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On finite permutation groups of rank 4

By

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Introduction

A general theory about permutation groups of arbitrary rank, in particular, of rank 3 was studied in D. G. Higman's papers [2] and [3]. In this note, making use of methods and results in the above papers, we shall be concerned with permutation groups of rank 4.

First, in sections 1 and 2 we have a general discussion about rank 4 permutation groups in the same way as that about rank 3 in Higman [2]. For example, Lemmas 1.1, 1.2 and 2.3 correspond to Lemmas 6 and 7 in [2].

Second, let G be a primitive permutation group of rank 4 with subdegrees 1, k, l and m. Then the next problem occurs naturally: When k, l and m are given, determine G. Some answers to this problem will be given in Theorem of section 3 (this corresponds to Theorem 1 in [2]) and some propositions of sections 1 and 3. For the most part, results obtained are of negative nature. In dealing with this problem, all the lemmas in sections 1, 2 and many relations among k, l and m, i.e., (4.1) and (4.2) in Higman [3] are used repeatedly.

Finally we may determine the primitive extension of rank 4 of alternating groups which act naturally, using results obtained so far, in the same way as T. Tsuzuku's paper [5]. However, this discussion will not be given in the present note.

All the calculations practised in this note are quite elementary, but results obtained are not so trivial.

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0. Notation and quoted results

The following notation will be fixed throughout this note. Let Ω be a set of n letters and let G be a transitive permutation group of rank 4 on Ω. For each $a\in\Omega$, G_a denotes the stabilizer of a. Ω decomposes into exactly 4 G_a-orbits,

$$\Omega=\{a\}+\Delta(a)+\Gamma(a)+\Lambda(a),$$

where the notation is chosen so that

$$\Delta(a)^g=\Delta(a^g),\ \Gamma(a)^g=\Gamma(a^g),\ \Lambda(a)^g=\Lambda(a^g)\ \text{ for all } a\in\Omega,\ g\in G.$$

We set $k=|\Delta(a)|$, $l=|\Gamma(a)|$ and $m=|\Lambda(a)|$, which are independent of choice of $a\in\Omega$. 1, k, l and m are called the subdegrees of G. Set $\Gamma_0(a)=\{a\}$, $\Gamma_1(a)=\Delta(a)$, $\Gamma_2(a)=\Gamma(a)$, $\Gamma_3(a)=\Lambda(a)$ and $\mu_{ij}^{(\alpha)}=|\Gamma_\alpha(b)\cap\Gamma_i(a)|$ for $b\in\Gamma_j(a)$. In (4.1) and (4.2) of [3], Higman and M. Suzuki have given relations among k, l, m and $\mu_{ij}^{(\alpha)}$. We call these relations parameters-relations. Moreover, the 4×4 matrix $M_\alpha=(\mu_{ij}^{(\alpha)})_{i,j}$ is called the intersection matrix of Γ_α. Let $f_0=1$, f_1, f_2 and f_3 be the degrees of the irreducible constituents of the permutation representation of G. Hence

$$n=1+f_1+f_2+f_3\,. \tag{0.1}$$

Since G is of rank 4, one of the next two cases occurs (see §16 in [6]).

I. Of three non-trivial G_a-orbits, only one orbit is self-paired and the other two orbits are paired (we may assume that $\Gamma(a)$ is self-paired and $\Delta(a)$, $\Lambda(a)$ are paired).

II. All the G_a-orbits are self-paired.

In Case I we can write paramenters-relations as follows.

	$\Delta(b)$	$\Gamma(b)$	$\Lambda(b)$	row sum	
$\Delta(a)$	λ	λ_1	λ	$k-1$	$b\in\Delta(a)$
	μ	μ_1	μ_2	k	$b\in\Gamma(a)$
	λ	ν_1	ν_2	k	$b\in\Lambda(a)$
$\Gamma(a)$	ν_1	λ'	λ_1	l	$b\in\Delta(a)$
	μ_1	μ'	μ_1	$l-1$	$b\in\Gamma(a)$
	λ_1	λ'	ν_1	l	$b\in\Lambda(a)$
$\Lambda(a)$	ν_2	ν_1	λ	m	$b\in\Delta(a)$
	μ_2	μ_1	μ	m	$b\in\Gamma(a)$
	λ	λ_1	λ	$m-1$	$b\in\Lambda(a)$

$$k=m,\ k\nu_1=l\mu_2,\ k\lambda'=l\mu_1,\ k\lambda_1=l\mu.$$

For example, $\lambda(\lambda_1, \lambda$ resp.) means $|\Delta(a)\cap\Delta(b)|$ $(|\Delta(a)\cap\Gamma(b)|$, $|\Delta(a)\cap\Lambda(b)|$ resp.) for $b\in\Delta(a)$ and $\lambda+\lambda_1+\lambda=k-1$. In this table nine parameters appear, but we see easily that two parameters λ_1, μ' and k, l, m determine the other seven perameters.

In Case II we can write parameters-relations as follows.

