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On a characterization of $PSL_4(p)$

By

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Let $PGL_r(p)$ and $PSL_r(p)$ denote the r-dimensional projective general linear and special linear groups over the prime field F_p of p elements respectively where $r \geqq 2$. Let $V(r, p)$ be the r-dimensional vector space over F_p. Then $PGL_r(p)$ can be considered as a permutation group on the set Ω of one-dimensional subspaces of $V(r, p)$. Any subgroup G of $PGL_r(p)$ containing $PSL_r(p)$ is called a group of Lie type of dimension $r-1$ over F_p (according to Ito [2]). Any group G of Lie type of dimension $r-1$ over F_p, $r \geqq 3$, has the following properties.

(1) (G, Ω) is a doubly transitive group of degree $p^{r-1}+p^{r-2}+\cdots+p+1=p^r-1/p-1$.

(2) For any two elements, say 0 and 1, of Ω, Ω is divided into four orbits $\{0\}$, $\{1\}$, Δ and Γ by the subgroup $G_{\{0,1\}}$ of G, the stabilizer of two elements 0 and 1, where Δ and Γ consist of $p-1$ and $p^{r-1}+p^{r-2}+\cdots+p^2$ elements respectively.

(3) The order of G is divisible by $p^{\frac{r(r-1)}{2}}$ exactly.

Now we consider to characterize groups of Lie type by these properties and we conjecture the following: A permutation group (G, Ω) which satisfies three properties (1), (2) and (3) is permutation-isomorphic to a group of Lie type of dimension $r-1$ over F_p.

For $r=3$ the conjecture is true. This was proved in [3], under some weaker assumption. The purpose of this note is to show that the conjecture is true for $r=4$, namely, we will prove the following.

Theorem. *Let (G, Ω) be a doubly transitive permutation group. Assume that*

(4) *Ω consists of p^2+p^2+p+1 elements where p is a prime number,*

(5) *for two elements of Ω, say 0 and 1, Ω is divided into four orbits $\{0\}$, $\{1\}$, Δ and Γ by $G_{\{0,1\}}$ where Δ and Γ consist of $p-1$ and p^3+p^2 elements respectively, and*

(6) *the order of G is divisible by p^4.*

Then (G, Ω) is permutation-isomorphic to a group of Lie type of dimension 3 over F_p.

We use the following notation. For a set π, $|\pi|=$the number of elements in π. For a permutation group (H, π), for a subgroup K of H, and for a subset Λ of π.

$$H_{\Lambda}=\{x\in H \mid \alpha^x=\alpha \text{ for any } \alpha\in\Lambda\},$$

which is called the stabilizer of Λ in (H, π),

$$H_{<\Lambda>}=\{x\in H \mid \Lambda^x=\Lambda\},$$

$$H_{\Lambda}=\frac{H_{<\Lambda>}}{H_{\Lambda}}$$

$$F(K)=\{\alpha\in\pi \mid \alpha^x=\alpha \text{ for any } x\in K\}.$$

Proof. First we remark that all the doubly transitive groups of degree $15(=2^3+2^2+2+1)$ have been determined (e.g. [1], p. 304) and we see that our theorem is true for the case $p=2$. In the following we can assume that $p\geqq 3$. Put $\bar{\Delta}=\{1\}\cup\Delta$. For any $\alpha\in\Delta$, since the length of any orbit of $(G_{\{0,1,\alpha\}}, \Gamma)$ is a multiple of p^2, we have that $\{0\}$, $\{\alpha\}$, $\bar{\Delta}-\{\alpha\}$ and Γ are the orbits of $(G_{\{0,\alpha\}}, \Omega)$, and that, for any $x\in G_{\{0\}}$ and for any $\alpha\in\bar{\Delta}^x$, $\{0\}$, $\{\alpha\}$, $\bar{\Delta}^x-\{\alpha\}$ and Γ^x are the orbits of $(G_{\{0,\alpha\}}, \Omega)$. Hence if $\bar{\Delta}\cap\bar{\Delta}^x\neq\phi$ for an element $x\in G_{\{0\}}$, then, for any element $\alpha\in\bar{\Delta}\cap\bar{\Delta}^x$, we have that $\bar{\Delta}-\{\alpha\}=\bar{\Delta}^x-\{\alpha\}$, that is $\bar{\Delta}=\bar{\Delta}^x$. Hence $(G_{\{0\}}, \Omega-\{0\})$ is imprimitive and

