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A connection between the Nevanlinna characteristic and behavior on a sequence of boundary arcs for meromorphic functions in the disk

By

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§1. Introduction

One of the achievements of the Nevanlinna theory for meromorphic functions was the recognition that the characteristic function, $T(r, f)$, is the proper generalization for meromorphic functions of $\log M(r, f)$, where $M(r, f)$ is the maximum modulus function for a holomorphic f.

Recently [7] we proved a series of theorems of a "Koebe" type, connecting the growth of the maximum modulus and the order with which a holomorphic f tends to 0 on a sequence of boundary arcs in the unit disk. For convenience we refer to this paper as $\mathscr{R}$.

The purpose of this paper is to show that the characteristic function $T(r, f)$ can be substituted for $\log M(r, f)$ in the theorems of $\mathscr{R}$ thereby effecting not only a generalization but also an extension to meromorphic f. In one corollary dealing with normal functions we also improve the corresponding result in $\mathscr{R}$.

It is difficult to decide whether to duplicate much of $\mathscr{R}$ or to refer the reader to this paper. The second saves space while the first saves the reader's sanity. On the grounds that reader's equilibrium is more important than brevity we shall try to make the paper as self contained as is reasonable. We will keep the notation as close as possible to that in $\mathscr{R}$ and clearly indicate where duplication occurs. In one or two instances we do not reproduce lengthy calculations whose absence will not be missed.

§2. Preliminaries

Our basic concern is with sequences of Jordan arcs in the unit disk D: $|z|<1$, which tend to the boundary C: $|z|=1$. Let $\{\gamma_n\}$ be such a sequence of Jordan arcs with each γ_n entirely contained in D. Define a four tuple of real numbers $(R_n, r_n, \theta_n, \alpha_n)$, called the parameters associated with γ_n, as follows:

$$R_n=\max|z|, \quad z\in\gamma_n;$$
(2.0)
$$r_n=\min|z|, \quad z\in\gamma_n.$$

If E_n is the closed circular sector of $|z|\leq R_n$ of minimum angle opening α_n containing γ_n, then E_n is of the form

$$0\leq|z|\leq R_n,\ \theta_n\leq\arg z\leq\theta_n+\alpha_n,\ 0\leq\theta_n<2\pi, \tag{2.1}$$

which defines θ_n and α_n. For convenience we assume that always $0\leq\alpha_n\leq\pi$ and that $\{\gamma_n\}$ is a sequence of boundary arcs in that

$$\frac{1}{2}\leq r_n\to 1, \quad n\to\infty. \tag{2.2}$$

For any given sequence $\{\gamma_n\}$ with associated parameters $\{(R_n, r_n, \theta_n, \alpha_n)\}$ define

$$F_n^{(\alpha)}:\ 0<|z|<R_n;\ \theta_n-\left(\frac{\alpha-\alpha_n}{2}\right)<\arg z<\theta_n+\left(\frac{\alpha+\alpha_n}{2}\right). \tag{2.3}$$

So $F_n^{(\alpha)}$ is the circular sector of $|z|<R_n$ of opening α which contains the interior of E_n in a symmetric fashion. Finally put

$$L_n^{(\alpha)}:\ \frac{1}{4}r_n<|z|<\frac{1}{2}r_n;\ \theta_n-\left(\frac{\alpha}{4}-\frac{\alpha_n}{2}\right)<\arg z<\theta_n+\left(\frac{\alpha}{4}+\frac{\alpha_n}{2}\right). \tag{2.4}$$

Then $L_n^{(\alpha)}$ is a wedge-shaped domain of opening $\frac{\alpha}{2}$ symmetric about the line bisecting the angle α_n. See Fig. 1 for the various domains.

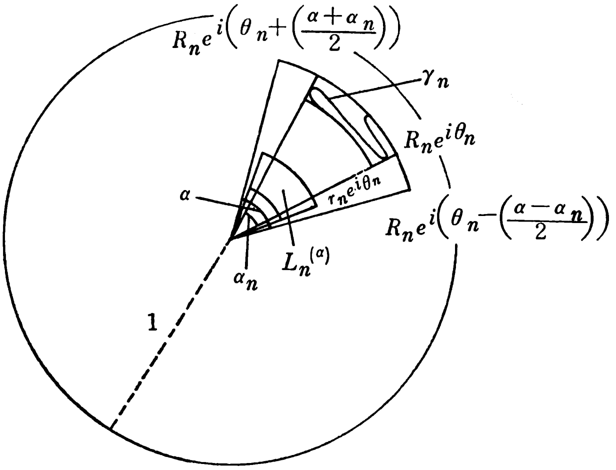

Fig. 1

We will use the notation E_n, $F_n^{(\alpha)}$ and $L_n^{(\alpha)}$ exclusively in section 4 as defined here and always relative to a given sequence $\{\gamma_n\}$. It is trivial but important to note that because $r_n<1$, $L_n^{(\alpha)}$ is always contained within the disk $|z|<\frac{1}{2}$, regardless of the sequence to which it is associated and regardless of our choice of α.

If $\underline{\lim}\ \alpha_n<0$, then $\{\gamma_n\}$ is a Koebe sequence. We do not consider such sequences but will treat them in a subsequent paper.

We shall be interseted in sequences $\{\gamma_n\}$ for which $\alpha_n\to 0$, $n\to\infty$, and which satisfy also a non-Euclidean hyperbolic distance condition.

For a,b $\in$ D put

$$\rho(a, b)=\frac{1}{2}\log\frac{|1-a\bar{b}|+|a-b|}{|1-a\bar{b}|-|a-b|};$$

and for a set $S\subseteq D$ let

$$\mathrm{HD}(S)=\sup\{\rho(a, b)\},\ a,\ b\in S,$$

which is the hyperbolic diameter of S.

Definition I. *We call a sequence of Jordan arcs $\{\gamma_n\}$ in D a positive hyperbolic diameter sequence, hereafter a* PHD *sequence, if*

$$0<\varliminf_{n\to\infty} \mathrm{HD}(\gamma_n)\leq\varlimsup_{n\to\infty} \mathrm{HD}(\gamma_n)<\infty. \tag{2.5}$$

It is easy to show that for a PHD sequence $\alpha_n\to 0$, $n\to\infty$, although this is not sufficient. In fact, a PHD sequence can be characterized by the behavior of its sequence of associated parameters. To this end let us call a sequence of Jordan arcs $\{\gamma_n\}$ in D with associated parameters $\{(R_n, r_n, \theta_n, \alpha_n)\}$ a radial-like sequence if

(2.6)

i) $0<\varliminf_{n\to\infty} \rho(R_n, r_n)\leq\varlimsup_{n\to\infty} \rho(R_n, r_n)<\infty$;

ii) $\varlimsup_{n\to\infty} \rho(R_n e^{i\theta_n}, R_n e^{i(\theta_n+\alpha_n)})<\infty$;

or an arc-like sequence if

(2.7)

i) $\lim_{n\to\infty} \rho(R_n, r_n)=0$

ii) $0<\varliminf_{n\to\infty} \rho(R_n e^{i\theta_n}, R_n e^{i(\theta_n+\alpha_n)})\leq\varlimsup_{n\to\infty} \rho(R_n e^{i\theta_n}, R_n e^{i(\theta_n+\alpha_n)})<\infty$.

(If γ_n is the segment of the radius $re^{i\theta_0}$ defined by $1-\frac{2}{n}\leq r\leq 1-\frac{1}{n}$ then $\{\gamma_n\}$ is a radial-like sequence; while if $\{\gamma_n\}$ is the arc of $|z|=1-\frac{1}{n}$ defined by $\theta_o\leq \arg z\leq\theta_o+\frac{1}{n}$ an easy calculation shows that this $\{\gamma_n\}$ is an arc-like sequence — hence the nomenclature for each family.)

With this terminology we state Proposition 1 or $\mathscr{R}$ without its proof which is an elementary case analysis.