	$\Delta(b)$	$\Gamma(b)$	$\Lambda(b)$	row sum	
$\Delta(a)$	λ	λ_1	λ_2	$k-1$	$b\in\Delta(a)$
	μ	μ_1	μ_2	k	$b\in\Gamma(a)$
	ν	ν_1	ν_2	k	$b\in\Lambda(a)$
$\Gamma(a)$	λ_1	λ'	λ_3	l	$b\in\Delta(a)$
	μ_1	μ'	μ_3	$l-1$	$b\in\Gamma(a)$
	ν_1	ν'	ν_3	l	$b\in\Lambda(a)$
$\Lambda(a)$	λ_2	λ_3	λ''	m	$b\in\Delta(a)$
	μ_2	μ_3	μ''	m	$b\in\Gamma(a)$
	ν_2	ν_3	ν''	$m-1$	$b\in\Lambda(a)$

$$k\lambda_1 = l\mu, \quad k\lambda' = l\mu_1, \quad k\lambda_2 = m\nu, \quad k\lambda'' = m\nu_2,$$
$$l\mu_3 = m\nu', \quad l\mu'' = m\nu_3, \quad k\lambda_3 = l\mu_2 = m\nu_1.$$

In this table 18 parameters appear, but as before we see that six parameters λ, μ, ν, λ', μ', ν' and k, l, m determine the other parameters.

Remark. In both cases, by (4.10) in Higman [3], any two intersection matrices commute with each other. This commutativity gives parameters-relations and some other relations (e.g., see (i) in Proposition 1.4).

Let A, B and C be incidence matrices for the orbitals Δ, Γ and Λ, respectively (2. in [3]).

Namely

$$A = (a_{ij}) \text{ where } a_{ij} = \begin{cases} 1 & \text{if } i \in \Delta(j) \\ 0 & \text{otherwise.} \end{cases}$$

$$B = (b_{ij}) \text{ where } b_{ij} = \begin{cases} 1 & \text{if } i \in \Gamma(j) \\ 0 & \text{otherwise.} \end{cases}$$

$$C = (c_{ij}) \text{ where } c_{ij} = \begin{cases} 1 & \text{if } i \in \Lambda(j) \\ 0 & \text{otherwise.} \end{cases}$$

The rows and columns of A, B and C are indexed by the points of Ω in some given order. Then, all the diagonal entries of A, B and C are 0's and by (2.1) in [3]

$$I + A + B + C = F \tag{0.2}$$

where F is the matrix with all entries 1. A (resp. B, C) has k (resp. l, m) 1's in each row and column.

For a subset X of Ω, G_X (resp. $G_{\{X\}}$) denotes the pointwise (resp. setwise) stabilizer of X.

Following results are often used.

Proposition 0.1. (Proposition 4.5 in C. C. Sims [4]). *If G is primitive and $k \leq l \leq m$, then $l \leq k^2$ and $m \leq kl$. If in addition $\Delta(a)$ is self-paired, then $l \leq (k-1)k$ and $m \leq (k-1)l$.*

Proposition 0. 2. (H. Wielandt [6], p. 51). *If G is primitive and one of k, l, m, say k is equal to 2, then $l=m=2$, G is a dihedral group of order $2\cdot 7$ and all the G_a-orbits are self-paired.*

1. Case I

Throughout this section, we assume that Case I holds. In this case we can treat G in the same way as rank 3 groups. Namely, as in rank 3, following lemmas hold (cf. Lemmas 6, 7 in [2]).

Lemma 1. 1. *The incidence matrix B for Γ satisfies*

$$(B-lI)\{B^2-(\mu'-\lambda')B-(l-\lambda')I\}=0.$$

Therefore, in addition to the eigenvalue l (of multiplicity 1) B has exactly the two distinct values s and t, where

$$\left\{\begin{matrix} s \\ t \end{matrix}\right\}=\frac{(\mu'-\lambda')\pm\sqrt{d}}{2}, \quad d=(\mu'-\lambda')^2+4(l-\lambda').$$

Lemma 1. 2. *Using the above notation, the following holds. At least one of f_1, f_2, f_3 is equal to*

$$\frac{2l+(\mu'-\lambda')(2k+l)\pm\sqrt{d}\,(2k+l)}{\pm 2\sqrt{d}}$$

Therefore

i) *if d is not a square, $2l+(\mu'-\lambda')(2k+l)=0$*

ii) *if d is a square, $\sqrt{d}$ divides $2l+(\mu'-\lambda')(2k+l)$*

and the eigenvalues (i.e. l, s and t) of B are integers.

Proof of Lemma 1. 1. We see easily $B\cdot{}^tB=lI+\lambda'A+\mu'B+\lambda'C$. Since $\Gamma(a)$ is self-paired, by (2. 3) in [3], we have

$$B^2=lI+\lambda'A+\mu'B+\lambda'C.$$

Therefore, by (0. 2)

$$\lambda'F=B^2-(\mu'-\lambda')B-(l-\lambda')I.$$

On the other hand, since $BF=lF$

$$(B-lI)\{B^2-(\mu'-\lambda')B-(l-\lambda')I\}=0.$$

Proof of Lemma 1.2. If B is similar to

$$\begin{pmatrix} l & & & \\ & \overbrace{\begin{matrix} s & & \\ & \ddots & \\ & & s \end{matrix}}^{f_1} & & \\ & & \overbrace{\begin{matrix} s & & \\ & \ddots & \\ & & s \end{matrix}}^{f_2} & \\ & & & \overbrace{\begin{matrix} t & & \\ & \ddots & \\ & & t \end{matrix}}^{f_3} \end{pmatrix}$$ (see §29 in [6]),

taking traces we have

$$0=l+s(f_1+f_2)+tf_3.$$

On the other hand, since $f_1+f_2+f_3=k+l+m=2k+l$,

$$f_3=\frac{l+s(2k+l)}{s-t}=\frac{2l+(\mu'-\lambda')(2k+l)+\sqrt{d}\,(2k+l)}{2\sqrt{d}}$$

and so on.

