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On t-ideals of an integral domain

To Professor Y. Akizuki for celebration of his 70th birthday

By

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Introduction.

In the following R will be an integral domain, and K will be the field of quotients of R. By an ideal of R, we shall mean a non-zero fractional ideal of R. If an ideal $A\subset R$, then we say that A is an *integral* ideal of R. Let A and B be ideals of R, then we shall define $A:B=\{x\,|\,x\in K, Bx\subset A\}$. In the special case where $A=R$, $R:B$ is often denoted by B^{-1}, and we shall write $(B^{-1})^{n}$ by B^{-n}, for brevity. $(A^{-1})^{-1}$ is often denoted by A_{v}, and we shall define $A_{t}=\bigcup_{B\subset A}B_{v}$ where B is a finitely generated ideal. Then it is clear that $A\subset A_{t}\subset A_{v}$.

Definition. Let A be an ideal of R. If $A=A_{t}$, then we say that A is a *t-ideal* of R. If $A=A_{v}$, then we say that A is a *V-ideal* of R. If $A=A_{v}$ and $AA^{-1}=A$, then we say that A is an *F-ideal* of R.

If $A_{t}=B_{t}$, then we say that A is *t-equal* to B and write $A\overset{t}{\sim}B$. If $A^{-1}=B^{-1}$, that is, $A_{v}=B_{v}$, then we say that A is *quasi-equal* to B and write $A\sim B$.

In [1], K. E. Aubert has introduced the following problem.

Problem. *Is a Krull ring characterized by the fact that any* *of its proper t-ideals can be written as a t-product of a prime t-ideals?*

In this paper, we shall obtain the following theorem which shows that this problem has a positive answer.

Theorem. *The following three conditions are equivalent to each other:*

(1) *Any integral ideal A of R which is not t-equal to R satisfies a t-equality of the following type:*

$$A \overset{t}{\sim} \mathfrak{p}_1^{r_1}\mathfrak{p}_2^{r_2}\cdots\mathfrak{p}_n^{r_n},$$

where $\mathfrak{p}_i$ $(i=1,2,\cdots,n)$ are prime t-ideals in R and r_i $(i=1,2,\cdots,n)$ are positive integers, and $\mathfrak{p}_i$, r_i $(i=1,2,\cdots,n)$ are uniquely determined up to the order.

(2) *Any ideal A of R satisfies a t-equality of the following type:*

$$A \overset{t}{\sim} \mathfrak{p}_1^{r_1}\mathfrak{p}_2^{r_2}\cdots\mathfrak{p}_n^{r_n},$$

where $\mathfrak{p}_i$ $(i=1,2,\cdots,n)$ are prime ideals in R and r_i $(i=1,2,\cdots,n)$ are integers, and $\mathfrak{p}_i$, r_i $(i=1,2,\cdots,n)$ are uniquely determined up to the order and factors which are t-equal to R.

(3) *R is a Krull ring.*

§1. *t*-ideals

Lemma 1. *Let A be an ideal of R. Then $(A_t)_t = A_t$.*

Proof. Let $a \in (A_t)_t$. Then there exists a finitely generated ideal B such that $B \subset A_t$ and $B_v \ni a$. Put

$$B = Rb_1 + Rb_2 + \cdots + Rb_m, \quad b_i \in A_t \ (i=1,2,\cdots,m),$$

then there exist finitely generated ideals B_i $(i=1,2,\cdots,m)$ such that $B_i \subset A$ and $(B_i)_v \ni b_i$. Hence

$$B \subset (B_1)_v + (B_2)_v + \cdots + (B_m)_v,$$
$$B_v \subset \{(B_1)_v + (B_2)_v + \cdots + (B_m)_v\}_v = (B_1 + B_2 + \cdots + B_m)_v.$$

Since $B_1+B_2+\cdots+B_m$ is a finitely generated ideal, $a\in A_t$.