Proposition I. *A sequence of Jordan arcs $\{\gamma_n\}$ in D is a* PHD *sequence if and only if each subsequence contains either a radial-like subsequence or an arc-like subsequence (or both).*

For a function f defined in D and taking values in the extended plane W we want to define how fast f tends to zero on a sequence of Jordan arcs.

Definition 2. *Let f be defined in D taking values in the extended plane W. Let $\{\gamma_n\}$ be a sequence of Jordan arcs in D, $\{A_n\}$ a sequence of positive numbers, and $s \geq 0$. We say f has s-exponential order $\{A_n\}$ on $\{\gamma_n\}$ if*

$$|f(z)| \leq \exp\frac{-A_n}{(1-|z|)^s}, \qquad z \in \gamma_n,\ n=1, 2, \ldots .$$

As mentioned in the introduction we want to compare the order of a meromorphic f on a sequence of arcs $\{\gamma_n\}$ with the growth of the Nevanlinna characteristic function of f over various subdomains in D. If $F \subseteq D$ is a domain for which the characteristic function exists for a meromorphic f relative to a point $z_0 \in F$ we denote this function by $T(z_0, F, f)$. Certainly if F is a subdomain of D bounded by finitely many analytic Jordan arcs (i.e. a regular subdomain) the characteristic function exists. We shall have occasion to consider the characteristic function over slightly more general subdomains but we will treat the problem of existence at that time. If F is the subdomain $|z|<r$, $0<r<1$, and $z_0=0$, we use the customary notation $T(r, f)$.

It is easy to prove, but we assume without proof, the monotonicity of $T(z_0, F, f)$ in that if F and G are subdomains with $F \subset G \subset D$ and $z_0 \in F$ (and the characteristic function exists for each domain) then

$$T(z_0, F, f) \leq T(z_0, G, f). \tag{2.8}$$

We refer the reader to the book by Sario and Noshiro [4] as a reference for a general treatment of value distribution theory, and of course to Nevanlinna [3] for the more classical approach.

We conclude with one further definition followed by a comment.

Definition 3. *A simple continuous curve $\gamma=\gamma(t)$, $0 \leq t < 1$, lying in D is said to be a boundary path if $\lim_{t \to 1} |\gamma(t)| = 1$; and a boundary path at $\tau \in C$ if $\lim_{t \to 1} \gamma(t) = \tau$.*

One further convention we adopt. Most of the arguments used in this work involve a limiting process, and we are not interested in the first N_0 terms. Rather than keeping a score of the various indices we sometimes use the phrase relative to some sequence "such and such a property holds eventually for the sequence" to replace "there is an integer N_0 such that the property is true for all members of the sequence with index greater than N_0." As long as we use this phrase only finitely often and are otherwise reasonably careful no problem arises.

§3. A form of the Schmidt-Milloux Theorem

We shall need a form of the Schmidt-Milloux theorem as one of the cornerstones for our main result. If γ is a boundary path at $\tau\in C$ we define $\omega(z, \gamma, D-\gamma)$ to be the harmonic measure at z of γ relative to $D-\gamma$.

Theorem A. *Let γ be a boundary path at a point $\tau\in C$. If $\min_{z\in\gamma}|z|=a$ then for $z\in D-\gamma$,*

$$\omega(z, \gamma, D-\gamma)\geq\frac{2}{\pi}\arcsin\frac{(1-a^2)\ (1-|z|^2)}{16} \tag{3.0}$$

This formulation is obtained from the usual form in which $a=0$ by the routine device of mapping D onto D by a linear transformation which takes γ onto a boundary path with $a=0$ and using the conformal invariance of the harmonic measure. Some obvious estimates then produce the above inequality. For a nice statement of the inequality in the usual form see Tsuji [12, Theorem VIII. 11]

We need also the following result from the folklore. If $\{z_n\}$ and $\{z'_n\}$ are two sequences in D such that

1) $\lim_{n\to\infty}|z_n|=\lim_{n\to\infty}|z'_n|=1$;

2) $\overline{\lim_{n\to\infty}}\,\rho(z_n, z'_n)<\infty$;

then

(3.1) $$0<\varliminf_{n\to\infty}\frac{1-|z_n|}{1-|z'_n|}\leq\varlimsup_{n\to\infty}\frac{1-|z_n|}{1-|z'_n|}<\infty.$$

If in addition $\varliminf_{n\to\infty}\rho(z_n, z'_n)>0$, so also

(3.2) $$0<\varliminf_{n\to\infty}\frac{|z_n-z'_n|}{1-|z_n|}\leq\varlimsup_{n\to\infty}\frac{|z_n-z'_n|}{1-|z_n|}<\infty.$$

Because of the obvious symmetry in this result, under suitable hypothesis we may substitute one of the expressions for another in any order argument without hesitation. For a proof see among others Rung [6, pg. 45].

§4. Main Theorem

Suppose we are given a non-constant meromorphic function in D which tends to some value, a, on a PHD sequence of arcs. Is there a connection between the possible growth of the characteristic function and the order with which the function tends to a on the PHD sequence? This question was answered affirmatively in $\mathscr{R}$ except that the characteristic function was replaced by the less general $\log \mathrm{M}(r)$. Theorem 1 below gives an affirmative answer for the characteristic function.

Our situation is this. We have a meromorphic f defined in D which tends to a value a, possibly infinite, on a PHD sequence $\{\gamma_n\}$, under the order condition that $f-a$ (or $\frac{1}{f}$ if $a=\infty$) has 1-exponential order $\{A_n\}$ on $\{\gamma_n\}$. We now proscribe a sequence of domains $F_n^{(\alpha)}$ such that $\bar{F}_n^{(\alpha)}\supseteq E_n$, where E_n is the minimum wedge containing γ_n. [See (2.1), (2.3) and (2.4).] Succinctly, we enclose each γ_n symmetrically in a pie shaped $F_n^{(\alpha)}$ which has a fixed angle opening α. Our aim is to estimate $\log|f(z)-a|$ for z in the subdomain $L_n^{(\alpha)}$ in terms of the quantities A_n and $\mathrm{T}(z_0, F_n^{(\alpha)}, f)$. The rather obvious route is via the Poisson-Jensen formula.

The next several section will be devoted to some technical housework in order to form the proper Poisson-Jensen representation needed for Theorem 1.

Let S be a subdomain of the extended plane W sufficiently well-behaved to support a Green's function $G(z, a, S)$ with pole at $a \in S$. It is well known that the Poisson-Jensen formula is valid for each of the domains $F_n^{(\alpha)}$. The explicit formula is stated for example in Petrendo [5].

Let $\{a_i\}_{i=1}^k$ and $\{b_i\}_{i=1}^l$ be the zeroes and poles of f in $F_n^{(\alpha)}$ with each point repeated as often as its multiplicity. Let $z \in F_n^{(\alpha)}$ be chosen (assuming z is not a zero or pole of f) and parameterizing the boundary of $F_n^{(\alpha)}$ by $\zeta = \zeta(t)$, $0 \leq t \leq 1$, we have

$$\begin{aligned}\log|f(z)| = &\left[\frac{1}{2\pi}\int_0^1 \log^+|f(\zeta(t))|d\omega(z, \zeta(t), F_n^{(\alpha)}) + \sum_{i=1}^{l} G(z, b_i, F_n^{(\alpha)})\right] \\ &-\left[\frac{1}{2\pi}\int_0^1 \log^+\frac{1}{|f(\zeta(t))|}d\omega(z, \zeta(t), F_n^{(\alpha)}) + \sum_{i=1}^{k} G(z, a_i, F_n^{(\alpha)})\right] \\ = &\ \mathrm{T}(z, F_n^{(\alpha)}, f) - \mathrm{T}\left(z, F_n^{(\alpha)}, \frac{1}{f}\right).\end{aligned} \tag{4.0}$$

If z is a pole of order λ then the corresponding Laurent coefficient c_λ replaces $f(z)$ in the left side of (4.0) and the right side contains the term $\frac{1}{2\pi}\int_0^1 \log|\zeta(t) - z| d\omega(z, \zeta(t), F_n^{(\alpha)})$ instead of the Green's function with pole at z. Since z will always lie in $L_n^{(\alpha)}$ this is a bounded term. Our arguments will apply equally well if z is a zero or pole so for simplicity of notation we assume that z is neither a zero nor a pole.