Lemma 1.3. *G is primitive on Ω if and only if $\mu'\neq l-1$ and $0<\lambda'<l$.*

Proof. Let G be primitive. If $\mu'=l-1$, then $a\cup\Gamma(a)=b\cup\Gamma(b)$ for $b\in\Gamma(a)$. Let g be an element in G such that $a^g=b$. Then $(a\cup\Gamma(a))^g=a^g\cup\Gamma(a^g)=b\cup\Gamma(b)=a\cup\Gamma(a)$. Therefore $G_a\subsetneqq G_{\{a\cup\Gamma(a)\}}\subsetneqq G$, which contradicts the primitivity of G. Thus $\mu'\neq l-1$. If $\lambda'=0$, then $\mu_1=0$ since $k\lambda'=l\mu_1$, which means $\mu'=l-1$ since $\mu_1+\mu'+\mu_1=l-1$. This is contrary to the first assertion. If $\lambda'=l$, then $\Gamma(a)=\Gamma(b)$ for $b\in\Delta(a)$. As before, taking an element g such that $a^g=b$, we have $\Gamma(a)^g=\Gamma(a)$. Hence $G_a\subsetneqq G_{\{\Gamma(a)\}}\subsetneqq G$, a contradiction.

In reality, the present lemma is easily seen from (4.8) and the first equality of (4.1) in [3].

Using above lemmas, we obtain following propositions (see also Proposition 3.1).

Proposition 1.1. *There exists no primitive group satisfying Case I and $l>k(k-1)$.*

Proof. Let G be a primitive group satisfying the above condition. By Propositions 0.2 and 3.1, we have $k>2$. Parameters-relation $k\lambda_1=l\mu$, $l>k(k-1)$ and $\lambda_1\leqq k-1$ imply $\lambda_1=\mu=0$. Hence we have $2\lambda=k-1$ since $\lambda+\lambda_1+\lambda=k-1$. Now, suppose $\mu_2\neq 0$. Then $k\nu_1=l\mu_2$, $l>k(k-1)$ and $\nu_1\leqq k$ imply $\mu_2=1$ and $\nu_1=k$. So $\lambda=0$ since $\lambda+\nu_1+\nu_2=k$. Hence $k-1=2\lambda=0$, which is a contradiction. Thus $\mu_2=0$ and so $\nu_1=0$. Therefore, since $\lambda_1+\lambda'+\nu_1=l$, we have $\lambda'=l$, which is contrary to Lemma 1.3.

Similarly we have at once

Proposition 1.2. *There exists no primitive group satisfying Case I and $k>l(l-1)/2$.*

Next two propositions correspond to Theorem 1 in [2].

Proposition 1.3. *There exists no transitive group satisfying Case I and $l=k(k-1)$.*

Proof. Let G be a transitive group with the given condition. From the values of subdegrees G is primitive and $k>2$ by Proposition 0.2. $k\nu_1=l\mu_2$ and $\nu_1\leqq k$ mean $\mu_2=0$ or 1. Similarly $k\lambda_1=l\mu$ and $\lambda_1\leqq k-1$ imply $\mu=0$ or 1. Therefore we have the next possibilities.

μ	μ_2	$\mu_1=k-\mu-\mu_2$	$\mu'=l-1-2\mu_1$	$\lambda'=(l/k)\mu_1=(k-1)\mu_1$	$\mu'-\lambda'$	$l-\lambda'$	
0	0	k	k^2-3k-1	$k(k-1)$	$-(2k+1)$	0	(1)
	1	$k-1$	k^2-3k+1	$(k-1)^2$	$-k$	$k-1$	(2)
1	0						
	1	$k-2$	k^2-3k+3	$(k-1)(k-2)$	1	$2(k-1)$	(3)

Set $d=(\mu'-\lambda')^2+4(l-\lambda')$.

Case (1): In this case $d=(2k+1)^2$ and so, by Lemma 1.2, $\sqrt{d}=2k+1$ divides $2k(k-1)-(2k+1)\{2k+k(k-1)\}=-k\{k(2k+1)$

$+3\}$, which is impossible. Thus case (1) cannot happen.

Case (2): $d=k^2+4(k-1)$. Since $2k(k-1)-k\{2k+k(k-1)\}\neq 0$, d must be a square by Lemma 1.2. Set $d=c^2$ $(c>0)$. Then $4(k-1)=(c-k)(c+k)$. Since $c-k$ and $c+k$ are even or odd simultaneously, $c-k$ is even and we can set $c-k=2e$ where e is a positive integer. Thus $c+k=2(e+k)$ and so $k-1=e(e+k)$, which is a contradiction. Thus case (2) cannot occur.

