom{4-j}{k}c_j c_1^k \quad \textit{and} \quad p_i=\sum_{j+k=2i}(-1)^{i+j}c_j c_k.$$

From these relations it follows directly

(2.4) $$R(p_1)=p_1, \qquad R(\bar{p}_2)=\bar{p}_2 \quad \text{for } \bar{p}_2=p_2-p_1^2,$$
$$R(p_3)=p_3-\bar{p}_2 p_1-c_4 p_1, \quad R(c_4)=-c_4+\bar{p}_2$$

and $$R(p_4)=p_4+\bar{p}_2^2+c_4\bar{p}_2.$$

Put

$$q_3=p_3+c_4 p_1 \quad \text{and} \quad q_4=p_4-c_4\bar{p}_2$$

then it follows from (2.4)

(2.5) p_1, $\bar{p}_2$, q_3 *and* q_4 *are invariant under* R.

First we prove

Lemma 2.1. *The invariant subalgebra* $H^*(BT;\ Z_3)^{\Phi(\mathrm{F}_4)}$ *is* *generated by the elements* p_1, $\bar{p}_2$, $\bar{p}_5=p_4p_1+p_3\bar{p}_2$, $\bar{p}_9=p_3^3-p_4p_3p_1^2+p_3^2\bar{p}_2p_1-p_4\bar{p}_2p_1^3$ *and* $\bar{p}_{12}=p_4^3+p_4^2\bar{p}_2^2+p_4\bar{p}_2^4$ *having the only relation* $r_{15}=\bar{p}_5^3+\bar{p}_5^2\bar{p}_2^2p_1-\bar{p}_{12}p_1^3-\bar{p}_9\bar{p}_2^3$. *Thus*

$$H^*(BT;\ Z_3)^{\Phi(F_4)}=Z_3[p_1, \bar{p}_2, \bar{p}_5, \bar{p}_9, \bar{p}_{12}]/(r_{15}).$$

Proof. An arbitrary element f of $Z_3[p_1, p_2, p_3, c_4]$, $p_4=c_4^2$, is written uniquely in a form

$$f=f_0+c_4f_1 \qquad \text{for} \quad f_0, f_1\in Z_3[p_1, \bar{p}_2, q_3, q_4].$$

If f is invariant: $R(f)=f$, then it follows from (2.4) and (2.5)

$$2(c_4+\bar{p}_2)f_1=0 \qquad \text{hence} \quad f_1=0.$$

Thus we have $Z_3[p_1, p_2, p_3, c_4]^R=Z_3[p_1, \bar{p}_2, q_3, q_4]$, and by (2.2)

$$(2.6) \qquad H^*(BT;\ Z_3)^{\Phi(F_4)}=Z_3[p_1, p_2, p_3, p_4]\cap Z_3[p_1, \bar{p}_2, q_3, q_4],$$

The generators of the lemma are invariant since $\bar{p}_5=q_4p_1+q_3\bar{p}_2$, $\bar{p}_9=q_3^3+q_3^2\bar{p}_2p_1-q_4q_3p_1^2$ and $\bar{p}_{12}=q_4^3+q_4^2\bar{p}_2^2$. The relation $r_{15}=0$ is directly checked. Thus

$$Z_3[p_1, \bar{p}_2, \bar{p}_5, \bar{p}_9, \bar{p}_{12}]/(r_{15})\subset H^*(BT;\ Z_3)^{\Phi(F_4)}.$$

On the other hand, an arbitrary element f of $Z_3[p_1, \bar{p}_2, q_3, q_4]$ is written uniquely in a form

$$f=g+c_4h \qquad \text{for} \quad g, h\in Z_3[p_1, p_2, p_3, p_4],$$

and also f and h are written uniquely in forms

$$f=\sum q_3^iq_4^jf_{ij}, \quad h=\sum p_3^ip_4^jh_{ij} \qquad (i, j=0, 1, 2)$$

for some $f_{ij}, h_{ij}\in Z_3[p_1, \bar{p}_2, \bar{p}_9, \bar{p}_{12}]$. Then we have

$$h_{00}=p_1f_{10}-\bar{p}_2f_{01}, \qquad h_{01}=p_1f_{11}+\bar{p}_2f_{02}-\bar{p}_2p_1^2f_{21}+\bar{p}_2^2p_1f_{12},$$
$$h_{10}=-p_1f_{20}-\bar{p}_2f_{11}, \quad h_{02}=p_1f_{12}+\bar{p}_2p_1^2f_{22},$$
$$h_{20}=-\bar{p}_2f_{21} \quad \text{and} \quad h_{12}=-p_1f_{22}.$$

If f belongs to $Z_3[p_1, p_2, p_3, p_4]$ then $h=0$, and $h_{ij}=0$. It follows that $f_{12}=f_{21}=f_{22}=0$ and that there exist $g_1, g_2\in Z_3[p_1, \bar{p}_2, \bar{p}_9, \bar{p}_{12}]$ such that $f_{01}=p_1g_1$, $f_{10}=\bar{p}_2g_1$, $f_{02}=p_1^2g_2$, $f_{11}=-\bar{p}_2p_1g_2$ and f_{20}

$=\bar{p}_2^2 g_2$. Thus $f=g+\bar{p}_5 g_1+\bar{p}_5^2 g_2$, and the lemma is proved by (2.6).

Consider the following ideals A' and A'' of $H^*(BT;\, \boldsymbol{Z}_3)^{\Phi(\mathrm{F}_4)}$:

$$(2.7)\qquad A'=(p_1, \bar{p}_2, \bar{p}_5^2) \quad \textit{and} \quad A''=(p_1^2, \bar{p}_2 p_1, \bar{p}_2^2, \bar{p}_5 p_1, \bar{p}_5 \bar{p}_2, \bar{p}_5^2).$$

The following lemma will be necessary in the next section.