We are interested in applying the Poisson-Jensen formula to the domains $F_n^{(\alpha)} - \gamma_n$. Some difficulties arise because γ_n is assumed to be only a continuous path. There are several ways around this difficulty; we adopt the simplest sufficient for our needs. We replace $F_n^{(\alpha)} - \gamma_n$ by a new domain of similar shape but whose boundary is free of zeroes and poles of f by forming first a subdomain $H_n^{(\alpha)}$ of $F_n^{(\alpha)}$, whose radial sides are parallel to those of $F_n^{(\alpha)}$; whose circular side is concentric with $|z|=R_n$, and whose sides are at a distance ε_n from the corresponding side of $F_n^{(\alpha)}$, with $\varepsilon_n \to 0$, $n \to \infty$.

Because of this condition we can assume that $L_n^{(\alpha)} \subseteq H_n^{(\alpha)}$ and by our choice of the sequence $\{\varepsilon_n\}$ we can also require that the sequence $\{\gamma_n^*\}$, where γ_n^* is that part of γ_n in $H_n^{(\alpha)}$, is yet again a PHD sequence.

We now alter γ_n^* slightly by replacing it with an arc γ_n', (if necessary) so that

1) $\{\gamma_n'\}$ is still a PHD sequence;
2) f is free of zeroes and poles on each γ_n';
3) each γ_n' meets $H_n^{(\alpha)}$ only at one endpoint and otherwise lies entirely in $H_n^{(\alpha)}$;
4) on $\{\gamma_n'\}$, f has 1-exponential order $\left\{\frac{A_n}{2}\right\}$.

This is all possible by elementary methods.

The Poisson-Jensen formula is now applicable to $H_n^{(\alpha)} - \gamma_n'$. If we continuously parameterize the boundary of this domain (remembering that γ_n' has two sides) by $\zeta = \zeta(t)$, $0 \le t \le 1$; let $(a_i\}_{i=1}^a$, $\{b_i\}_{i=1}^b$, be the zeroes and poles of f in $H_n^{(\alpha)} - \gamma_n'$; and choosing any $z \in H_n^{(\alpha)} - \gamma_n'$ not a zero or pole, we have

$$\begin{aligned}\log|f(z)| = &\left[\frac{1}{2\pi}\int_0^1 \log^+ |f(\zeta(t))|\, d\omega(z, \zeta(t), H_n^{(\alpha)} - \gamma_n')\right. \\ &\qquad \left. + \sum_{i=1}^b G(z, b_i, H_n^{(\alpha_n)} - \gamma_n')\right] \\ &- \left[\frac{1}{2\pi}\int_0^1 \log^+ \left|\frac{1}{f(\zeta(t))}\right| d\omega(z, \zeta(t), H_n^{(\alpha)} - \gamma_n')\right. \\ &\qquad \left. + \sum_{i=1}^a G(z, a_i, H_n^{(\alpha)} - \gamma_n')\right] \\ = &\ \mathrm{T}(z, H_n^{(\alpha)} - \gamma_n', f) + \mathrm{T}\left(z, H_n^{(\alpha)} - \gamma_n', \frac{1}{f}\right).\end{aligned} \tag{4.1}$$

Utilizing the subharmonic properties of the quantities above we have for $z \in H_n^{(\alpha)} - \gamma_n'$, the two important inequalities

$$\mathrm{T}(z, H_n^{(\alpha)} - \gamma_n', f) \le \mathrm{T}(z, F_n^{(\alpha)}, f) \tag{4.2}$$

and for $z \in F_n^{(\alpha)}$, with $R_n = \max|\xi|$, $\xi \in F_n^{(\alpha)}$,

$$\mathrm{T}(z, F_n^{(\alpha)}, f) \leq \mathrm{T}(z, R_n, f). \tag{4.3}$$

For the particulars we again refer the reader to Noshiro and Sario [10].

Now (4.2) is essential in keeping the notation as simple as possible. Because of this inequality we are able to retain the domains $F_n^{(\alpha)}$ and $F_n^{(\alpha)}-\gamma_n$ and to assume that (i) the Poisson-Jensen formula is valid for $F_n^{(\alpha)}-\gamma_n$; (ii) each γ_n meet $F_n^{(\alpha)}$ only at one endpoint and is otherwise entirely contained in $F_n^{(\alpha)}$; and (iii) f has 1-exponential order $\{A_n\}$ on $\{\gamma_n\}$.

With these housekeeping chores finished we are free to give

Theorem 1. *Let f be meromorphic in D. Suppose $\{\gamma_n\}$ is a* PHD *sequence in D such that for some $a\in W$, $f-a$ has 1-exponential order $\{A_n\}$ on $\{\gamma_n\}$ (or if $a=\infty$, $\frac{1}{f}$ has 1-exponential order $\{A_n\}$ on $\{\gamma_n\}$). If, for some $0<\alpha<\pi$, and some sequence $\{z_n\}$, $z_n\in L_n^{(\alpha)}$ (defined as in (2.4)),*

$$\varliminf_{n\to\infty} \frac{\mathrm{T}(z_n, F_n^{(\alpha)}, f)}{A_n}=0 \tag{4.4}$$

then $f=a$. (Note if $\mathrm{T}=0$ we assume $A_n\to+\infty$.)

Proof: We divide the proof into two cases according as to whether $\{\gamma_n\}$ contains a radial-like subsequence, or an arc-like subsequence. As mentioned before we will assume no z_n is a pole of $f-a$. If infinitely many z_n are zeroes of $f-a$ there is nothing to prove.

Case 1. $\{\gamma_n\}$ contains an arc-like subsequence, say $\{\gamma_j\}$. We can assume $a=0$. Applying the Poisson-Jensen formula to $F_j^{(\alpha)}-\gamma_j$

$$\begin{aligned}\log|f(z_j)| &= \mathrm{T}(z_j, F_j^{(\alpha)}-\gamma_j, f)-\mathrm{T}\left(z_j, F_j^{(\alpha)}-\gamma_j, \frac{1}{f}\right)\\ &\leq \mathrm{T}(z_j, F_j^{(\alpha)}-\gamma_j, f)-\frac{1}{2\pi}\int_{Bd(F_j^{(\alpha)}-\gamma_j)} \log^+\left|\frac{1}{f(\zeta)}\right| d\omega(z_j, \zeta, F_j^{(\alpha)}-\gamma_j).\end{aligned} \tag{4.5}$$

Since for $\zeta\in\gamma_j$, eventually

(4.6) $$\log^+\left|\frac{1}{f(\zeta)}\right| = \log\left|\frac{1}{f(\zeta)}\right| \geq \frac{A_j}{(1-r_j)},$$

and is otherwise non-negative,

(4.7) $$\log|f(z_j)| \leq \mathrm{T}(z_j, F_j^{(\alpha)} - \gamma_j, f) - \frac{1}{2\pi}\cdot\frac{A_j}{1-r_j}\int_{\gamma_j} d\omega(z_j, \zeta, F_j^{(\alpha)} - \gamma_j)$$

$$= \mathrm{T}(z_j, F_j^{(\alpha)}, f) - \frac{A_j}{1-r_j}\omega(z_j, \gamma_j, F_j^{(\alpha)} - \gamma_j).$$

(Now (4.7) can be thought of as a kind of generalized two-constant theorem for meromorphic functions.)