Case (3): $d=8k-7$. By Lemma 1.2, d must be a square and $\sqrt{d}=\sqrt{8k-7}$ divides $2k(k-1)+1\cdot\{2k+k(k-1)\}=k(3k-1)$. Since $(8k-7, k)$ divides 7 and $(8k-7, 3k-1)$ divides 13, it follows that $8k-7\,|\,7^2\cdot 13^2$. Hence $k=7$, 22 or 1036. In case $k=22$, we may assume that

$$\begin{cases} f_1+f_2=2\cdot 9\cdot 11 \\ \qquad\; f_3=4\cdot 7\cdot 11 \end{cases} \quad\text{or}\quad \begin{cases} f_1+f_2=4\cdot 7\cdot 11 \\ \qquad\; f_3=2\cdot 9\cdot 11. \end{cases}$$

But these are contrary to a theorem of Frame, Theorem 30.1 in [6]. Similarly $k=1036$ is excluded, too. In case $k=7$, by the same reason we can eliminate except the case $f_1=f_2=19$ and $f_3=18$. But this exception is also excluded in the following way. The intersection matrix of Δ is

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 7 & 0 & 1 & 0 \\ 0 & 6 & 5 & 6 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

and so its characteristic polynomial is

$$(1.1) \qquad (x-7)(x^3+2x^2+2x+1).$$

By (4.11) and (4.12) in [3], this polynomial is the minimum polynomial of the incidence matrix A for Δ. Let 7, s, t and u be the characteristic roots of A. Then, since A is similar to

$$\begin{pmatrix} 7 & & & & & & & & & \\ & s & & & & & & & & \\ & & \ddots & & & & & & & \\ & & & s & & & & & & \\ & & & & t & & & & & \\ & & & & & \ddots & & & & \\ & & & & & & t & & & \\ & & & & & & & u & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & u \end{pmatrix},$$

we have

$$7+19s+19t+18u=0.$$

On the other hand, by (1.1) $s+t+u=-2$ and so $u=-31$. But this is not a root of (1.1), which is a contradiction.

Proposition 1.4. *There exists no transitive group satisfying Case I and* $k=l(l-1)/2$.

Proof. Let G be a transitive group with the given condition. As in the first part of the proof of the previous proposition, we have $l=3$, 7 or 57.

(i) $l=57$: Let M_1, M_2 be intersection matrices of Δ, Γ, respectively. Then

$$M_1=\begin{pmatrix} 0 & 0 & 0 & 1 \\ k & \lambda & \mu & \lambda \\ 0 & \nu_1 & \mu_1 & \lambda_1 \\ 0 & \nu_2 & \mu_2 & \lambda \end{pmatrix} \quad M_2=\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & \lambda_1 & \mu_1 & \nu_1 \\ 57 & \lambda' & \mu' & \lambda' \\ 0 & \nu_1 & \mu_1 & \lambda_1 \end{pmatrix}$$

where $k=57\cdot 28$, $\mu_1=28$, $\lambda'=1$, $\mu'=0$. By (4.10) in [3], any two intersection matrices are commutative and so

$$M_1 M_2=M_2 M_1.$$

Hence, considering (3, 4)-entries of the both sides, we have

$$\nu_1^2+28+\lambda_1^2=57+2\lambda.$$

On the other hand, since $\lambda_1+\nu_1=56$ and $2\lambda=57\cdot 28-1-\lambda_1$ by parameters-relations, we get $\lambda_1=24$ and $\lambda=(57\cdot 28-1-24)/2$, which is not an integer. This is a contradiction.

(ii) $l=7$: Let us consider $G_a^{\Gamma(a)}$. By Theorem 1 in P. J. Cameron [1], this is not doubly transitive. Hence, $|G_a^{\Gamma(a)}|=$ (1) 7, (2) $7\cdot 2$ or (3) $7\cdot 3$. However, in case of (1) and (2), by Theorem 18.4 in [6] $|G_a|$ is not divisible by 3, which contradicts $k=7\cdot 3$. In case of (3), $|G_a|=7^x\cdot 3^y$ and $|G|=2h$ where $h=7^x\cdot 3^y\cdot 5^2$. Hence G contains a normal subgroup H of order h. Since G is primitive, H is transitive and so $5^2\cdot 2$ must divide h, which is a contradiction.

(iii) $l=3$: As in (ii) or by [7], this case is excluded, too.

2. Case II

Throughout this section we assume that Case II holds. We have easily

$$A^2=A\cdot {}^tA=kI+\lambda A+\mu B+\nu C,$$
$$AB=BA=\lambda_1 A+\mu_1 B+\nu_1 C.$$

Substituting $C=F-I-A-B$((0.2)) for above equalities, we have

$$A^2=(\lambda-\nu)A+(\mu-\nu)B+(k-\nu)I+\nu F, \tag{2.1}$$

$$AB=(\lambda_1-\nu_1)A+(\mu_1-\nu_1)B-\nu_1 I+\nu_1 F. \tag{2.2}$$

Multiplying (2.1) by A and using $AF=kF$,

$$A^3=(\lambda-\nu)A^2+(\mu-\nu)AB+(k-\nu)A+\nu kF.$$

Substituting (2.1) and (2.2) for above,

$$\begin{aligned} A^3=\{(\lambda-\nu)^2+(\mu-\nu)(\lambda_1-\nu_1)+k-\nu\}A \\ +(\lambda-\nu+\mu_1-\nu_1)(\mu-\nu)B \\ +\{(\lambda-\nu)(k-\nu)-(\mu-\nu)\nu_1\}I \\ +\{(\lambda-\nu)\nu+(\mu-\nu)\nu_1+k\nu\}F. \end{aligned} \tag{2.3}$$

Cancelling $(\mu-\nu)B$ from (2.1) and (2.3), we have

$$\begin{aligned} &\{(\mu-\nu)\nu_1+k\nu-\nu(\mu_1-\nu_1)\}F \\ &=A^3-(\lambda-\nu+\mu_1-\nu_1)A^2 \\ &\quad-\{k-\nu+(\mu-\nu)(\lambda_1-\nu_1)-(\lambda-\nu)(\mu_1-\nu_1)\}A \\ &\quad+\{(k-\nu)(\mu_1-\nu_1)+(\mu-\nu)\nu_1\}I. \end{aligned}$$

This equality and $AF=kF$ conclude

Lemma 2. 1. *The incidence matrix A for Δ satisfies $(A-kI)\cdot g(A)=0$, where*

$$\begin{aligned} g(x)=x^3-(\lambda-\nu+\mu_1-\nu_1)x^2 \\ -\{k-\nu+(\mu-\nu)(\lambda_1-\nu_1)-(\lambda-\nu)(\mu_1-\nu_1)\}x \\ +\{(k-\nu)(\mu_1-\nu_1)+(\mu-\nu)\nu_1\}. \end{aligned}$$

(In the coefficients of $g(x)$, only six parameters appear. From now on s, t and u denote the roots of $g(x)=0$.)