Lemma 2.2. $\boldsymbol{Z}_2[p_1, p_2, p_3, p_4]$ *is additively isomorphic to the direct sum of* $H^*(BT;\, \boldsymbol{Z}_3)^{\Phi(\mathrm{F}_4)}$, $s^8 A'$ *and* $s^{16} A''$ *where* s^t *increases the dgree by* $t(=8$ *or* $16)$.

Proof. The Poincaré polynomials of the three direct summands are

$$P_1=(1+x^{20}+x^{40})(1-x^4)^{-1}(1-x^8)^{-1}(1-x^{36})^{-1}(1-x^{48})^{-1},$$
$$P_2=x^8(P_1-(1+x^{20})(1-x^{36})^{-1}(1-x^{48})^{-1})$$

and
$$P_3=x^{16}(P_1-(1+x^4+x^8+x^{20})(1-x^{36})^{-1}(1-x^{48})^{-1}).$$

Then $P_1+P_2+P_3=(1-x^4)^{-1}(1-x^8)^{-1}(1-x^{12})^{-1}(1-x^{16})^{-1}$ is the Poincaré polynomial of $\boldsymbol{Z}_3[p_1, p_2, p_3, p_4]$, and the lemma follows.

3. Proof of Theorem I.

The natural map $\rho : BT \to BF_4$ is the composition of the natural map $\rho : BT \to B\mathbf{Spin}(9)$ and the projection p of the bundle (1.2). Under the identification

$$H^*(BT;\, \boldsymbol{Z}_3)^{\Phi(\mathbf{Spin}(9))}=H^*(B\mathbf{Spin}(9);\, \boldsymbol{Z}_3)=\boldsymbol{Z}_3[p_1, p_2, p_3, p_4],$$

it follows from Lemma 2.1

$$(3.1)\qquad \operatorname{Im} p^* \subset \boldsymbol{Z}_3[p_1, \bar{p}_2, \bar{p}_5, \bar{p}_9, \bar{p}_{12}]/(r_{15}) \subset \boldsymbol{Z}_3[p_1, p_2, p_3, p_4]$$
$$\textit{for}\quad p^* : H^*(BF_4;\, \boldsymbol{Z}_3) \to H^*(B\mathbf{Spin}(9);\, \boldsymbol{Z}_3).$$

Denote by $(E_r^{*,*})$ the mod 3 cohomology spectral sequence associated with the fibering (1.2). Let w be a generator of $H^8(\boldsymbol{\Pi};\, \boldsymbol{Z}_3)$. Then the spectral sequence satisfies the following properties:

$$E_2^{*,*}=H^*(BF_4;\, \boldsymbol{Z}_3)\otimes\{1, w, w^2\},$$
$$E_r^{*,*}=E_r^{*,0}+E_r^{*,8}+E_r^{*,16} \qquad (r=2, 3, \cdots, \infty),$$

$$E_2^{*,*}=E_9^{*,*},\ H(E_9^{*,*})=E_{10}^{*,*}=E_{17}^{*,*},\ H(E_{17}^{*,*})=E_{18}^{*,*}=E_\infty^{*,*},$$
$$E_\infty^{*,0}\cong D^{*,0},\quad E_\infty^{*,8}\cong D^{*,8}/D^{*,0},\quad E_\infty^{*,16}\cong D^{*,16}/D^{*,8}$$
$$\text{for}\quad \operatorname{Im} p^*=D^{*,0}\subset D^{*,8}\subset D^{*,16}=H^*(B\,\mathbf{Spin}(9);\ \mathbf{Z}_3).$$

Let $x_9\in H^9(B\mathbf{F}_4;\ \mathbf{Z}_3)$ be the transgression image of w, then the differential d_9 in $E_9^{*,*}$ is given by

(3.2) $d_9(b\otimes 1)=0,\ d_9(b\otimes w)=b\cdot x_9\otimes 1$
and $d_9(b\otimes w^2)=-b\cdot x_9\otimes w$ *for* $b\in H^*(B\mathbf{F}_4;\ \mathbf{Z}_3)$.

We shall discuss the following assertions.

(3.3) (i) *There exist* $x_i\in H_i(B\mathbf{F}_4;\ \mathbf{Z}_3)$ *for* $i=4, 8, 20, 36, 48$ *such that*
$p^*(x_4)=p_1,\ p^*(x_8)=\bar{p}_2,\ p^*(x_{20})=\bar{p}_5,\ p^*(x_{36})=\bar{p}_9$ *and* $p^*(x_{48})=\bar{p}_{12}$.
(ii) $x_4\otimes w,\ x_8\otimes w,\ x_{20}^2\otimes w,\ x_{20}x_4\otimes w^2,\ x_{20}x_8\otimes w^2$ *and* $x_{20}^2\otimes w^2$ *are permanent cycles.*

(3.2) implies

(3.4) $x_9x_4=0$ *and* $x_9x_8=0$ *provided the assertion* (3.3), (ii) *for* $x_4\otimes w$ *and* $x_8\otimes w$ *respectively.*

Obviously $x_9^2=0$. By (3.2), $x_4\otimes w^2$, $x_8\otimes w^2$ and $x_9\otimes w^2$ are d_9-cycles, and we can define elements $x_i\in H^i(B\mathbf{F}_4;\ \mathbf{Z}_3)$ for $i=21$, 25 and 26 by

(3.5) $x_{21}\otimes 1=d_{17}(x_4\otimes w^2),\quad x_{25}\otimes 1=d_{17}(x_8\otimes w^2)$
and $x_{26}\otimes 1=d_{17}(x_9\otimes w^2)$.

First we prove the following

Lemma 3.1. *If the assertion* (3.3) *holds for total degree* $\leq n$, *then Theorem I holds for degree* $\leq n$.