Lemma 2. *Let A, B be ideals of R, then $A_tB_t\subset(AB)_t$.*

Proof. Let $\alpha\in A_tB_t$, and

$$\alpha=a_1b_1+a_2b_2+\cdots+a_nb_n,\quad a_i\in A_t,\ b_i\in B_t\ (i=1,2,\cdots,n).$$

Then there exist finitely generated ideals X_i, Y_i $(i=1,2,\cdots,n)$ such that $X_i\subset A$, $Y_i\subset B$, $(X_i)_v\ni a_i$, $(Y_i)_v\ni b_i$. Put

$$X=X_1+X_2+\cdots+X_n,\quad Y=Y_1+Y_2+\cdots+Y_n,$$

then

$$\alpha\in X_vY_v\subset(XY)_v\subset(AB)_t.$$

Lemma 3. *Let A, B, C, D be ideals of R. If $A\overset{t}{\sim}B$, $C\overset{t}{\sim}D$, then $AC\overset{t}{\sim}BD$.*

Proof. $A_t=B_t$, $C_t=D_t$ and $A_tC_t=B_tD_t$. Hence

$$(AC)_t=(A_tC_t)_t=(B_tD_t)_t=(BD)_t.$$

Proposition 1. *If A is a maximal integral t-ideal of R, then A is a prime ideal.*

Proof. We assume that $BC\subset A$, $B\not\subset A$. Then $A\subsetneqq A+B$ and $(A+B)_t=R$. Put $D=AC+BC=(A+B)C\subset A$. Then $D_t\supset(A+B)_tC_t=C_t$. Hence $A=A_t\supset D_t\supset C_t$. Hence $A\supset C$.

§2. Ascending chain condition

Lemma 4. *Let A, B be ideals of R. If $A\overset{t}{\sim}B$, then $A\sim B$.*

Proof. $A\subset A_t\subset A_v$. Hence $A_v=(A_t)_v=(B_t)_v=B_v$.

Lemma 5. *If A is a V-ideal of R, then A is a t-ideal of R.*

Proof. $A=A_v\supset A_t\supset A$.

Lemma 6. *Let R satisfy the ascending chain condition for* *integral V-ideals. Let A be an ideal of R, then* $A_v = A_t$. *If A is a t-ideal of R, then A is a V-ideal of R.*

Proof. By Lemma 1 of [3], there exists a finitely generated ideal B such that $B \subset A$, $A_v = B_v$. On the other hand, it is clear that $B_t = B_v$. Hence $B_t = B_v = A_v \supset A_t \supset B_t$. Hence $A_v = A_t$. If A is a t-ideal of R, then $A = A_v$.

Proposition 2. *The following two conditions are equivalent to each other*:

(1) *R satisfies the ascending chain condition for integral V-ideals.*

(2) *R satisfies the ascending chain condition for integral t-ideals.*

Proof. It follows from Lemma 5 and Lemma 6.

Lemma 7. *Let* $A_1, A_2, \cdots, A_n, \cdots$ *be t-ideals of R and* $A_1 \subset A_2 \subset \cdots \subset A_n \subset \cdots$. *Then* $\cup A_i = A$ *is a t-ideal of R.*

Proof. Let B be a finitely generated ideal such that $B \subset A$. Then there exists A_m such that $B \subset A_m$. Hence $B_v = B_t \subset A_m$. Hence $B_v \subset A$.

Proposition 3. *We shall introduce the following two conditions.*

(1) *Every integral t-ideal of R is a finite R-module.*

(2) *R satisfies the ascending chain condition for integral t-ideals.*

Then (1)⇒(2), *but the inverse is false.*

Proof. It is clear that (1)⇒(2) by Lemma 7. If R satisfy the condition (2), then every t-ideal of R is a V-ideal of R. In [3], we have showed the example such that R satisfies the ascending chain condition for integral V-ideals, but every integral V-ideal of R is not necessarily a finite R-module. Hence the condition (2) does not imply the condition (1).

§3. Theorem

The following lemmas 8, 9 are known, but we shall recall them for the reader's convenience.

Lemma 8. *Let R be completely integrally closed in K and $\mathfrak{p}$ be a prime ideal of R. If $\mathfrak{p}$ is not quasi-equal to R, then $\mathfrak{p}$ is a V-ideal of R and the height of $\mathfrak{p}$ is 1.*

Proof. It is clear that the height of $\mathfrak{p}$ is 1 by Lemma 7 of [3]. Since $\mathfrak{p}\sim\mathfrak{p}_v$, $\mathfrak{p}_v\mathfrak{p}^{-1}$ and $\mathfrak{p}\mathfrak{p}^{-1}$ are integral ideals, and $\mathfrak{p}\mathfrak{p}^{-1}\sim R$. Since $\mathfrak{p}_v\mathfrak{p}^{-1}\mathfrak{p}=\mathfrak{p}\mathfrak{p}^{-1}\mathfrak{p}_v$ and $\mathfrak{p}\mathfrak{p}^{-1}\not\subset\mathfrak{p}$, $\mathfrak{p}_v\subset\mathfrak{p}$. Hence $\mathfrak{p}=\mathfrak{p}_v$.