From the above lemma, the minimum polynomial of A is $(x-k)\cdot$(a divisor of $g(x)$). But if this polynomial is $(x-k)\cdot$(a linear divisor of $g(x)$), we may assume that A is similar to

$$\begin{pmatrix} k & & & \\ & s & & \\ & & \ddots & \\ & & & s \end{pmatrix} \qquad \text{(see §29 in [6])}.$$

Taking traces we have $0=k+(n-1)s$, which is a contradiction since s is an algebraic integer. Thus the minimum polynomial of A is $(x-k)\cdot$(a quadratic divisor of $g(x)$) or $(x-k)g(x)$.

Lemma 2. 2. *If the minimum polynomial of A is $(x-k)\cdot$(a quadratic divisor of $g(x)$), then*

$$(\nu-\mu)\{k^2-(\lambda+\mu_1+1)k+(\lambda+1)\mu_1-\lambda_1\mu\}=0.$$

Proof. We may assume that the minimum polynomial of A is $(x-k)(x-s)(x-t)$ and so

$$A^3-(k+s+t)A^2+(ks+st+tk)A-kstI=0.$$

Substituting (2. 1) and (2. 3) for the above, we have

$$\begin{aligned} (2.4) \qquad & \{(\lambda-\nu)(\lambda-\nu-k-s-t)+(\mu-\nu)(\lambda_1-\nu_1) \\ & \qquad\qquad +k-\nu+ks+st+tk\}A \\ & +(\mu-\nu)(\lambda-\nu+\mu_1-\nu_1-k-s-t)B \\ & +\{(k-\nu)(\lambda-\nu-k-s-t)-(\mu-\nu)\nu_1-kst\}I \end{aligned}$$

$$+\{(\lambda-\nu-\nu_1-s-t)\nu+\mu\nu_1\}F=0.$$

$(1,1)$ entry of the above is

$$(k-\nu)(\lambda-\nu-k-s-t)-(\mu-\nu)\nu_1$$
$$-kst+(\lambda-\nu-\nu_1-s-t)\nu+\mu\nu_1=0$$

and so

$$(2.5)\qquad st=\lambda-k-s-t.$$

Now let A have (i,j) entry 1 and let B have (i',j') entry 1. Then the (i,j) and (i',j') entries of (2.4) are

$$(\lambda-\nu)(\lambda-\nu-k-s-t)+(\mu-\nu)(\lambda_1-\nu_1)+k-\nu+ks+st+tk$$
$$+(\lambda-\nu-\nu_1-s-t)\nu+\mu\nu_1=0,$$

and

$$(\mu-\nu)(\lambda-\nu+\mu_1-\nu_1-k-s-t)+(\lambda-\nu-\nu_1-s-t)\nu+\mu\nu_1=0,$$

that is (using (2.5)),

$$(s+t)(k-1-\lambda)=\lambda(k+\nu-\lambda)+(\nu_1-k+1)\nu-\lambda_1\mu+\lambda_1\nu-\nu\nu_1-\lambda$$

and

$$(s+t)\mu=(\lambda-\nu+\mu_1-k)\mu-(\mu_1-k)\nu.$$

Cancelling $s+t$ from the above two, we get a desired result.

Lemma 2.3. *If the minimum polynomial of A is $(x-k)g(x)$, then we have followings.*

(i) $f_1+f_2+f_3=k+l+m$

$sf_1+tf_2+uf_3=-k$

$s^2f_1+t^2f_2+u^2f_3=k(l+m+1)$.

(ii) *$k=l=m$ or at least one of s, t and u is an integer.*

(iii) *If f_1, f_2 and f_3 are all different, then s, t and u are all integers.*

Proof. (i) The first equality is obtained at once from (0.1). Since we may assume that A is similar to

$$\begin{pmatrix} k & & & \\ & \overbrace{\begin{matrix} s & & \\ & \ddots & \\ & & s \end{matrix}}^{f_1} & & \\ & & \overbrace{\begin{matrix} t & & \\ & \ddots & \\ & & t \end{matrix}}^{f_2} & \\ & & & \overbrace{\begin{matrix} u & & \\ & \ddots & \\ & & u \end{matrix}}^{f_3} \end{pmatrix} \qquad \text{(see §29 in [6])},$$

taking traces we get the second equality.

$$A^2 = kI + \lambda A + \mu B + \nu C = \begin{pmatrix} k & & * \\ & k & \\ * & & \ddots & \\ & & & k \end{pmatrix} \text{ is similar to}$$

$$\begin{pmatrix} k^2 & & & \\ & \overbrace{\begin{matrix} s^2 & & \\ & \ddots & \\ & & s^2 \end{matrix}}^{f_1} & & \\ & & \overbrace{\begin{matrix} t^2 & & \\ & \ddots & \\ & & t^2 \end{matrix}}^{f_2} & \\ & & & \overbrace{\begin{matrix} u^2 & & \\ & \ddots & \\ & & u^2 \end{matrix}}^{f_3} \end{pmatrix}$$

and so, taking traces we have the third equality.