The remainder of the proof for this case consists in estimating $\omega(z_j, \gamma_j, F_j^{(\alpha)} - \gamma_j)$ exactly as is done in the proof of Case 1 of Theorem 1 of $\mathscr{R}$ (from (4.4) to (4.14)). The notation is identical. We sketch the estimate procedures and leave the details in $\mathscr{R}$.

We first map $F_j^{(\alpha)}$ onto $|w|<1$ and using the conformal invariance of the harmonic measure, the Schmidt-Milloux theorem in amended form (3.0), and some elementary but lengthy estimates we find that

(4.8) $$\omega(z_j, \gamma_j, F_j^{(\alpha)} - \gamma_j) \geq \frac{2}{\pi}\arcsin C'_\alpha(R_j - r_j)$$

$$\geq C_\alpha(R_j - r_j).$$

If we put (4.8) into (4.7) we obtain

(4.9) $$\log|f(z_j)| \leq \mathrm{T}(z_j, F_j^{(\alpha)}, f) - A_j C_\alpha\left(\frac{R_j - r_j}{1-r_j}\right)$$

$$= -A_j\left[C_\alpha\left(\frac{R_j - r_j}{1-r_j}\right) - \frac{\mathrm{T}(z_j, F_j^{(\alpha)}, f)}{A_j}\right].$$

As $j \to \infty$, the first term in the brackets has a positive lower limit — by the properties of a radial-like sequence in conjunction with (3.2). The second term tends to zero by hypothesis and so, since $A_j \to +\infty$

(4.10) $$\lim_{j\to\infty}\log|f(z_j)| = -\infty.$$

The sequence $\{z_j\}$ has a point of accumulation, ζ_0, in $|z| \leq \frac{1}{2}$

and so $f(\zeta_0)=0$. But the Poisson representation shows that any point in a small disk about ζ_0 is the limit of a sequence $\{z'_j\}$, with z'_j close to z_j for which $\mathrm{T}(z'_j, F_j^{(\alpha)}-\gamma_j, f)$ has the same order of growth as does $\mathrm{T}(z_j, F_j^{(\alpha)}-\gamma_j, f)$ so that the hypothesis is satisfied for this sequence and we conclude that f is identically zero in D.

If $a=\infty$, then we replace f by $\frac{1}{f}$ and since $\mathrm{T}(z_n, F_n^{(\alpha)}, f)$ and $\mathrm{T}\left(z_n, F_n^{(\alpha)}, \frac{1}{f}\right)$ have the same order of growth the proof is complete in the radial-like case.

Case 2. $\{\gamma_n\}$ contains an arc-like subsequence, which we again label $\{\gamma_j\}$. As in Case 1 we can assume $a=0$, and apply the Poisson-Jensen formula to z_j, again assuming that z_j is neither a pole nor a zero of f,

$$\log|f(z_j)| \leq \mathrm{T}(z_j, F_j^{(\alpha)}-\gamma_j, f) - \frac{1}{2\pi}\int_{Bd(F_j^{(\alpha)}-\gamma_j)} \log^+\left|\frac{1}{f(\zeta)}\right| d\omega(z_j, \zeta, F_j^{(\alpha)}-\gamma_j). \tag{4.11}$$

By the *identical* argument as in Case 1 we obtain

$$\log|f(z_j)| \leq \mathrm{T}(z_j, F_j^{(\alpha)}-\gamma_j, f) - \frac{A_j}{1-r_j}\omega(z_j, \gamma_j, F_j^{(\alpha)}-\gamma_j). \tag{4.12}$$

We must now estimate the harmonic measure but instead of using the deep Schmidt-Milloux result we can invoke simpler ideas as in $\mathscr{R}$. We modify the arcs $\{\gamma_j\}$ (as in $\mathscr{R}$) by selecting a subarc $\gamma'_j \subseteq \gamma_j$, which extends from one radial boundary of E_j to the other, meeting these boundaries only at its endpoints. Note that $\{\gamma'_j\}$ is still an arc-like sequence. Let $q_j^{(1)}$ and $q_j^{(2)}$ be the radial segments of the boundary of E_j extending from these and points to $|z|=R_j$. (One or both of these segments may reduce to a point.) Let $G_j^{(\alpha)}$ be the domain bounded by the radial boundaries of $F_j^{(\alpha)}$; the two arcs of $|z|=R_j$ from these rays to the rays bounding E_j; the segments $q_j^{(1)}$, $q_j^{(2)}$; and γ'_j. See Fig. 2.

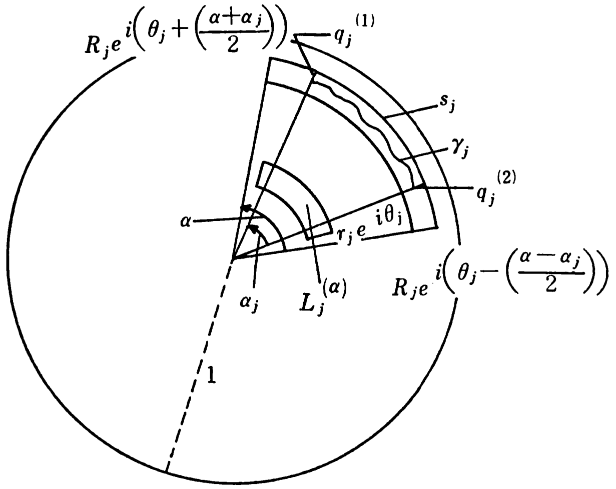

Fig. 2

By Carleman's Gebietserweiterung

$$\omega(z_j,\ \gamma_j,\ F_j^{(\alpha)}-\gamma_j)\geq\omega(z_j,\ \gamma'_j,\ G_j^{(\alpha)}). \tag{4.13}$$

If we let s_j be the arc of $|z|=R_j$ bounding E_j another application of the Gebietserweiterung gives

$$\omega(z_j,\ \gamma'_j\cup q_j^{(1)}\cup q_j^{(2)},\ G_j^{(\alpha)})\geq\omega(z_j,\ s_j,\ F_j^{(\alpha)}). \tag{4.14}$$

The additivity of the harmonic measure gives, with (4.14),

$$\omega(z_j,\ \gamma'_j,\ G_j^{(\alpha)})\geq\omega(z_j,\ s_j,\ F_j^{(\alpha)})-\omega(z_j,\ q_j^{(1)},\ G_j^{(\alpha)})-\omega(z_j,\ q_j^{(2)},\ G_j^{(\alpha)}). \tag{4.15}$$

We now estimate each of the terms on the right which is done in Lemmata 1 and 2 of $\mathscr{R}$.

We outline the technique of these two lemmas and state the results.

By the first lemma we obtain

$$\omega(z_j,\ s_j,\ F_j^{(\alpha)})\geq C_\alpha|R_je^{i\theta_j}-R_je^{i(\theta_j+\alpha_j)}|, \tag{4.16}$$

where C_α is a positive constant depending only on α and the inequality is valid for any $z_j\in L_j^{(\alpha)}$. To see this first map $F_j^{(\alpha)}$ onto the upper half disk of radius R_j by a root transformation.

Reflect the harmonic measure across the real axis to all of $|z|<R_j$

and use the Poisson formula to arrive at (4.16).

The second lemma says that given $\varepsilon>0$ there is an integer $J=J(\varepsilon)$ such that if $j>J$

$$\begin{aligned}\omega(z_j,\ q_j^{(1)},\ G_j^{(\alpha)})&\leq\varepsilon(R_j-r_j)\\ \omega(z_j,\ q_j^{(2)},\ G_j^{(\alpha)})&\leq\varepsilon(R_j-r_j).\end{aligned}\tag{4.17}$$

Here we first use the Gebietserweiterung to give, setting $D_j=|z|<R_j$,

$$\omega(z_j,\ q_j^{(1)},\ G_j^{(\alpha)})\leq\omega(z_j,\ q_j^{(1)},\ D_j-q_j^{(1)}).\tag{4.18}$$

One can compute this last harmonic measure by first mapping D_j linearly onto itself so that $q_j^{(1)}$ is now a radius and after taking the square root, one is left with the same harmonic measure as appears in the classical Phragmen-Lindelöf theorem. The arc-like property—in particualr (2.7(i))—gives the proper estimate, which is duplicated for $q_j^{(2)}$.