Proof. The following discussions are considered for total degree $\leq n$. Consider subgroups A, A' and A'' of $H^*(B\mathbf{F}_4;\ \mathbf{Z}_3)$ which are given by

$$A=\boldsymbol{Z}_3[x_4, x_8, x_{36}, x_{48}]\otimes\{1, x_{20}, x_{20}^2\},$$
$$A'=A-\boldsymbol{Z}_3[x_{36}, x_{48}]\otimes\{1, x_{20}\}$$
and $$A''=A'-\boldsymbol{Z}_3[x_{36}, x_{48}]\otimes\{x_4, x_8\}.$$

By (3.3), (i) and (3.1), we see that $\mathrm{Im}\, p^*=p^*(A)$, p^* is injective on A, and in the spectral sequence $A\otimes 1$ is not bounded and

$$A\otimes 1=E_\infty^{*,0} \qquad (\text{for } * \leq n).$$

$A'\otimes w$ is the product of $A\otimes 1$ and $\{x_4\otimes w, x_8\otimes w, x_{20}^2\otimes w\}$. It follows from (3.3), (ii) that $A'\otimes w$ is permanent cycle. Similarly $A''\otimes w^2$ is permanent cycle by (3.3), (ii) and by that $x_4^2\otimes w^2$ (and $x_8x_4\otimes w^2$, $x_8^2\otimes w^2$) are permanent cycles if $x_4\otimes w$ (and $x_8\otimes w$) are so. Obviously $A''\otimes w^2$ is not bounded. Thus we have an inclusion

$$A''\otimes w^2\subset E_\infty^{*,16} \qquad (*+16 \leq n).$$

Assume that $a\otimes w\in A'\otimes w$ is bounded. Then, by (3.2), $a=-b\cdot x_9$ for some b, and $p^*(a)=0$ by $p^*(x_9)=0$. Since p^* is injective on $A'\subset A$, we have that $A'\otimes w$ is not bounded and

$$A'\otimes w\subset E_\infty^{*,8} \qquad (*+8 \leq n).$$

$H^*(B\mathbf{Spin}(9); \boldsymbol{Z}_3)=\boldsymbol{Z}_3[p_1, p_2, p_3, p_4]$ is additively isomorphic to the direct sum of $E_\infty^{*,0}$, $E_\infty^{*-8,8}$ and $E_\infty^{*-16,16}$. The three direct summands of Lemma 2.2 is isomorphic to $A\otimes 1$, $A'\otimes w$ and $A''\otimes w^2$ respectively. Then it follows from Lemma 2.2 the equalities

(3.6) $$A\otimes 1=E_\infty^{*,0}, \quad A'\otimes w=E_\infty^{*,8} \quad \textit{and} \quad A''\otimes w=E_\infty^{*,16}$$
for total degree $\leq n$.

Now we assume that Theorem I is true for degree $<n$, and compute d_9 and $E_{17}=E_{10}=H(E_9)$ by (3.2) and (3.4). Then we have

$$E_{17}^{n-17,16}=(A''+B'')^{n-17}\otimes w^2$$

for $$B''=\boldsymbol{Z}_3[x_{36}, x_{48}]\otimes[\{x_4, x_8\}+\boldsymbol{Z}_3[x_{26}]\otimes\{x_9, x_{20}x_9, x_{21}x_9, x_{25}x_9\}]$$
and $$\mathrm{Im}\, d_9(\text{in } E_9^{n,0})\cong E_9^{n-9,8}/(d_9E_9^{n-18,16}\oplus A'\otimes w)=(B')^{n-9}\otimes w$$
for $$B'=\boldsymbol{Z}_3[x_{26}, x_{36}, x_{48}]\otimes\{1, x_{20}, x_{21}, x_{25}\}.$$

By the properties of the spectral sequence we have exact sequences

$$0 \longrightarrow (B')^{n-9} \xrightarrow{\cdot x_9} H^n(BF_4; Z_3) \longrightarrow E_{17}^{n,0} \longrightarrow 0$$

and
$$0 \longrightarrow (B'')^{n-17} \xrightarrow{g} E_{17}^{n,0} \longrightarrow (A)^n \longrightarrow 0,$$

where g is given by $d_{17}(b \otimes w^2) = g(b) \otimes 1$. By (3.5)

$$g(B'') \oplus B' \cdot x_9 = Z_3[x_{26}, x_{36}, x_{48}]$$
$$\otimes \{x_9, x_{21}, x_{25}, x_{26}, x_{20}x_9, x_{21}x_9, x_{25}x_9, x_{26}x_{20}\}$$

and $H^*(BF_4; Z_3)$ is additively isomorphic to $A \oplus g(B'') \oplus B' \cdot x_9$. This shows the first statement $Z_3[x_{36}, x_{48}] \otimes C \cong H^*(BF_4; Z_3)$ of Theorem I. Obviously the generators x_9, x_{21}, x_{25} and x_{26} vanishes under ρ^*. Thus the ideal generated by these elements is contained in the kernel of ρ^*. The kernel contains $g(B'') \oplus B' \cdot x_9$. Since ρ^* is injective on A, we have that the kernel of ρ^* coincides with the ideal. Consequently the lemma is proved by induction on n. We have also proved

(3.7) $$\operatorname{Ker} \rho^* = Z_3[x_{26}, x_{36}, x_{48}]$$
$$\otimes \{x_9, x_{21}, x_{25}, x_{26}, x_{20}x_9, x_{21}x_9, x_{25}x_9, x_{26}x_{20}\}.$$

Next we shall prove (3.3) by dividing into three steps.