Lemma 9. *Let R be completely integrally closed in K and $\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_r, \mathfrak{p}'_1, \mathfrak{p}'_2, \cdots, \mathfrak{p}'_s$ be prime ideals in R which are not quasi-equal to R. If $\mathfrak{p}_1\mathfrak{p}_2\cdots\mathfrak{p}_r\sim\mathfrak{p}'_1\mathfrak{p}'_2\cdots\mathfrak{p}'_s$, then $\mathfrak{p}_1\mathfrak{p}_2\cdots\mathfrak{p}_r\equiv\mathfrak{p}'_1\mathfrak{p}'_2\cdots\mathfrak{p}'_s$.*[1]

Proof. By Lemma 8 of [3], there are integral ideals $A_{i\mu}$, $B_{j\nu}$ such that $\mathfrak{p}_i\sim A_{i1}A_{i2}\cdots A_{ik}$, $\mathfrak{p}'_j\sim B_{j1}B_{j2}\cdots B_{jl}$ $(i=1, 2, \cdots, r;\ j=1, 2, \cdots, s)$ and such that $\Pi A_{i\mu}\equiv\Pi B_{j\nu}$. By Lemma 6 of [3], $\mathfrak{p}_i\sim A_{i\mu}$, $A_{i1}\cdots A_{i\mu-1}A_{i\mu+1}\cdots A_{ik}\sim R$. Let $A_{i\mu}=B_{j\nu}$, then $\mathfrak{p}_i=\mathfrak{p}'_j$ by Lemma 8. Hence $r=s$ and $\mathfrak{p}_i=\mathfrak{p}'_{j_i}$ $(i=1, 2, \cdots, r)$ where $j_1, j_2, \cdots, j_r$ is a permutation of $1, 2, \cdots, r$.

Theorem. *The following three conditions are equivalent to each other:*

(1) *Any integral ideal A of R which is not t-equal to R satisfies a t-equality of the following type:*

$$A\overset{t}{\sim}\mathfrak{p}_1^{r_1}\mathfrak{p}_2^{r_2}\cdots\mathfrak{p}_n^{r_n},$$

where $\mathfrak{p}_i$ $(i=1, 2, \cdots, n)$ are prime t-ideals in R and r_i $(i=1, 2, \cdots, n)$

1) $C_1C_2\cdots C_r\equiv D_1D_2\cdots D_s$ means that $r=s$ and $C_i=D_{j_i}$ $(i=1, 2, \cdots, r)$ where $j_1, j_2, \cdots, j_r$ is a permutation of $1, 2, \cdots, r$.

are positive integers, and $\mathfrak{p}_i$, r_i $(i=1, 2, \cdots, n)$ *are uniquely determined up to the order.*

(2) *Any ideal* A *of* R *satisfies a t-equality of the following type*:

$$A \overset{t}{\sim} \mathfrak{p}_1^{r_1}\mathfrak{p}_2^{r_2}\cdots\mathfrak{p}_n^{r_n},$$

where $\mathfrak{p}_i$ $(i=1, 2, \cdots, n)$ *are prime ideals in* R *and* r_i $(i=1, 2, \cdots, n)$ *are integers, and* $\mathfrak{p}_i$, r_i $(i=1, 2, \cdots, n)$ *are uniquely determined up to the order and factors which are t-equal to* R.

(3) R *is a Krull ring.*

Remark. *The following three conditions are equivalent to each other*: (*cf.* [3])

(3) R *is a Krull ring.*

(4) *Any ideal* A *of* R *satisfies a quasi-equality of the following type*:

$$A \sim \mathfrak{p}_1^{r_1}\mathfrak{p}_2^{r_2}\cdots\mathfrak{p}_n^{r_n},$$

where $\mathfrak{p}_i$ $(i=1, 2, \cdots, n)$ *are prime ideals in* R *and* r_i $(i=1, 2, \cdots, n)$ *are integers, and* $\mathfrak{p}_i$, r_i $(i=1, 2, \cdots, n)$ *are uniquely determined up to the order and factors which are quasi-equal to* R.