Combining (4.15) with (4.16) and (4.17) gives

$$\omega(z_j,\ \gamma_j',\ G_j^{(\alpha)})\geq(C_\alpha-2\varepsilon)(R_j-r_j).\tag{4.19}$$

If we use this inequality in (4.13) and then refer to (4.12) we have for sufficiently large j

$$\begin{aligned}\log|f(z_j)|&\leq\mathrm{T}(z_j,F_j^{(\alpha)}-\gamma_j,f)-A_j(C_\alpha-2\varepsilon)|R_je^{i\theta_j}-R_je^{i(\theta_j+\alpha_j)}|\\ &=-A_j\left[(C_\alpha-2\varepsilon)(|R_je^{i\theta_j}-R_je^{i(\theta_j+\alpha_j)}|)-\frac{\mathrm{T}(z_j,F_j^{(\alpha)}-\gamma_j,f)}{A_j}\right].\end{aligned}\tag{4.20}$$

By the arc-like property and by choosing ε small we obtain that

$$\lim_{j\to\infty}\log|f(z_j)|=-\infty;\tag{4.21}$$

and our proof concludes as in Case 1.

Because of the inequality (2.8) and the equivalence of the characteristic function vis-a-vis the point chosen – so long as the points lie in a compact subset of D, we can replace $\mathrm{T}(z_n, F_n^{(\alpha)}, f)$ by $\mathrm{T}(R_n, f)$ in Theorem 1.

If f tends to a value $a\in W$ on a boundary arc γ with a certain order we can infer infer information about its characteristic growth since a boundary arc produces PHD sequences in abundance.

Corollary 1. *Let f be meromorphic in D, and γ be a boundary arc on which f tends to $w_0 \in W$.*

If f is of bounded characteristic in D and for $z \in \gamma$ satisfies

$$|f(z)-w_0| \leq \exp\left(-\frac{A(|z|)}{1-|z|}\right), \tag{4.22}$$

where $A(|z|) \to +\infty$ as $|z| \to 1$; or if f is a normal function in D and for $z \in \gamma$ satisfies for some $\varepsilon > 0$ and some positive constant A_f,

$$|f(z)-w_0| \leq \exp\left(-\frac{A_f}{(1-|z|)^{1+\varepsilon}}\right), \tag{4.23}$$

then $f = w_0$. (If $w_0 = \infty$, we make the usual substitution of $\frac{1}{f(z)}$ for $f(z)-w_0$.)

Remark 1. The definition of a normal function originates in Lehto and Virtanen [2], although the ideas in less complete form are found in Noshiro [4], and Seidel and Walsh [11].

Remark 2. The first result generalizes the theorem for bounded functions proved by Gavrilov [1] while the second generalizes Corollary 7 in that the boundary path is not required to be a non-tangential path.

Proof. In both cases we seek a PHD sequence with proper order. Thus select any sequences $\{R_n\}$ with $0 < R_n < R_{n+1} \to 1$, $n \to \infty$. In easy fashion one can determine a PHD sequence $\{\gamma_n\}$ with the properties

(4.24)
i) $\gamma_n \subseteq \gamma$, all n;
ii) $\{\gamma_n\}$ has associated parameters $\{(R_n, r_n)\}$.

Suppose f has bounded characteristic, setting

$$A_n = \inf A(|z|), \quad z \in \gamma_n,$$

we have that $A_n \to \infty$, $n \to \infty$. Thus f has 1-exponential order $\{A_n\}$ on $\{\gamma_n\}$ so Theorem 1 prevails and $f = w_0$.

If f is a normal function recall that

$$\mathrm{T}(r, f) \leq C_f \log \frac{1}{1-r^2}, \qquad 0<r<1, \tag{4.25}$$

(Lehto and Virtanen [2, pg. 58]. If we put $A(|z|) = (1-|z|)^{-\varepsilon}$, then f has 1-exponential order $\left\{\frac{1}{(1-r)^{\varepsilon}}\right\}$ on $\{\gamma_n\}$, and

$$\lim_{n\to\infty} \mathrm{T}(R_n, f)(1-r_n)^{\varepsilon} \leq \lim_{n\to\infty} C_f \left(\log \frac{1}{1-R_n^2}\right)(1-r_n)^{\varepsilon} = 0.$$

This is so because of the PHD property and (3.1).

Thus Theorem 1 holds and again we conclude $f = w_0$.

§5. Behavior of f away from its zeroes

There is a reasonably direct way to obtain PHD sequences in a meromorphic function f tends to a value on a sequence approaching C but does not take on the limit value "near" any point of the sequence. To be precise define for a set $A \subseteq D$ and a point $b \in D$, $\rho(b, A) = \inf \rho(b, a)$, $a \in A$. Also let $Z(f) = \{z \in D \mid f(z) = 0\}$ and for $0 < \delta < \infty$

$$K_\delta(f) = \{z \in D \mid \rho(z, Z(f)) \geq \delta\}.$$

Suppose a meromorphic f tends to a value $w \in W$ along a sequence $\{z_n\}$, with $\lim_{n\to\infty} |z_n| = 1$. Suppose also that every $z_n \in K_\delta(f-w)$ (or $K_\delta\left(\frac{1}{f}\right)$, if $w = \infty$) Thus in each disk $N(z_n, \delta) = \{z \in D \mid \rho(z_n, z) < \delta\}$, f omits w. That is $f(N(z_n, \delta))$ is a domain on the Riemann surface which does not lie over the point w. By lifting a rectilinear segment $t(f(z_n) - w) + w$, $0 \leq t_0 < t \leq 1$, into $f(N(z_n, \delta))$ —altering the segment slightly to avoid any algebraic branch points — we obtain under f^{-1} a curve γ_n in $N(z_n, \delta)$ which extends from z_n (or a point close to z_n if $f'(z_n) = 0$) to the boundary of $N(z_n, \delta)$ and so $\{\gamma_n\}$ is certainly a PHD sequence. Moreover on each γ_n, $|f(z) - w| \leq |f(z_n) - w|$. If we let $R_n = |z_n|$ and $R_n' = \max |z|$, $z \in \gamma_n$, then

$$R_n < R'_n, \text{ and } \rho(R_n, R'_n) \leq \delta.$$

Enclosing each γ_n in its corresponding $F_n^{(\alpha)}$ for any choice of $0<\alpha<\pi$, with n sufficiently large, Theorem 1 can be reformulated to give

Theorem 2. *Let f be meromorphic in D. For some value $w \in W$ and some value $0<\delta<\infty$ suppose $\{z_n\}$ is a sequence such that*

$$\text{i) } z_n \in K_\delta(f-w) \left(or\ K_\delta\left(\frac{1}{f}\right) \ \ if\ w=\infty\right);$$

$$\text{ii) } |z_n| \to 1, \text{ and } f(z_n) \to w, \ n \to \infty.$$

Set $A_n = -(1-|z_n|)\log|f(z_n)-w|$ (or $A_n = (1-|z_n|)\log|f(z_n)|$ if $w=\infty$). If $A_n \to +\infty$, $n\to\infty$, and for some choice of $0<\delta<\pi$, and $\zeta_n \in L_n^{(\alpha)}$,

$$\varliminf_{n\to\infty} \frac{T(\zeta_n, F_n^{(\alpha)}, f)}{A_n} = 0,$$

then $f=w$.

Remark. This theorem for holomorphic f and with $T(\zeta_n, F_n^{(\alpha)}, f)$ replaced by the logarithm of the maximum modulus occurs in Rung [7, Theorem 1].