(iii) From the proof of (C) of Theorem 30.1 in Wielandt [6], it follows that all the eigenvalues of A are integers.

(ii) Suppose $g(x)$ is irreducible over the rational field.

Then, by (iii), some two of f_1, f_2 and f_3 are equal (say $f_1 = f_2$) and so we have by (i)

$$sf_1 + tf_1 + uf_3 = -k.$$

On the other hand, by Lemma 2.1

$$sf_1 + tf_1 + uf_1 = (\lambda - \nu + \mu_1 - \nu_1) f_1 .$$

Hence $u(f_1 - f_3)$ is a rational number and so $f_1 = f_3$ since u is irrational. Thus $f_1 = f_2 = f_3$, which implies $k = l = m$ by Theorem 30.2 in Wielandt [6]. This completes the proof.

A similar argument as in Lemma 1. 3 shows

Lemma 2. 4. *G is primitive on Ω if and only if* $\lambda\neq k-1$, $\mu\neq k$, $\nu\neq k$; $\lambda'\neq l$, $\mu'\neq l-1$, $\nu'\neq l$; $\lambda''\neq m$, $\mu''\neq m$, $\nu''\neq m-1$.

3. On primitive permutation groups of rank 4 with given subdegrees

Continuing propositions of section 1, in this section we consider the next problem: When we are given k, l and m, determine primitive permutation groups of rank 4 which have 1 and such k, l, m as subdegrees. In dealing with this problem, our procedure is: First, making use of parameters-relations and Lemma 2. 4 we determine six parameters appeared in the coefficients of $g(x)$ (see Lemma 2. 1) as exactly as possible. Next, in view of Lemma 2. 3, we examine an integral root of $g(x)=0$ and compute the values of f_1, f_2, f_3. Only the cases that such values are integers are remained. Computations are quite elementary, but routine and tedious.

Here we remark

Proposition 3. 1. *Let* 1, k, l *and* m ($k\leqq l\leqq m$) *be the subdegrees of a primitive permutation group of rank* 4, *then* $l\leqq k(k-1)$ *and* $m\leqq k(k-1)^2$.

In fact, if $\Delta(a)$ is self-paired, the conclusion is immediate from Proposition 0. 1. If $k=1$, then $l=m=1$ since $l\leqq k^2$, $m\leqq kl$, and so G is of order 4 and not primitive. Thus $k\neq 1$. If $k=2$, then by Proposition 0. 2 $l=m=2$, all the G_a-orbits are self-paired. If $\Delta(a)$ and $\Gamma(a)$ are paired, then $k=l$ and so the conclusion is at once since $m\leqq kl$ and $k\neq 1$, 2. If $\Delta(a)$ and $\Lambda(a)$ are paired, then $k=l=m$.

Lemma 3. 1. *Let G be a transitive group of rank* 4 *on* Ω *with subdegrees* 1, k(*arbitrary*), $l=k(k-1)$ *and* $m=k(k-1)^2$. *Then* $k=2$, $|\Omega|=7$ *and G is a dihedral group of order* 14.

Proof. From the values of subdegrees G is primitive on Ω. By Proposition 0.2, it suffices to show that $k=2$. In the following, suppose $k>2$. Then we have $k<l<m$ and so all the G_a-orbits are self-paired (i.e., Case II holds). We shall determine the values of six parameters λ, μ, ν, λ_1, μ_1 and ν_1. Parameters-relations $k\lambda_2=m\nu$ and $\lambda_2\leqq k-1$ imply $\nu=0$, $\lambda_2=0$. Similarly, from $k\lambda_1=l\mu=k(k-1)\mu$ and $\lambda+\lambda_1+\lambda_2=k-1$, it follows that (1) $\lambda=0$, $\mu=1$, $\lambda_1=k-1$ or (2) $\mu=0$, $\lambda_1=0$. But, in case (2) $\lambda+\lambda_1+\lambda_2=k-1$ implies $\lambda=k-1$, which contradicts the primitivity of G by Lemma 2.4. Thus case (1) must hold. From $m\nu_1=k\lambda_3$, we have $(k-1)^2\nu_1=\lambda_3=l-\lambda_1-\lambda'\leqq l-\lambda_1=(k-1)^2$ and so $\nu_1=0$ or 1.

$$\nu_1=0 \xrightarrow[(m\nu_1=k\lambda_3)]{} \lambda_3=0 \xrightarrow[(\lambda_2+\lambda_3+\lambda''=m)]{} \lambda''=m:$$

contradicts Lemma 2.4.

$$\nu_1=1 \xrightarrow[((k-1)^2\nu_1=\lambda_3)]{} \lambda_3=(k-1)^2 \xrightarrow[(k\lambda_3=l\mu_2)]{} \mu_2$$
$$=k-1 \xrightarrow[(\mu+\mu_1+\mu_2=k)]{} \mu_1=0.$$

Thus we have

$$\lambda=0,\ \mu=1,\ \nu=0,\ \lambda_1=k-1,\ \mu_1=0 \text{ and } \nu_1=1.$$

Therefore, by Lemma 2.2, the minimum polynomial of A is $(x-k)g(x)$ where $g(x)=x^3+x^2-2(k-1)x-(k-1)$. By Lemma 2.3. (ii) $g(x)=0$ has an integral root s. Hence

$$\frac{s^2(s+1)}{2s+1}=k-1.$$

Since $(s, 2s+1)=1$ and $(s+1, 2s+1)=1$, we have $2s+1=\pm 1$, i.e., $k=1$, which is a contradiction. Thus Lemma 3.1 is proved.