Lemma 3.2. (3.3) *holds for total degree* ≤ 35. *By a suitable choice of the generator* w, p_3 *and* p_4 *represent* $-x_4 \otimes w$ *and* $x_8 \otimes w$ *respectively.*

Proof. The existence of x_4 is very easy. By (3.1)

$$p^*(H^{12}(BF_4; Z_2)) = D^{12,0} \subset \{\bar{p}_2 p_1, p_1^3\}.$$

Then $E_\infty^{4,8} = D^{4,8}/D^{12,0} = H^{12}(B\mathbf{Spin}(9); Z_3)/D^{12,0}$ contains non-trivial class of p_3. Since $E_2^{4,8}$ has only one generator $x_4 \otimes w$, $E_2^{4,8} = E_\infty^{4,8}$ and $-x_4 \otimes w$ is a permanent cycle represented by p_3 mod $\{\bar{p}_2 p_1, p_1^3\}$, by changing the sign of w if it is necessary.

Next assume that $\bar{p}_2$ is not a p^*-image. Then as above, $\bar{p}_2$ represents $1 \otimes w$ mod $D^{8,0}$, up to sign. So, $\bar{p}_2 p_1$ represents $x_4 \otimes w$

mod $D^{12,0}=\{\bar{p}_2p_1, p_1^3\}$ which contradicts to the above result. Therefore $\bar{p}_2$ is a p^*-image, and the existence of x_8 follows.

Now, as in the proof of the previous lemma, $D^{16,0}=(A)^{16}$ and $H^{16}(B\mathbf{Spin}(9); \mathbf{Z}_3)/D^{16,0}=\{p_4, p_3p_1\}$. Since $d_9(1\otimes w^2)=-x_9\otimes w\neq 0$, we have $E_\infty^{0,16}=0$, $H^{16}(B\mathbf{Spin}(9); \mathbf{Z}_3)=D^{8,8}$ and $E_\infty^{8,8}=D^{8,8}/D^{16,0}=\{p_4, p_3p_1\}$. On the other hand, $E_2^{8,8}=\{x_8\otimes w, x_4^2\otimes w\}$ and p_3p_1 represents $-x_4^2\otimes w$. It follows that $x_8\otimes w$ is a permanent cycle and that p_4 represents $(sx_8+tx_4^2)\otimes w$ for some $s, t\in\mathbf{Z}_3$.

Finally, p_4p_1 and $-sp_3\bar{p}_2-tp_3p_1^2$ represent the same element $(sx_8x_4+tx_4^3)\otimes w$ mod $D^{20,0}$. Thus $p_4p_1+sp_3\bar{p}_2+tp_3p_1^2$ belongs to $\mathrm{Im}\,p^*$. By (3.1) we have $s=1$, $t=0$, and that p_4 represents $x_8\otimes w$, and also the existence of x_{20} such that $p^*x_{20}=p_4p_1+p_3\bar{p}_2=\bar{p}_5$.

Consequently, (3.3) is proved for total degree≤ 35.

Lemma 3.3. (3.3) *holds for total degree* $n\leq 43$ *and* $n=48$.

Proof. Consider the discussions in the proof of Lemma 3.1 for the cases $n=36, 40, 48$. Then we see $E_\infty^{n,0}\subset H^n(BT; \mathbf{Z}_3)^{\Phi(\mathrm{F}_4)}$, $E_\infty^{n-8,8}\subset(A')^{n-8}\otimes w=E_{10}^{n-8,8}$ and $E_\infty^{n-16,16}\subset(A'')^{n-16}\otimes w^2=E_{10}^{n-16,16}$. It follows from Lemma 2.2 that the equalities hold in the above three inclusions. This proves (3.3).

Lemma 3.4. (3.3) *holds for all degree.*

Proof. The proof of the above lemma valids for the cases $n=44$ and $n=56$ provided that d_9 is injective on $(A-A')^{n-8}\otimes w$, that is, $d_9(x_{36}\otimes w)\neq 0$ and $d_9(x_{48}\otimes w)\neq 0$.

Assume that $d_9(x_{36}\otimes w)=0$, then $x_{36}\otimes w$ is a permanent cycle and represented by an element $f\in H^{44}(B\mathbf{Spin}(9); \mathbf{Z}_3)$. Since $x_{36}x_4\otimes w$ is represented by both of fp_1 and $-\bar{p}_9p_3$, $fp_1+\bar{p}_9p_3\in\mathrm{Im}\,p^*$. The coefficient of p_3^4 in $fp_1+\bar{p}_9p_3$ is 1, but such an element is not contained in $\mathrm{Im}\,p^*\subset\mathbf{Z}_3[p_1, \bar{p}_2, \bar{p}_5, \bar{p}_9, \bar{p}_{12}]/(r_{15})$. Thus we have $d_9(x_{36}\otimes w)\neq 0$. Similarly $d_9(x_{48}\otimes w)\neq 0$.

Consequently we have proved all the assertions of (3.3).

Proof of Theorem I. By Lemma 3.4, (3.3) holds. Then Lemma 3.1 implies Theorem I.

Remark. We have insisted to prove Theorem I without use of cohomology operations. The use of cohomology operations simplifies the proof of the theorem as follows. The existence of x_4 implies the existence of x_8 and x_{20} by use of $\mathcal{P}^1$ and $\mathcal{P}^3\mathcal{P}^1$. The existence of x_{36} implies that of x_{48} by use of $\mathcal{P}^3$. The assertions in (3.3), (ii) are equivalent to $x_9x_4=x_9x_8=x_{20}^2x_9=0$ and $x_{21}x_{20}\equiv x_{25}x_{20}\equiv 0 \bmod (x_9)$ except the last assertion for $x_{20}^2\otimes w^2$. Then the first relation $x_9x_4=0$ implies the others by applying suitable coholomogy operations as is seen in section 5.