$\bar{\Delta}$ is the unique non-trivial block (of imprimitivity) of $(G_{\{0\}}, \Omega-\{0\})$ which contains 1, and therefore any element of $\Omega-\{0\}$ is contained in a unique non-trivial block of $(G_{\{0\}}, \Omega-\{0\})$. Let α be an element of Γ. Let H_α be the subgroup of G generated by the set of all Sylow p-subgroups of $G_{\{0,1,\alpha\}}$ and let P be a subgroup of $G_{\{0,1\}}$ which is conjugate to H_α in G. Then, since P fixes at least one element of Γ and $G_{\{0,1\}}$ is transitive on Γ, there is an element x of $G_{\{0,1\}}$ such that $P^x \subseteq G_{\{0,1,\alpha\}}$, and, by the definition of H_α, we have $P^x = H_\alpha$. Hence, by a theorem of Witt ([4], Theorem 9.4), $N(H_\alpha)$, the normalizer of H_α in G, is doubly transitive on $F(H_\alpha)$. Since each element of Γ is contained exactly in one non-trivial block of $(G_{\{0\}}, \Omega-\{0\})$, we have $|F(H_\alpha)\cap\Gamma| \equiv 0 \pmod p$. Put $|F(H_\alpha)\cap\Gamma| = tp$. Since $G_{\{0,1\}}$ is transitive on Γ, we have that, for any α and $\beta\in\Gamma$, $|F(H_\alpha)| = |F(H_\beta)|$ and that either $F(H_\alpha)=F(H_\beta)$ or $F(H_\alpha)\cap F(H_\beta) = \{0\}\cup\{1\}\cup\Delta$, and so $t \mid p(p+1)$,

(7) $[G_{\{0,1\}} : N(H_\alpha)\cap G_{\{0,1\}}] = \dfrac{p(p+1)}{t}$, and

(8) $[G : N(H_\alpha)\cap G_{\{0,1\}}] \equiv 0$
$\mod (tp+p+1)(tp+p) (= [N(H_\alpha) : N(H_\alpha)\cap G_{\{0,1\}}])$.

Then we have $|N(H_\alpha)^{F(H_\alpha)}| \equiv 0 \pmod{p^2}$ from (7), because, for a Sylow p-subgroup Q of $G_{\{0,1\}}$, the length of any orbit of (Q, Γ) is a multiple of p^2. Assume that $t<p-1$. Then $(N(H_\alpha)^{F(H_\alpha)}, F(H_\alpha))$ is a doubly transitive group of degree $1+(t+1)p$ where $t+1<p$, and $|N(H_\alpha)^{F(H_\alpha)}| \equiv 0 \pmod{p^2}$. Hence $(N(H_\alpha)^{F(H_\alpha)}, F(H_\alpha))$ is either alternating or symmetric by Proposition 1 in [3] which contradicts the assumption (5). Assume that $t=p-1$. Then we have $p=3$, $|F(H_\alpha)| = 1+p^2$ and $|N(H_\alpha)^{F(H_\alpha)}| \equiv 0 \pmod{p^3}$. Hence $(N(H_\alpha)^{F(H_\alpha)}, F(H_\alpha))$ is either alternating or symmetric by Proposition 2 in [3] which contradicts the assumption (5). Assume that $t=p+1$. Then, from (7) and (8), we have

$$(p^3+p^2+p+1)(p^3+p^2+p) \equiv 0 \mod p^2+2p+1$$

which is a contradiction. Hence $t=up$ and $u \mid p+1$. Then, from (7) and (8),

$$(p^3+p^2+p+1)(p^3+p^2+p)s \equiv 0 \mod up^2+p+1$$

where $us=p+1$, and then we have

$$(s-1)(u-1) \equiv 0 \mod p^2+s.$$

Hence $u=1$ or $s=1$. If $s=1$, then $|G| \not\equiv 0 \pmod{p^4}$ which is a contradiction. Hence $u=1$. Then, from (7) and (8), we have

$$[G : N(H_\alpha)] = \frac{[G : N(H_\alpha) \cap G_{\{0,1\}}]}{[N(H_\alpha) : N(H_\alpha) \cap G_{\{0,1\}}]} = p^3+p^2+p+1.$$

Thus $N(H_\alpha)$ is a subgroup of G of index p^3+p^2+p+1 and $N(H_\alpha)$ is not faithful on $F(H_\alpha)$, an orbit of $(N(H_\alpha), \Omega)$. Hence (G, Ω) is a group of Lie type of dimension 3 over F_p by a result of Ito [2].

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References

[1] Carmichael, R. D. "Introduction to the Theory of Groups of Finite Order", Ginn, Boston; reprinted 1956, Dover, New York.

[2] Ito, N. On a class of doubly, but not triply transitive permutation groups, Arch. Math. **18** (1967), 564-570.

[3] Tsuzuku, T. On doubly transitive permutation groups of degree $1+p+p^2$ where p is a prime number, J. Algebra **8** (1968), 143-147.

[4] Wielandt, H. "Finite Permutation Groups". Academic Press, New York and London, 1964.