As a corollary we have a next result, which corresponds to Theorem 1 in Higman [2].

Theorem. *Let G be a primitive permutation group of rank* 4 *on Ω with subdegrees* 1, *k, l and m where k and l are arbitrary and $m=k(k-1)^2$. Then $k=2$, $|\Omega|=7$ and G is a dihedral group of order* 14.

Proof. It suffices to show that $k=2$. If $(k\leqq)m\leqq l$, then $k=2$ by Proposition 3.1. Similarly, if $l\leqq k(\leqq m)$, we have $k=l$. Thus we may assume that $k\leqq l\leqq m$. If all the G_a-orbits are not self-paired, by Propositions 1.1 and 3.1 we have $k=2$, which contradicts Proposition 0.2. That is, all the G_a-orbits are self-paired. Hence, by Proposition 0.1 l must be equal to $k(k-1)$ and by the previous lemma we have a desired conclusion.

Similar arguments as in Lemma 3.1 yield following propositions. (These are necessary for determining the primitive extension of rank 4 of alternating groups which act naturally).

Proposition 3.2. *There exists no primitive permutation group of rank* 4 *with subdegrees* 1, k, l, m *such that*

(i) $l=k(k-1)$, $m=k(k-1)(k-2)$

(ii) $l=k(k-1)/2$, $m=k(k-1)(k-2)$

(iii) $l=k(k-1)/2$, $m=k(k-1)(k-2)/2$

(iv) $l=k(k-1)$, $m=k(k-1)(k-2)/3$

where in all the cases k is arbitrary.

Proof. By Wong [7], $k\neq 3$.

(i) and (ii) Omitted.

(iii) Suppose that there exists a group G satisfying condition (iii). If $k=4$, then the degree of G is equal to 23. Hence, by Therorems 11.6 and 11.7 in [6] G is a Frobenius group and the order h of Frobenius complement is a divisor of $23-1$, while h must be a multiple of 4. This is a contradiction. Thus $k\geqq 5$ and in our usual way we have the next possibilities (of course, all the G_a-orbits are self-paired since $k<l<m$).

λ	μ	ν	λ_1	μ_1	ν_1	
$(k-1)/2$	1	0	$(k-1)/2$	1	1	(1)
0	2		$k-1$	0		(2)

By Lemmas 2.1 and 2.2 the minimum polynomial of A is $(x-k)g(x)$ where

$$g(x)=\begin{cases} x^3-\dfrac{k-1}{2}x^2-\dfrac{3(k-1)}{2}x+1 & \text{(case (1))} \\ x^3+x^2-(3k-4)x-k+2 & \text{(case (2))} \end{cases}$$

By Lemma 2.3. (ii) in both cases $g(x)=0$ has at least one integral root s and so case (1) cannot happen. In case (2) the equality

$$9k=3s^2+2s+11+\frac{s+7}{3s+1}$$

holds and $(s+7, 3s+1)$ divides $2^2\cdot 5$ and so $3s+1$ divides $2^2\cdot 5$. Thus we have $s=3$, $k=5$ or $s=-7$, $k=16$. But, by Lemma 2.3. (i) these cannot occur either.

(iv) Suppose that there exists a group G satisfying condition (iv). If $k=4$, then $\lambda=1$, $\mu=0$, $\nu=1$, $\lambda_1=0$, $\mu_1=2$, $\nu_1=3$ or $\lambda=0$, $\mu=1$, $\nu=0$, $\lambda_1=3$, $\mu_1=1$, $\nu_1=3$ and in our usual way we have a contradiction. If $k=5$, then the degree of G is $2\cdot 23$ and this is contrary to Theorem 31.2 in [6]. Thus we have $k\geqq 6$ and as before there are the following possibilities (of course, Case II holds).

λ	μ	ν	λ_1	μ_1	ν_1	
0	1	0	$k-1$	$(2k-1)/3$	1	(1)
				$(k+1)/3$	2	(2)
				1	3	(3)

The minimum polynomial of A is $(x-k)g(x)$ where

$$g(x)=\begin{cases} x^3-\dfrac{2(k-2)}{3}x^2-2(k-1)x+\dfrac{2k(k-2)}{3}+1 & \text{(case (1))} \\ x^3-\dfrac{k-5}{3}x^2-(2k-3)x+\dfrac{k(k-5)}{3}+2 & \text{(case (2))} \\ (x+1)(x^2+x-2k+3) & \text{(case (3))} \end{cases}$$

In all the cases $g(x)=0$ has an integral root s by Lemma 2.3. (ii).

Case (3): Set $s=-1$, $t=(-1+\sqrt{8k-11})/2$ and $u=(-1-\sqrt{8k-11})/2$. Then by Lemma 2.3. (i) we have $f_2+f_3=$

$k^2(k-1)(k+1)/3(2k-3)$, which is an integer. This and $k\geqq 6$ imply $k=9$, which is contrary to Lemma 2.3 (i).