4. Cohomology operations

In the first half of this section we shall prove the following

Lemma 4.1. *For a generator x_4 of $H_4(BF_4;\ Z_3)$ we have, up to sign, $x_9=-\delta\mathcal{P}^1x_4$, $x_{21}=-\mathcal{P}^3\delta\mathcal{P}^1x_4$, $x_{25}=\mathcal{P}^1x_{21}$ and $x_{26}=\delta x_{25}=-\delta\mathcal{P}^4\delta\mathcal{P}^1x_4$.*

Proof. Let $\widetilde{BF}_4$ be a 4-connective fibre space over BF_4. $\widetilde{BF}_4$ is a fibre of a fibering

(4.1) $\qquad \widetilde{BF}_4\longrightarrow BF_4\longrightarrow K(Z, 4).$

Let $\widetilde{F}_4$ be the loop space of $\widetilde{BF}_4$. Since F_4 is equivalent to the loop space of BF_4, we see that $\widetilde{F}_4$ is a 3-connective fibre space over F_4. The coholomogy of $\widetilde{F}_4$ was computed in [8: Th. 2.5] and the result is

$$H^*(\widetilde{F}_4;\ Z_3)=Z_3[y_{18}]\otimes\Lambda(y_{11}, \mathcal{P}^1y_{11}, \delta y_{18}, \mathcal{P}^1\delta y_{18}).$$

Consider a contractible fibering over $\widetilde{BF}_4$ with a fibre $\widetilde{F}_4$. By dimensional reasons, y_{11} and y_{18} are transgressive. Let y_{12} and y_{19} be transgression images of y_{11} and y_{18} respectively. Then we have

(4.2) *The natural homomorphism $Z_3[y_{12}, \mathcal{P}^1y_{12}, \delta y_{19}, \mathcal{P}^1\delta y_{19}]\otimes\Lambda(y_{19})\to H^*(\widetilde{BF}_4;\ Z_3)$ is bijective for degree≤ 54.*

This can be proved by use of the comparision theorem [10], but we need (4.2) only for degree≤ 26 and whence (4.2) is an easy exercise of spectral sequence.

Now let $(E_r^{*,*})$ be the mod 3 cohomology spectral sequence associated with the fibering (4.1) converging to $H^*(BF_4; Z_3)$ and having

$$E_2^{*,*}=H^*(Z,4;Z_3)\otimes H^*(\widetilde{BF}_4;Z_3),$$

where, by [6] for $u\in H^4$,

$$H^*(Z,4;Z_3)=Z_3[u,\mathcal{P}^1u,\mathcal{P}^3\mathcal{P}^1u,\delta\mathcal{P}^4\delta\mathcal{P}^1u,\cdots]$$
$$\otimes\Lambda(\delta\mathcal{P}^1u,\delta\mathcal{P}^3\mathcal{P}^1u,\mathcal{P}^4\delta\mathcal{P}^1u,\cdots).$$

By checking the degrees, we see that $E_2^{s,26-s}=0$ unless $s=26$, and $E_2^{26,0}$ is generated by $\delta\mathcal{P}^4\delta\mathcal{P}^1u\otimes 1$. On the other hand $H^{26}(BF_4; Z_3)$ is generated by x_{26}, by Theorem I. This shows that up to sign x_{26} is the image $\delta\mathcal{P}^4\delta\mathcal{P}^1x_4$ of $\delta\mathcal{P}^4\delta\mathcal{P}^1u$. It follows that $\delta\mathcal{P}^4\delta\mathcal{P}^1x_4=\delta\mathcal{P}^1(\mathcal{P}^3\delta\mathcal{P}^1x_4)\neq 0$, $\mathcal{P}^3\delta\mathcal{P}^1x_4\neq 0$ and $\delta\mathcal{P}^1x_4\neq 0$. By Theorem I, $H^i(BF_4; Z_3)$ has only one generator x_i for $i=9, 21, 25, 26$. Therefore the lemma is proved.

Next we compute the reduced power operations in $H^*(BT; Z_3)^{\Phi(F_4)}$ by means of the methods in [5]. The reduced powers of p_i are computed directly or computing those of c_i at first and then applying the Cartan formula to the second equation of (2.3). The results are stated as follows.

(4.3) (i) $\mathcal{P}^1p_1=-p_2-p_1^2,\qquad \mathcal{P}^2p_1=p_1^3.$

(ii) $\mathcal{P}^1p_2=-p_2p_1,\qquad \mathcal{P}^2p_2=-p_2^2+p_2p_1^2,$
$\mathcal{P}^3p_2=p_4p_1+p_3p_2-p_3p_1^2-p_2^2p_1,\qquad \mathcal{P}^4p_2=p_2^3.$

(iii) $\mathcal{P}^1p_3=p_4-p_3p_1,\qquad \mathcal{P}^2p_3=-p_4p_1-p_3p_2+p_3p_1^2,$
$\mathcal{P}^3p_3=p_4p_2-p_3^2-p_3p_2p_1,\qquad \mathcal{P}^4p_3=p_4p_3-p_3^2p_1+p_3p_2^2,$
$\mathcal{P}^5p_3=-p_4^2+p_4p_3p_1-p_4p_2^2-p_3^2p_2,\qquad \mathcal{P}^6p_3=p_3^3.$

(iv) $\mathcal{P}^1p_4=-p_4p_1,\qquad \mathcal{P}^2p_4=-p_4p_2+p_4p_1^2,$
$\mathcal{P}^3p_4=-p_4p_3-p_4p_2p_1,\qquad \mathcal{P}^4p_4=-p_4^2-p_4p_3p_1+p_4p_2^2,$

$$\mathcal{P}^5 p_4 = -p_4^2 p_1 - p_4 p_3 p_2, \quad \mathcal{P}^6 p_4 = -p_4^2 p_2 + p_4 p_3^2,$$
$$\mathcal{P}^7 p_4 = -p_4^2 p_3, \qquad \mathcal{P}^8 p_4 = p_4^3.$$

Then the reduced powers of the generators p_1, $\bar{p}_2$, $\bar{p}_5$, $\bar{p}_9$, $\bar{p}_{12}$ can be computed by use of the Cartan formula. By sequences of many routine computations we have the following

Proposition 4. 2.