Just as in Corollary 1, $T(\zeta_n, F_n^{(\alpha)}, f)$ can be replaced by $T(R'_n, f)$. Further simplification is possible if $T(r, f)$ satisfies some growth condition, say of the form

$$\text{(5.0)} \qquad T(r, f) \leq \left(\frac{1}{1-r}\right)^s, \ 0 \leq s < \infty.$$

Since $\rho(R_n, R'_n) \leq \delta$, one can replace $T(R'_n, f)$ by $T(|z_n|, f)$.

Corollary 2. *Let f be a non-constant meromorphic function satisfying* (5.0). *Then for any value $w \in W$, and any $0<\delta<\infty$, there exists a positive constant C, depending on f, w, and δ, such that for $z \in K_\delta (f-w) \left(or\ z \in K_\delta\left(\frac{1}{f}\right)\ if\ w=\infty\right)$*

i) *if w is finite, we have that*

$$(1-|z|)^{s+1} \log|f(z)-w| \geq -C > -\infty;$$

while

ii) *if* $w=\infty$,

$$(1-|z|)^{s+1}\log|f(z)|\leq C<+\infty.$$

Proof. Suppose w si finite and assume no such C exists. Then for some sequence $\{z_n\}$, and some value $0<\delta<\infty$, $z_n\in K_\delta(f-w)$, all n and

$$(5.1)\qquad (1-|z_n|)^{s+1}\log|f(z_n)-w|\to-\infty,\ n\to\infty.$$

If we set $A_n=-(1-|z_n|)\log|f(z_n)-w|$, then (5.1) and (5.0) combine to give

$$0\leq\varliminf_{n\to\infty}\frac{\mathrm{T}(|z_n|,f)}{A_n}\leq\lim_{n\to\infty}\frac{1}{(1-|z_n|)^s}\left(\frac{1}{1-|z_n|\log|f(z_n)-w|}\right)=0,$$

and so Theorem 2 implies f is constant contrary to assumption.

As we noted earlier in (4.25) if f is a normal function then $\mathrm{T}(r,f)\leq C_f\log\frac{1}{1-r^2}$, $0\leq r<1$. So the characteristic function satisfies (5.0) for *any* choice of $0<s<\infty$, and Corollary 4 holds for a normal meromorphic f for any $s>0$, although the choice of the constant C depends now also upon s. This remark and Corollary 4 both generalize and improve Corollary 1 of Rung [8].

§6. Similar theorems near points on the boundary

To this point we have needed an estimate for the characteristic function on fairly large subdomains of D. Estimates of the characteristic function near a point on C would not be of any use. There are cases in which estimates of the characteristic function near a point of C are available. If a function has a non-tangential limit at a point on the boundary then clearly the characteristic function is bounded when restricted to a domain bounded by two hypercycles at this point, and it is true, as we shall prove, that a meromorphic normal function has this property at every point of C, whether or not it has a non-tangential limit at the point.

With little extra work we are able to obtain identity theorems as in the prior sections, except relative to domains impinging on C at a single point instead of an arc as previously required.

The proof of Case 1 of Theorem 1 contains all the necessary estimates and all that is required is to change our point of view. We now consider a PHD sequence $\{\gamma_j\}$ where each γ_j is contained in a domain of the form $F_j^{(\alpha)}$ only this time we suppose that γ_j is contained in $F_j^{(\alpha)}$ with the exception of one endpoint which is at the origin. Then tilt the $F_j^{(\alpha)}$ so that the vertex (and consequently the γ_j's) approaches C and we have the situation of the theorems in Gavrilov [1]. We are not being entirely accurate. Actually we find it more convenient to use domains bounded by arcs of circles rather than triangular domains. Let us proceed to the details.

For a complex number a and real values $0<R<\infty$, $0\leq\theta<2\pi$, $0<\alpha<\pi$, first put $a'=a+Re^{i\theta}$, then let C_1 and C_2 be the distinct circles of the same radius, each of which meets a and a' and which meet at a with angle α. If B is the perpendicular bisector of the line segment from a to a' then $F(a, R, \theta, \alpha)$ will denote the domain bounded by C_1, C_2 and B, which contains the point $\alpha+\frac{R}{4}e^{i\theta}$. We shall be concerned with sequences of such domains $\{F(a_n, R_n, \theta_n, \alpha_n)\}$. In the sequel we restrict $\{R_n\}$ and $\{\alpha_n\}$ to be constant sequences which allows somewhat less complicated statements for the results.

Definition 4. *Let* $\{\gamma_n\}$ *be a sequence of Jordan arcs in* D *and* $\{F(a_n, R, \theta_n, \alpha)\}$ *a sequence of domains as defined above. We say that* $\{\gamma_n\}$ *travels in* $\{F(a_n, R, \theta_n, \alpha)\}$ *if*

(6.0)
i) $F(a_n, R, \theta_n, \alpha)\subseteq D$, *all* n;

ii) *For some value* $\varepsilon>0$, $\gamma_n\subseteq F(a_n, R, \theta_n, \alpha-\varepsilon)$, *all* n,

except for one endpoint which coincides with a_n.

Note that $\{\gamma_n\}$ may travel in many different sequences $\{F(a_n, R, \theta_n, \alpha)\}$.

Given such a $F(a_n, R, \theta_n, \alpha)$ let $L_n^{(\alpha)}$ be the subdomain of $F\left(a_n, R, \theta_n, \frac{\alpha}{2}\right)$

between the circles $\left|\frac{z-a_n}{z-a'_n}\right|=\left(\frac{1}{4}\right)^{\alpha/\pi}$ and $\left|\frac{z-a_n}{z-a'_n}\right|=\left(\frac{1}{2}\right)^{\alpha/\pi}$. These domains are equivalent to the corresponding domains L_n defined in (2.4) relative to Theorem 1.

Theorem 3. *Let $\{\gamma_n\}$ be a* PHD *sequence travelling in $\{F(a_n, R, \theta_n, \alpha)=F_n^{(\alpha)}\}$. Suppose f is a meromorphic function such that for some $w\in W$, $f-w$ has $\frac{\pi}{\alpha}$-exponential order $\{A_n\}$ on $\{\gamma_n\}$ $\left(\text{or } \frac{1}{f} \text{ has this order if } w=\infty\right)$. If for some sequence $\{z_n\}$, $z_n\in L_n^{(\alpha)}$,*

$$\varliminf_{n\to\infty} \frac{T(z_n, F_n^{(\alpha)}, f)}{A_n}=0 \tag{6.1}$$

then $f=w$.

Remark. That the characteristic function exists over the domains $F_n^{(\alpha)}$ is clear because they are conformal equivalents to the old domains $F_n^{(\alpha)}$ of Theorem 1. Again we need to use the Poisson-Jensen formula for the domains $F_n^{(\alpha)}-\gamma_n$ but we make the same arguments to justify this as given prior to Theorem 1. We need to be a bit more careful in selecting a slightly smaller domain — if necessary — because of the need to preserve the travelling property. If the endpoint a_n of γ_n is a singularity of f we choose a new arc γ'_n close to γ_n and satisfying basically the same exponential order in order to obtain a new $F_n^{(\alpha)}$ contained in the original and such that the revised γ'_n still travel in the new $F_n^{(\alpha)}$ — although the value ε may be different.

Proof. Our proof parallels the proof of Theorem 1 (and agrees in the main with the proof of Theorem 3 of $\mathscr{R}$). For completeness we sketch the pertinent ideas.

As usual we can assume $w=0$. Choose a subsequence $n_i=j$ for which the limit in (6.1) holds. Thus

$$\lim_{j\to\infty} \frac{T(z_j, F_j^{(\alpha)}, f)}{A_j}=0. \tag{6.2}$$

Let

(6.3) $$r_j=\min|z|,\quad z\in\gamma_j.$$

With this value r_j, if $z\in\gamma_j$, for all j,

(6.4) $$\log|f(z)|\leq -\frac{A_j}{(1-r_j)^{\pi/\alpha}}\,.$$

Now choose a point $b_j\in\gamma_j$ such that

(6.5) $$\frac{1}{4}\,\mathrm{HD}(\gamma_j)\leq\rho(a_j,\ b_j)\leq\mathrm{HD}(\gamma_j),\ \text{all}\ j.$$

This is clearly possible.