Case (2): Since

$$3g(s)=k^2-(s^2+6s+5)k+3s^3+5s^2+9s+6=0,$$

it follows that

$$(s^2+6s+5)^2-4(3s^3+5s^2+9s+6)=(s^2+1)^2+24s(s+1)$$

is a square (say d^2, $d\geqq 0$). Moreover, since s is neither 0 nor -1, we have

$$0<24s(s+1)=(d-(s^2+1))(d+s^2+1)$$

and so we can set $2c=d-(s^2+1)$ where c is a positive integer. Hence

$$(6-c)s^2+6s-(c^2+c)=0.$$

If $c\neq 6$, then $3^2+(6-c)(c^2+c)$ must be a square and so $c=4$, $s=2$ or -5, $k=17$. If $c=6$, then $s=7$ and $k=79$ or 17. But, since $\mu_1=(k+1)/3$ must be an integer, $k\neq 79$. Thus we have $k=17$ at any rate. Hence $g(x)=(x-2)(x-7)(x+5)$ and put $s=2$, $t=7$, $u=-5$, getting $f_3=17\cdot 250/7$ by Lemma 2.3 (i). But this is a contradiction since f_3 must be an integer.

Case (1): Follow case (2).

Thus Proposition 3.2 is established.

In the same way as above, we have following propositions.

Proposition 3.3. *If there exists a primitive permutation group of rank* 4 *with subdegrees* 1, k, l, m *such that*

(i) $l=k(k-1)$, $m=k(k-1)(k-2)/2$

or

(ii) $l=k(k-1)/2$, $m=k(k-1)(k-2)/3$.

then $k=5$.

Remark. For $k=5$, we have the following.

	l	m	n	λ	μ	ν	λ_1	μ_1	ν_1	s	t	u	f_1	f_2	f_3	$\frac{n^2\cdot 5\cdot l\cdot m}{f_1\cdot f_2\cdot f_3}$
(i)	20	30	56	0	1	0	4	1	2	-3	$1-\sqrt{2}$	$1+\sqrt{2}$	15	20	20	$2^5\cdot 7^2$
(ii)	10	20	36	0	0	1	0	1	2	-3	-1	2	9	10	16	30^2

However the author doesn't know if a group of type (i) for $k=5$ exists. On the other hand, Mr. E. Bannai and Mr. H. Enomoto*) have kindly informed that the automorphism group of the symmetric group of degree 6 operating by conjugation on the set of the Sylow 5-subgroups gives an example of type (ii) for $k=5$.

Proposition 3. 4. *Let G be a primitive permutation group of rank 4 with subdegrees 1, k (arbitrary), $l=k(k-1)$, $m=\binom{k}{3}=k(k-1)(k-2)/6$. Then $k=5$ or 6 and in fact such G exist.*

Remark. The case $k=5$ is quite the same as type (ii) for $k=5$ above. As such G for $k=6$, there exists $PSL(2, 19)$ operating by right multiplication on the cosets of a subgroup isomorphic to the alternating group of degree 5. In the latter case, the values of f_1, f_2, f_3 are 18, 18, 20 and this group gives a counterexample to Frame's conjecture (B) on p. 89 of [6] since

$$57^2\cdot\frac{6\cdot 30\cdot 20}{18\cdot 18\cdot 20}=5\cdot 19^2$$

is not a square.**)

Proposition 3. 5. (*cf.* Prop. 1. 3) *Let G be a transitive group of rank 4 with subdegrees 1, k (arbitrary), $l=k(k-1)$, $m=k$ and suppose that all the G_a-orbits are self-paired. Then $k=2$ and G is a dihedral group of order 14.*

Proposition 3. 6. *There exists no primitive permutation group G of rank 4 such that G_a acts doubly transitive on $\Delta(a)$, all the G_a-orbits are self-paired and the subdegrees are 1, $|\Delta(a)|=k$*

*) The author wishes to thank both of them.

**) Professor N. Ito has kindly informed the author that this had already been known in P. M. Neuman: Primitive permutation groups of degree $3p$

(arbitrary), $l=k(k-1)/2$, $m=k$.

Proposition 3.7. *Let G be a primitive permutation group of rank 4 such that G_a acts doubly transitive on $\Delta(a)$ and the subdegrees are* 1, $|\Delta(a)|=k$, $l=k(k-1)/2=\binom{k}{2}$, $m=k(k-1)(k-2)/6=\binom{k}{3}$. *Then $k=7$ and in fact such G exists.*

Remark. As such G for $k=7$, we have a primitive rank 4 extension of the symmetric or alternating group of degree 7 with a regular normal subgroup. It will be seen in a subsequent paper, which deals with primitive extensions of rank 4 of alternating groups.

In the proofs of the last two propositions (in Prop. 3.7, for $k\geqq 6$ and $k\neq 8$), from the 2-transitivity of $G_a^{\Delta(a)}$ we may assume that $\lambda=0$, $\lambda_1=k-1$ and $\lambda_2=0$ (see the proof of Theorem 1 in Cameron [1]). Probably, however, the assumption of 2-transitivity of $G_a^{\Delta(a)}$ in Proposition 3.7 may be omitted.

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Added in proof: According to our usual method, in Prop. 3.7, the case $k=23$ also remains besides the case $k=7$. This careless mistake was pointed out by Mr. H. Enomoto, and he has informed the author that the case $k=23$ cannot occur. His method is graph-theoretical. Moreover, he has pointed out that the assumption of 2-transitivity of $G_a^{\Delta(a)}$ is omitted. Here the author wishes to thank Mr. H. Enomoto.