(i) $\mathcal{P}^1 p_1 = -\bar{p}_2 + p_1^2$, $\quad \mathcal{P}^1 \bar{p}_2 = \bar{p}_2 p_1$, $\quad \mathcal{P}^1 \bar{p}_5 = 0$,
$\mathcal{P}^1 \bar{p}_9 = -\bar{p}_5^2$ *and* $\mathcal{P}^1 \bar{p}_{12} = 0$.

(ii) $\mathcal{P}^3 p_1 = 0$, $\quad \mathcal{P}^3 \bar{p}_2 = \bar{p}_5 - \bar{p}_2^2 p_1$, $\quad \mathcal{P}^3 \bar{p}_5 = \bar{p}_5(-\bar{p}_2 + p_1^2) p_1$,
$\mathcal{P}^3 \bar{p}_9 = \bar{p}_{12} + (-\bar{p}_9 p_1 + \bar{p}_5^2)(\bar{p}_2 + p_1^2)$,
$\mathcal{P}^3 \bar{p}_{12} = -\bar{p}_{12}(\bar{p}_2 + p_1^2) p_1$.

(iii) $\mathcal{P}^9 p_1 = \mathcal{P}^9 \bar{p}_2 = 0$,
$\mathcal{P}^9 \bar{p}_5 = \bar{p}_{12}(-\bar{p}_2 + p_1^2) + \bar{p}_9 \bar{p}_5 + \bar{p}_9 \bar{p}_2^2 p_1 + \bar{p}_5^2(-\bar{p}_2^2 + \bar{p}_2 p_1^2)$,
$\mathcal{P}^9 \bar{p}_9 = -\bar{p}_{12} \bar{p}_5 p_1 + \bar{p}_{12}(\bar{p}_2^2 p_1^2 + p_1^6) - \bar{p}_9^2 + \bar{p}_9 \bar{p}_5(\bar{p}_2 p_1^2 + p_1^4)$
$\qquad - \bar{p}_9(\bar{p}_2^2 + p_1^4)^2 p_1 + \bar{p}_5^2 \bar{p}_2(\bar{p}_2^3 + (\bar{p}_2 + p_1^2)^2 p_1^2)$,
$\mathcal{P}^9 \bar{p}_{12} = -\bar{p}_{12} \bar{p}_9 - \bar{p}_{12} \bar{p}_5(\bar{p}_2^2 + \bar{p}_2 p_1^2 - p_1^4) - \bar{p}_{12}(\bar{p}_2^2 + p_1^4)^2 p_1$.

For $t = 1, 3, 9$ and for $i = 1, 2, 5, 9, 12$, denote by

$$\mathcal{P}^t(\bar{p}_i) = f_{t,i}(p_1, \bar{p}_2, \bar{p}_5, \bar{p}_9, \bar{p}_{12}), \qquad (\bar{p}_1 = p_1) \tag{4. 3}$$

the formulas of the above proposition. By the naturality of $\mathcal{P}^t$, the difference

$$\mathcal{P}^t(x_{4i}) - f_{t,i}(x_4, x_8, x_{20}, x_{36}, x_{48})$$

vanishes under p^*. $\mathrm{Ker}\, p^* = \mathrm{Ker}\, \rho^*$ can be read off by (3. 7). Then we have

Corollary. 4. 3. (i) *The formulas in Theorem II hold for* $\mathcal{P}^1(x_4)$, $\mathcal{P}^1(x_8)$, $\mathcal{P}^1(x_{20})$, $\mathcal{P}^1(x_{36})$, $\mathcal{P}^3(x_4)$, $\mathcal{P}^3(x_8)$, $\mathcal{P}^3(x_{20})$, $\mathcal{P}^3(x_{36})$, $\mathcal{P}^9(x_4)$, $\mathcal{P}^9(x_8)$ *and* $\mathcal{P}^9(x_{48})$.

(ii) *For some coefficients* $a, b, c, d \in \boldsymbol{Z}_3$ *the following relations hold*:

$$\mathcal{P}^1(x_{48}) = a \cdot x_{26}^2,$$

$$\mathcal{P}^3(x_{48}) = -x_{48}(x_8+x_4^2)x_4+b\cdot x_{26}x_{25}x_9,$$

$$\mathcal{P}^9(x_{20}) = f_{9,5}(x_4, x_8, x_{20}, x_{36}, x_{48})+c\cdot x_{26}x_{21}x_9,$$

and $$\mathcal{P}^9(x_{36}) = f_{9,9}(x_4, x_8, x_{20}, x_{36}, x_{48})+d\cdot x_{26}^2x_{20}.$$

5. Proof of Theorems II and III

By Theorem I and (3.7)

$$H^n(BF_4; Z_3)=0$$

for $n=5, 13, 17, 18, 22, 27, 33, 37, 38, 41, 42, 49, 50,$

thus the following trivialities follow.