Apply the Poisson-Jensen formula in $F_j^{(\alpha)}-\gamma_j$ to obtain, after the usual estimates including (6.4),

(6.6) $$\log|f(z_j)|\leq\mathrm{T}(z_j,\ f,\ F_j^{(\alpha)})-\frac{A_j}{(1-r_j)^{\pi/\alpha}}\,\omega(z_j,\ \gamma_j,\ F_j-\gamma_j).$$

We proceed just as in Case 1 of the proof of Theorem 1 after we obtained (4.7). The notation is identical with Case 1 and the details of the proof follow exactly the proof of Theorem 2 of $\mathscr{R}$ where the notation is also the same. Carry $F_j^{(\alpha)}$ onto D_w: $|w|<1$, by

(6.7) $$w_j^*(z)=w(h_j(z))=\frac{i-\left(\dfrac{1+h_j(z)}{1-h_j(z)}\right)^2}{i+\left(\dfrac{1+h_j(z)}{1-h_j(z)}\right)^2}$$

where

(6.8) $$h_j(z)=\left(e^{-i(\pi-\frac{\alpha}{2})}\left(\frac{z-a_j}{z-a_j}\right)\right)^{\pi/\alpha},\ a_j'=a_j+Re^{i\theta_j},\ \text{all}\ j.$$

If $L^{(\alpha)}:\ \frac{1}{4}<|w|<\frac{1}{2}\ ;\ \frac{\alpha}{4}<\arg w<\frac{3\alpha}{4}$, then by the definition of $L_j^{(\alpha)}$,

(6.9) $$L_j^{(\alpha)}=h_j^{-1}(L^{(\alpha)}),\quad \text{all}\ j.$$

We must again estimate $\omega(z,\ \gamma_j,\ F_j^{(\alpha)}-\gamma_j)$ for $z\in L_j^{(\alpha)}$. Because $r_j\to 1$ eventually $L_j^{(\alpha)}\subseteq F_j^{(\alpha)}-\gamma_j$, and we assume this to be the case for all j.

Let γ_j^* be the subarc of $w_j^*(\gamma_j)$ which connects $|w|=|w_j^*(b_j)|$ to $|w|=1$ and lies, except for endpoints, in this annulus. The Gebietserweiterung produces for $w \in D_w - w_j^*(\gamma_j)$, and all j,

$$(6.10) \qquad \omega(w, w_j^*(\gamma_j), D_w - w_j^*(\gamma_j^*)) \geq \omega(w, \gamma_j^*, D_w - \gamma_j^*).$$

Observe that by our construction of $L_j^{(\alpha)}$, there is a B_1 such that

$$(6.11) \qquad |w_j^*(L_j^{(\alpha)})| \leq B_1 < 1, \text{ all } j.$$

By use of Theorem A, the conformal invariance of the harmonic measure, and (6.10) and (6.11), it is true that

$$(6.12) \qquad \omega(z_j, \gamma_j, F^{(\alpha)}_j - \gamma_j) \geq \frac{2}{\pi} \arcsin \frac{(1-B_1^2)(1-|w_j^*(b_j)|^2)}{16}$$

for $z_j \in L_j^{(\alpha)}$.

A few calcalations yield

$$(6.13) \qquad 1 - |w_j^*(b_j)|^2 \geq \operatorname{Im}\left[\left(\frac{b_j - a_j}{b_j - a_j'}\right) e^{-i(\pi - \frac{\alpha}{2})}\right]^{\pi/\alpha} \left[1 - \left|\frac{b_j - a_j}{b_j - a_j'}\right|^{2\pi/\alpha}\right].$$

Unlike case (i) of Theorem 1 the last term in brackets in (6.13) contributes not at all to the order estimate because $\rho(a_j, b_j) \leq K_0$ implies $|b_j - a_j| \to 0, j \to \infty$, which in turn implies, because $|b_j - a_j'| \geq \frac{R}{2}$,

$$(6.14) \qquad 1 - \left|\frac{b_j - a_j}{b_j - a_j'}\right|^{2\pi/\alpha} \geq \frac{1}{2}, \qquad j \text{ sufficiently large.}$$

The first term is crucial. By considering the geometry of $F_j^{(\alpha)}$ $\left(\text{first transform it by } \left(\frac{z - a_j}{z - a_j'}\right) e^{-i(\pi - \frac{\alpha}{2})}\right)$ it is easy to see that, setting $\varphi_j = \arg\left(\left(\frac{b_j - a_j}{b_j - a_j'}\right) e^{-i(\pi - \frac{\alpha}{2})}\right)$,

$$(6.15) \qquad \begin{aligned} \operatorname{Im}\left(\left(\frac{b_j - a_j}{b_j - a_j'}\right) e^{-i(\pi - \frac{\alpha}{2})}\right)^{\pi/\alpha} &\geq \left(\frac{|b_j - a_j|}{R}\right)^{\pi/\alpha} \sin\left(\frac{\varphi_j \pi}{\alpha}\right) \\ &\geq \left(\frac{|b_j - a_j|}{R}\right)^{\pi/\alpha} \sin\frac{\pi\varepsilon}{2\alpha}. \end{aligned}$$

The last estimate (and value ε) is obtained from the definition of $\{\gamma_j\}$ travelling in $\{F_j^{(\alpha)}\}$. According to our definition of b_j in (6.5), Lemma A obtains and so eventually

(6.16) $$|a_j - b_j| \geq (1-|a_j|)t_0,\ t_0 > _0.$$

with $K_{\alpha,\varepsilon} = \frac{1}{2}\left(\frac{t_0}{R}\right)^{\pi/\alpha} \sin\frac{\pi\varepsilon}{2\alpha}$, (6.12) is now because of (6.13), (6.14), (6.15), (6.16), and the property of arcsin

(6.17) $$\omega(z_j, \gamma_j, F_j^{(\alpha)} - \gamma_j) \geq \frac{2}{\pi}\left(\frac{1-B_1^2}{16}\right) K_{\alpha,\varepsilon}\,(1-a|_j|)^{\pi/\alpha},\ z_j \in L_j^{(\alpha)},$$

and j sufficiently large.

Our last step is to notice because $\{\gamma_j\}$ is a PHD sequence

(6.18) $$1-|a_j| \geq (1-r_j)t_1,\quad t_1 > 0,\quad \text{all } j.$$

And so (6.6) us affected by (6.17) and (6.18) and becomes eventually for $z_j \in L_j^{(\alpha)}$, after setting $C_{\alpha,\varepsilon} = \left(\frac{1-B_1^2}{8\pi}\right) t_1^{\pi/\alpha} K_{\alpha,\varepsilon} > 0$,

$$\log|f(z_j)| \leq -A_j C_{\alpha,\varepsilon} + \mathrm{T}(z_j, F_j^{(\alpha)}, f) = -A_j\left[C_{\alpha,\varepsilon} - \frac{\mathrm{T}(z_j, F_j^{(\alpha)}, f)}{A_j}\right].$$

Because of (5.2) $\lim_{j\to\infty} f(z_j) = 0$, $z_j \in L_j^{(\alpha)}$. Since the $L_j^{(\alpha)}$ have a "limit domain" of similar form in D than f is identically zero on this limit domain and so must be zero throughout D. This completes the proof.

The limit in (6.1) cannot be relaxed to a positive value, nor can the PHD property be omitted. Examples are found in $\mathscr{R}$ [pg. 444–5].

The various corollaries in $\mathscr{R}$ after Theorem 3 remain valid with the function T replacing $\mathscr{M}$. Note that the order relation for f on the boundary arc γ in Corollary 5 of $\mathscr{R}$ must be changed as indicated in Rung [9]. The formulation in $\mathscr{R}$ is foolish.