(5) R *is completely integrally closed in* K *and satisfies the ascending chain condition for integral V-ideals.*

Proof of Theorem. We shall show that (3)$\Rightarrow$(2) and (3)$\Rightarrow$(1). By the condition (5) and Lemma 6, $A \sim B$ if and only if $A \overset{t}{\sim} B$. Hence (3)$\Rightarrow$(2) by the condition (4). It is clear that (3)$\Rightarrow$(1) by Lemma 9 of [3]. Next, we shall show that (1)$\Rightarrow$(3). We assume that there exists an F-ideal $N(\neq R)$ of R. Then there exists an element a of R such that $aN^{-1} \subset R$. Let

$$N \overset{t}{\sim} \mathfrak{p}_1^{r_1}\mathfrak{p}_2^{r_2}\cdots\mathfrak{p}_n^{r_n},$$

$$aN^{-1} \overset{t}{\sim} \mathfrak{p}_1'^{s_1}\mathfrak{p}_2'^{s_2}\cdots\mathfrak{p}_m'^{s_m},$$

$$aR \overset{t}{\sim} \mathfrak{p}_1''^{u_1}\mathfrak{p}_2''^{u_2}\cdots\mathfrak{p}_l''^{u_l},$$

then

$$aN^{-1}N = aN \overset{t}{\sim} \mathfrak{p}_1^{r_1}\mathfrak{p}_2^{r_2}\cdots\mathfrak{p}_n^{r_n}\mathfrak{p}_1'^{s_1}\mathfrak{p}_2'^{s_2}\cdots\mathfrak{p}_m'^{s_m},$$
$$aRN = aN \overset{t}{\sim} \mathfrak{p}_1^{r_1}\mathfrak{p}_2^{r_2}\cdots\mathfrak{p}_n^{r_n}\mathfrak{p}_1''^{u_1}\mathfrak{p}_2''^{u_2}\cdots\mathfrak{p}_l''^{u_l}.$$

This is a contradiction. Since R has no F-ideal ($\neq R$), R is completely integrally closed (cf. [2]). Let A be an integral ideal of R which is not t-equal to R. Then $A \overset{t}{\sim} \mathfrak{p}_1^{r_1}\mathfrak{p}_2^{r_2}\cdots\mathfrak{p}_n^{r_n}$ where $\mathfrak{p}_i$ ($i=1, 2, \cdots, n$) are prime t-ideals in R and r_i ($i=1, 2, \cdots, n$) are positive integers, and $\mathfrak{p}_i$, r_i ($i=1, 2, \cdots, n$) are uniquely determined up to the order. Hence $A \sim \mathfrak{p}_1^{r_1}\mathfrak{p}_2^{r_2}\cdots\mathfrak{p}_n^{r_n}$ by Lemma 4. We shall eliminate $\mathfrak{p}_i$ which is quasi-equal to R, then $A \sim \mathfrak{p}_1'^{s_1}\mathfrak{p}_2'^{s_2}\cdots\mathfrak{p}_m'^{s_m}$ where $\mathfrak{p}_i'$ ($i=1, 2, \cdots, m$) are prime ideals of height 1 in R by Lemma 8 and s_i ($i=1, 2, \cdots, m$) are positive integers, and $\mathfrak{p}_i'$, s_i ($i=1, 2, \cdots, m$) are uniquely determined up to the order by Lemma 9. Let B be an ideal of R. Then there exists an element b of R such that $bB \subset R$. Let

$$bB \sim \mathfrak{p}_1'^{r_1}\mathfrak{p}_2'^{r_2}\cdots\mathfrak{p}_n'^{r_n},$$
$$bR \sim \mathfrak{p}_1^{s_1}\mathfrak{p}_2^{s_2}\cdots\mathfrak{p}_m^{s_m}.$$

By Lemma 10 of [3],

$$b^{-1}R \sim \mathfrak{p}_1^{-s_1}\mathfrak{p}_2^{-s_2}\cdots\mathfrak{p}_m^{-s_m}.$$

Hence

$$B \sim \mathfrak{p}_1'^{r_1}\mathfrak{p}_2'^{r_2}\cdots\mathfrak{p}_n'^{r_n}\mathfrak{p}_1^{-s_1}\mathfrak{p}_2^{-s_2}\cdots\mathfrak{p}_m^{-s_m}.$$

Therefore R satisfies the condition (4) and the condition (3). In the same way, it is clear that (2)⇒(3).

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