(5.1) $$x_9x_4=x_9x_8=x_9^2=x_{25}x_8=x_{21}x_{20}=x_{21}^2=x_{25}^2=0.$$

(5.2) $$\delta x_4=\delta x_{21}=\delta x_{26}=\delta x_{36}=\delta x_{48}=0.$$

(5.3) $$\mathcal{P}^1(x_9)=0, \quad \mathcal{P}^3(x_{21})=\mathcal{P}^3(x_{25})=\mathcal{P}^3(x_{26})=0.$$

Proof of Theorem II. We choose the generators x_9, x_{21}, x_{25} and x_{26} such that they satisfy the equalities of Lemma 4.1. $\mathcal{P}^1x_4 = -x_8+x_4^2$ by Corollary 4.3, (i). Then, by (5.2),

$$x_9=-\delta\mathcal{P}^1x_4=(x_8-x_4^2)=\delta x_8 \quad \text{and} \quad x_{21}=-\mathcal{P}^3\delta\mathcal{P}^1x_4=\mathcal{P}^3x_9.$$

$\mathcal{P}^3x_8=x_{20}-x_8^2x_4$ by Corollary 4.3, (i) and $\mathcal{P}^2x_4=x_4^3$, $\mathcal{P}^9x_9=0$ by dimensional reasons. By Cartan formula, $\mathcal{P}^3x_4^2=(x_8-x_4^2)x_4^3$. By (5.1) and (5.2), $\delta\mathcal{P}^3x_4^2=0$ and $\delta(x_8^2x_4)=0$. Then, by use of Adem relation $\mathcal{P}^3\delta\mathcal{P}^1=\delta\mathcal{P}^3\mathcal{P}^1$, we have

$$x_{21}=-\mathcal{P}^3\delta\mathcal{P}^1x_4=-\delta\mathcal{P}^3\mathcal{P}^1x_4=\delta\mathcal{P}^3(x_8-x_4^2)=\delta x_{20}.$$

We have proved

(5.4) $$\delta x_8=x_9, \quad \delta x_{20}=x_{21}, \quad \delta x_{25}=x_{26}, \quad \mathcal{P}^1(x_{21})=x_{25},$$
$$\mathcal{P}^3(x_9)=x_{21} \quad \textit{and} \quad \mathcal{P}^9(x_9)=0.$$

By Adem relations $\mathcal{P}^1\mathcal{P}^1\mathcal{P}^3=-\mathcal{P}^5$ and $\mathcal{P}^1\delta\mathcal{P}^1\mathcal{P}^3=\delta\mathcal{P}^5+\mathcal{P}^5\delta$, we have $\mathcal{P}^1x_{25}=\mathcal{P}^1\mathcal{P}^1\mathcal{P}^3x_9=-\mathcal{P}^5x_9=0$ and $\mathcal{P}^1x_{26}=\mathcal{P}^1\delta\mathcal{P}^1\mathcal{P}^3\delta x_8=\delta\mathcal{P}^5x_9 =0$, i.e.,

(5.5) $$\mathcal{P}^1(x_{25})=\mathcal{P}^1(x_{26})=0.$$

By $\mathcal{P}^1\mathcal{P}^1=-\mathcal{P}^2$, (5.5) and (5.3) imply $\mathcal{P}^2x_9=\mathcal{P}^2x_{25}=\mathcal{P}^2x_{26}=0$. Then, by Cartan formula we have

(5.5)′ $\mathcal{P}^3(x_9f)=x_{21}f+x_9\mathcal{P}^3(f)$, $\mathcal{P}^3(x_{25}f)=x_{25}\mathcal{P}^3(f)$ *and* $\mathcal{P}^3(x_{26}f)=x_{26}\mathcal{P}^3(f)$.

For example, applying (5.5)′ to the relations $x_9x_4=x_9x_8=x_{25}x_8=0$ we have

(5.6) $x_{21}x_4=0$, $x_{21}x_8=-x_{20}x_9$ *and* $x_{25}x_{20}=0$.

Since δ and $\mathcal{P}^1$ are derivative, we have

$$0=\delta(x_{25}x_8)=x_{26}x_8-x_{25}x_9,$$

$$0=\delta(x_{25}x_{20})=x_{26}x_{20}-x_{25}x_{21},$$

$$0=\mathcal{P}^1(x_{21}x_4)=x_{25}x_4+x_{21}(-x_8+x_4^2)=x_{25}x_4-x_{21}x_8$$

and $$0=\delta(x_{25}x_4-x_{21}x_8)=x_{26}x_4+x_{21}x_9.$$

Therefore

(5.7) $x_{25}x_4=-x_{20}x_9$, $x_{26}x_4=-x_{21}x_9$, $x_{26}x_8=x_{25}x_9$ *and* $x_{25}x_{21}=x_{26}x_{20}$.

Finally consider the difference

$$x_{20}^3-(x_{48}x_4^3+x_{36}x_8^3-x_{20}^2x_8^2x_4)$$

which vanishes by p^* since the relation $r_{15}=0$ holds. By (3.7) the kernel of p^* for degree 60 is generated by $x_{26}x_{25}x_9$. Let the difference be $e\cdot x_{26}x_{25}x_9$ for some $e\in Z_3$. Then we have

$$0=\delta(x_{20}^3)=x_{21}x_{20}x_8^2x_4+x_{20}^2x_9x_8x_4+e\cdot x_{26}^2x_9=e\cdot x_{26}^2x_9.$$

It follows $e=0$ and the relation

(5.8) $x_{20}^3=x_{48}x_4^3+x_{36}x_8^3-x_{20}^2x_8^2x_4$.

Consequently, (5.1), (5.6), (5.7) and (5.8) cover all the relations of Theorem II, and by use of the relations each polynomial of the generators can be written in a form of Theorem I. Note the following relations:

(5.7)′ $x_{20}^2 x_9 = x_{26} x_{20}^2 = 0.$

The proof of Lemma 3.1 and the relation (5.8) show the last half of Theorem II.

Proof of Theorem III. First we shall prove

(5.9) $b=0$ *and* $d=0$ *in Corollary* 4.3, (ii).