§7. Applications to normal functions

A meromorphic function f in D is normal if and only if

(7.0) $$p(f)(z) \leq \frac{A_f}{1-|z|},\ z \in D$$

where $p(f)(z) = \frac{|f'(z)|}{1+|f(z)|^2}$. Details are found in Lehto and Virtanen [2, p. 55]. We shall be concerned with a less restrictive hypothesis namely that f be normal in hypercyclic domains at a point τ on C, i.e. domains of the form $F(\tau, \alpha, \theta_\tau, \beta)$, $0 \leq \beta < \pi$, and where we choose θ_τ so that $\tau + \alpha e^{i\theta_\tau} = -\tau$. In this situation a necessary and sufficient condition akin to (7.0) is that for each $0 \leq \beta < \frac{\pi}{2}$, if $z \in F(\tau, \alpha, \theta_\tau, \beta) = H(\tau, \beta)$,

$$\overline{\lim_{z \to \tau}}\, p(f)(z)(1 - |z|) < \infty. \tag{7.1}$$

See Lehto and Virtanen [2, Theorem 5].

As promised we show that if a meromorphic f satisfies (7.1) at a point $\tau \in C$ then the characteristic function is bounded on any $H(\tau, \beta)$. Precisely we mean that viewing $H(\tau, \beta)$ as a domain of the form $F(\tau, \alpha, \theta_\tau, \beta)$, and letting $z \in L^{(\beta)}$ then if f satisfies (7.1) at $\tau \in C$, $\mathrm{T}(z, f, H(\tau, \beta))$ is finite. If we map $H(\tau, \beta)$ conformally onto $|w| < 1$ by $w = w^*(z)$ defined in (6.7) and (6.8) (with $a = \tau$, $a_j = -\tau_j$, all j, $\alpha = \beta$, and $R = 2$) it is sufficient to show that the Nevanlinna characteristic of $g(w) = f(w^{*-1}(w))$ is finite. We use the spherical form, and so we want to estimate

$$S_g(r) = \iint_{|w| < r} p(g(w))^2 \, du dv, \; w = u + iv \tag{7.2}$$

Using the inequality in (6.13) and the fact that $z \in H(\tau, \beta)$ it is easy to show that if $|w| < r$, then $z = w^{*-1}(w)$ satisfies

$$|z| < 1 - \left(\frac{1 - r}{K}\right), \tag{7.3}$$

for some constant $K > 0$.
Because of the conformal invariance of the spherical area we aim to estimate $S_g(r)$ in $H(\tau, \beta)$, and for convenience we select $\tau = 1$. Let $H^*(1, \beta, r)$ be the domain bounded by the circle $\frac{|1 - z|}{|1 + z|} = \frac{1 - r}{K}$. For r sufficiently close to 1, and refering to (7.3) it is clear that

$$S_g(r) \leq \iint_{H^*(1, \beta, r)} (p(f(z))^2 \, dx dy, \; z = x + iy \tag{7.4}$$

To evaluate the integral on the right transform it by $z=\frac{1-\zeta}{1+\zeta}$, $\zeta=te^{i\varphi}$, and after some obvious estimates, including (7.1), obtain with $C(\beta)$ a positive constant depending on β,

$$S_g(r)\leq \int_{\beta/2}^{-\beta/2}\int_{1}^{\frac{1-r}{K}} \frac{C(\beta)t}{4t^2\cos^2\varphi}\mathrm{d}t\,\mathrm{d}\varphi$$

$$\leq \frac{-\beta C(\beta)}{4\cos^2\beta/2}\int_{1}^{\frac{1-r}{K}}\frac{\mathrm{d}t}{t}$$

$$\leq \frac{\beta C(\beta)}{4\cos^2\beta/2}\log\left(\frac{1-r}{K}\right).$$

It is now clear that

$$\int_0^1 S_g(r)\mathrm{d}r<\infty,$$

and so $\mathrm{T}(z, f, H(\tau, \beta))$ is finite for $z\in L^{(\beta)}$.

We owe this theorem in a sense to Tsuji who proved it by heavy computations for meromorphic f which omit (the same) 3 values in each $H(\tau, \beta)$, $0\leq\beta<\pi$, [12, Theorem VII. 12].

If f satisfies (7.1) and hence (7.4), the counting function restricted to $H(\tau, \beta)$ must be finite and so (7.1) implies that for any value $w\in W$, and each $0\leq\beta<\pi$

$$\sum_{z\in Z(f-w)\cap H(\tau,\beta)}(1-|z|)<\infty, \tag{7.5}$$

counting multiplicities and as usual letting $z\in Z\left(\frac{1}{f}\right)\cap H(\tau, \beta)$ if $w=\infty$.

Using (3.1), (7.5) and Hurwitz's theorem we can restate the above observation as

Theorem 4. *Let a meromorphic f satisfy (7.1) for some $\tau\in C$. Suppose $\{z_n\}$ is a sequence approaching τ inside a hypercyclic domain at τ and such that*

i) $f(z_n) \to w \in W$, $n \to \infty$;

ii) $\Sigma(1-|z_n|)=\infty$,

then w is in the cluster set along every curve approaching τ in a non-tangential fashion.

We now return to PHD sequences to state the following theorem which improves Theorem 5 of $\mathscr{R}$.

Theorem 6. *Let f be meromorphic in D. Let $\{\gamma_n\}$ be a* PHD *sequence travelling in $\{F(a_n, R, \theta_n, \alpha)\}$, $0<\alpha<\pi$, and suppose $a_n \to \tau \in C$, $n \to \infty$. If for some $w_0 \in W$ and some $\eta>0$, $A>0$, $f-w_0$* (or $\frac{1}{f}$ if $w_0=\infty$) *has $(1+\eta)$ exponential order $\{A\}$ on $\{\gamma_n\}$, then either $f \equiv w_0$, or there exists a sequence $\{z_n\}$ approaching τ non-tangentially with*

$$\lim_{n\to\infty} p(f)(z_n)(1-|z_n|)=\infty. \tag{7.6}$$

Proof. The geometry of travelling gives for some $0<\beta_0<\pi$, (with ε given by Definition 4 travelling) that, for n sufficiently large,

$$F\left(a_n, R, \theta_n, \alpha-\frac{\varepsilon}{2}\right) \subseteq H(\tau, \beta_0). \tag{7.7}$$

By choosing β_0 close enough to π so that $\frac{\pi}{\beta_0}<1+n$, we can now begin by supposing that (7.6) does not hold, which is to say that (7.1) does hold. Then by (7.7) the characteristic function is uniformly bounded on $F\left(a_n, R, \theta_n, \alpha-\frac{\varepsilon}{2}\right)$ for all n. Theorem 4 is then applicable with $A_n=A(1-|a_n|)^{(\pi/\beta_0)-(1+\eta)}$, and so $f \equiv w_0$. Otherwise f is not normal in $H(\tau, \beta_0)$ and the existence of the sequence $\{z_n\}$ is assured.

Remark 1. The mirror result in $\mathscr{R}$ [Theorem 5] added the condition that $\rho(\gamma_n, \gamma_{n+1}) \leq m$, all n.

Remark 2. Except for the above result, the theorems in $\mathscr{R}$ concerned with normal functions do not have generalizations via Theorem

3. The reason for this (unhappy) state of affairs is that the hypothesis all guarantee that f has angular limit and so is bounded in angles.

Remark 3. It would be desireable to abandon the requirement in Theorem 3, and its corollaries, that $\{\gamma_n\}$ must approach τ in a travelling manner. It would be nice to allow them to approach τ within some hypercyclic region but otherwise be free to assume any shape they desire. This would allow results of the type given in Theorem 2 except now relative to the characteristic behavior on hypercyclic domains. But we have not been able to show either the necessary of the travelling condition nor to prove Theorem 3 without it. On this somewhat gloomy note we conclude.

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