By Adem relation $\mathcal{P}^2\delta\mathcal{P}^1 = \delta\mathcal{P}^3 - \mathcal{P}^3\delta$,

$$\delta\mathcal{P}^3 x_{48} = \mathcal{P}^2\delta\mathcal{P}^1 x_{48} + \mathcal{P}^3\delta x_{48} = a\mathcal{P}^2\delta x_{26}^2 = 0.$$

On the other hand,

$$\begin{aligned}\delta\mathcal{P}^3 x_{48} &= \delta(-x_{48}(x_8 + x_4^2)x_4 + b\cdot x_{26}x_{25}x_9)\\ &= -x_{48}x_9x_4 + b\cdot x_{26}^2 x_9 = b\cdot x_{26}^2 x_9.\end{aligned}$$

It follows that $b=0$. Similarly, using Adem relation $\mathcal{P}^8\delta\mathcal{P}^1 = \delta\mathcal{P}^9 - \mathcal{P}^9\delta$ and computing $\delta(f_{9,9}(x_4, x_8, x_{20}, x_{36}, x_{48})) = 0$, we have

$$\begin{aligned}0 &= \mathcal{P}^8(x_{21}x_{20}) = \mathcal{P}^8\delta(-x_{20}^2) = \mathcal{P}^8\delta\mathcal{P}^1 x_{36} = \delta\mathcal{P}^9 x_{36}\\ &= \delta(f_{9,9}) + d\cdot\delta(x_{26}^2 x_{20}) = d\cdot x_{26}^2 x_{21},\end{aligned}$$

and $d=0$.

Next we shall prove

(5.10) $\mathcal{P}^9(x_{21}) = -x_{48}x_9 + x_{36}x_{21}$, $\mathcal{P}^9(x_{25}) = x_{36}x_{25} - x_{26}^2 x_9$ *and* $\mathcal{P}^9(x_{26}) = x_{36}x_{26}$.

Since $\mathcal{P}^1 x_{20} = 0$, Adem relation $\mathcal{P}^8\delta\mathcal{P}^1 = \delta\mathcal{P}^9 - \mathcal{P}^9\delta$ implies $\mathcal{P}^9 x_{21} = \mathcal{P}^9\delta x_{20} = \delta\mathcal{P}^9 x_{20} = \delta(f_{9,5}) + c\cdot\delta(x_{26}x_{21}x_9) = \delta(f_{9,5})$, and $\delta(f_{9,5}) = -x_{48}x_9 + x_{36}x_{21}$ by (5.1), (5.2), (5.4). Thus the first formula is proved.

Since $\mathcal{P}^3 x_{21} = 0$, Adem relation $\mathcal{P}^9\mathcal{P}^1 - \mathcal{P}^1\mathcal{P}^9 = \mathcal{P}^3\mathcal{P}^7 = \mathcal{P}^3\mathcal{P}^4\mathcal{P}^3$ implies

$$\begin{aligned}\mathcal{P}^9 x_{25} &= \mathcal{P}^9\mathcal{P}^1 x_{21} = \mathcal{P}^1\mathcal{P}^9 x_{21} = \mathcal{P}^1(-x_{48}x_9 + x_{36}x_{21})\\ &= -a\cdot x_{26}^2 x_9 - x_{20}^2 x_{21} + x_{36}x_{25} = -a\cdot x_{26}^2 x_9 + x_{36}x_{25}.\end{aligned}$$

The coefficient a will be fixed in later.

Since $\mathcal{P}^1 x_{25} = 0$, Adem relation $\mathcal{P}^8\delta\mathcal{P}^1 = \delta\mathcal{P}^9 - \mathcal{P}^9\delta$ implies

$$\mathcal{P}^9 x_{26} = \mathcal{P}^9\delta x_{25} = \delta\mathcal{P}^9 x_{25} = \delta(x_{36}x_{25} - a\cdot x_{26}^2 x_9) = x_{36}x_{26},$$

and the last formula of (5.10).

Finally we shall prove

(5.11) *$a=1$ and $c=1$ in Corollary* 4.3, (ii).

By Adem relation $\mathcal{P}^{13}=\mathcal{P}^1\mathcal{P}^3\mathcal{P}^9-\mathcal{P}^4\mathcal{P}^8\mathcal{P}^1$, (5.5), (5.5)′ (5.6) and by (5.7)

$$\begin{aligned} x_{26}^3 &= \mathcal{P}^{13}x_{26}=\mathcal{P}^1\mathcal{P}^3\mathcal{P}^9 x_{26}-\mathcal{P}^4\mathcal{P}^8\mathcal{P}^1 x_{26} \\ &= \mathcal{P}^1\mathcal{P}^3(x_{36}x_{26})=\mathcal{P}^1(\mathcal{P}^3(x_{36})x_{26}) \\ &= \mathcal{P}^1(x_{48}x_{26}+x_{36}(x_8+x_4^2)x_{21}x_9+x_{20}^2(x_{25}x_9-x_{21}x_9x_4)) \\ &= \mathcal{P}^1(x_{48}x_{26})=a\cdot x_{26}^3. \end{aligned}$$

Therefore $a=1$. Also we have, by Adem relation $\mathcal{P}^{10}=\mathcal{P}^1\mathcal{P}^9$,

$$\begin{aligned} x_{20}^3 &= \mathcal{P}^{10}x_{20}=\mathcal{P}^1\mathcal{P}^9 x_{20}=\mathcal{P}^1(f_{9,5}+c\cdot x_{26}x_{21}x_9) \\ &= x_{26}^2(-x_8+x_4^2)-x_{48}x_4^3-x_{20}^3-x_{20}^2x_8^2x_4-x_{36}x_8^3 \\ &\qquad +x_{20}^2(x_8x_4(-x_8+x_4^2)+x_8(-x_4^3))+c\cdot x_{26}x_{25}x_9 \\ &= (c-1)x_{26}x_{25}x_9-x_{48}x_4^3-x_{36}x_8^3-x_{20}^3+x_{20}^2x_8^2x_4. \end{aligned}$$

Then by use of the relation (5.8) we have $(c-1)x_{26}x_{25}x_9=0$, and $c=1$.

Consequently all the relations in Theorem III are established by Corollary 4.3, (5.3), (5.4), (5.5), (5.9), (5.10) and (5.11).

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