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Equivariant completion

By

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0. Introduction.

Let k be an algebraically closed field of arbrtrary characteristic. Let G be a linear algebraic group and let X be an algebraic vairety on which there is given a regular action, i.e. there is a morphism $\sigma: G\times X \ni (g, x) \to \sigma(g, x) = gx \in X$ satisfying $ex = x$ (e being the unit element of G), $(g_1g_2)x = g_1(g_2x)$ for every point x of X and any elements g_1, g_2 of G. We are assuming that G, X and σ are defined over k. In this paper, we shall show the following three results.

a) If G is a connected linear algebraic group (resp. a torus group) and if X is a normal variety on which there is given a regular action of G, then X has an open covering which consists of G-stable quasi-projective (resp. affine) open subsets of X (cf. Lemma 8 and Corollary 2). Furthermore, if X is a normal quasi-projective variety on which G acts regularly, then we may assume that the action is linear, i.e. there exist a projective embedding $\varphi: X \to \mathbf{P}^n$ and a projective representation $\rho: G \to \mathrm{PGL}(n)$ such that $\varphi(gx) = \rho(g)\varphi(x)$ for every g of G and every x of X (cf. Theorem 1).

Therefore, combining these results, we see that every regular action of connected linear algebraic group (resp. a torus group) on a normal variety is obtained by patching finitely many linear actions on normal quasi-projective (resp. affine) varieties.

b) Let X be a variety on which connected linear algebraic group G acts regularly. Then X has an equivariant Chow cover, i.e. there exist a quasi-projective variety $\tilde{X}$ on which G acts regularly, a G-

birational projective, surjective morphism $\varphi: \tilde{X} \to X$ and a non-empty G-stable open subset U of X such that $\varphi|\varphi^{-1}(U) \tilde{\to} U$ is an isomorphism (cf. Theorem 2).

c) M. Nagata proved in [4] that every algebraic variety X is embedded in a complete algebraic varitey $\bar{X}$ as an open subset. We shall generalize this beautiful result in the following way. Let G be a linear algebraic group (not necessarily connected) and let X be a normal variety on which there is given regular action of G. Then, there exists a complete variety $\bar{X}$ on which a regular action of G is given such that X is embdded in $\bar{X}$ as an open subset and the regular action of G on $\bar{X}$ is an extension of the given regular action of G on X (cf. Theorem 3). We call such a complete variety $\bar{X}$ a G-completion (or equivariant completion) of X.

Notations and conventions.

We shall fix a universal domain Ω. For every algebraic variety X and every subfield K of Ω, the set of all K-rational points of X is denoted by $X(K)$. If X is an algebraic variety defined over $k(\subset \Omega)$, then the field of rational functions of X defined over k is denoted by $k(X)$. Furthermore, if X is affine, the ring of regular functions of X defined over k is denoted by $k[X]$.

Let G be a linear algebraic group and let X be an algebraic variety on which there is given a regular action of G. We assume that G, X and the action are defined over k. For every f of $k(X)$ and every g of G, we shall define $f^g(x) = f(g^{-1}x)$ where x is a generic point of X over $k(g)$. Then we have that $f^{g_1 g_2} = (f^{g_2})^{g_1}$ for every g_1 and g_2 of G.

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1. Invertible regular functions.

In this section, we shall prepare some lemmas on invertible regular functions.

Lemma 1. *Let G be a connected linear algebraic group defined* *over an algebraically closed field k and let K be an extension field of k. If f is an invertible regular function defined over K on G, then there is an element c of K and a rational character λ defined over k of G such that $f=c\lambda$.*

Proof. See the proof of Theorem 3.4 of [7].

Lemma 2. *Let X be a variety defined over k on which G acts regularly and let f be an invertible regular function defined over k of X. Then there is a rational character λ of G such that $f^g=\lambda(g^{-1})f$ for any element g of G.*

Proof. Let x be a generic point of X over k and let $\sigma: G\ni g\to g\cdot x\in X$ be the operation. Put $f'=f\circ\sigma$. Then f' is an invertible regular function defined over $k(x)$ on G. By virtue of Lemma 1, we have an element d of $k(x)$ and a rational character λ defined over k on G such that $f'=d\lambda$. Hence, $(f')^g(g')=f'(g^{-1}g')=d\lambda(g^{-1}g')=d\lambda(g^{-1})\lambda(g')=\lambda(g^{-1})d\lambda(g')=\lambda(g^{-1})f'(g')$ for any elements g and g'. If we put $g'=e$, then we have that $(f')^g(e)=\lambda(g^{-1})f'(e)$ and that $f^g(x)=f(g^{-1}x)=f'(g^{-1})=(f')^g(e)=\lambda(g^{-1})f'(e)=\lambda(g^{-1})f(x)$.

q.e.d.

Lemma 3. *Let X be a variety defined over k and let K be an extension field of k. If f is an invertible regular function defined over K on X, then there is an element c of K and an invertible regular function f' defined over k on X such that $f=cf'$.*

Proof. We may assume that X is normal and its completion $\bar{X}$ is normal. If we regard f as a rational function on $\bar{X}$, then the components of (f) (the divisor of f on $\bar{X}$) is contained in $\bar{X}-X$. Since $\bar{X}-X$ is k-closed, $(f)=E$ for some k-rational divisor E. Thus, by virtue of Cor. 2 ([8] p. 265), there is an element c of K and a rational function f' defined over k such that $f=c.f'$. It is obvious that f' is an invertible regular function. q.e.d.

The next lemma 4 is interesting, however we shall not use it below.

Lemma 4. *Let X be a variety defined over k. There are finitely many invertible regular functions $f_1,\dots,f_r$ defined over k on X such that every invertible regular function f on X is written uniquely in the following form*; $f=c\prod_{i=1}^{r} f_i^{n_i}$ where c is a non-zero element of k and n_i $(i=1, 2,\dots, r)$ are integers.

Proof. We may assume that X is normal and that its completion $\bar{X}$ is normal. Let E_i $(i=1, 2,\dots, s)$ be the irreducible components of codimension 1 of $\bar{X}-X$. For every invertible regular function f on X, $(f)=\sum_{i=1}^{s} l_iE_i$. Put $H=\{l=(l_1,\dots, l_s)\mid (f)=\sum_{i=1}^{s} l_iE_i,\ f\in\Gamma(X, O_X^*)\}$ $(\subset Z^{\oplus s})$. H is a torsion free Z-submodule of $Z^{\oplus s}$. Therefore, there are finitely many invertible regular functions $\{f_i\}_{1\le i\le r}$ on X such that $(f)=\sum_{i=1}^{r} n_i(f_i)(\{n_i\}_{1\le i\le r}$ are uniquely determined) for every invertible regular function f on X. q.e.d.

2. Quasi-projective case.

Let G be a connected linear algebraic group defined over k and let X be a normal quasi-projective algebraic variety defined over k on which G acts regularly. Under this circumstance, we shall prove in this section that there is a G-linearlizable ample line bundle on X (cf. [3]) i.e., there is a projective embedding $\psi: X\to \mathbf{P}^n$ and a group representation $\rho: G\to PGL(n)$ such that $\rho(g)\cdot\psi(x)=\psi(g\cdot x)$ for every $g\in G$ and $x\in X$.

At first, we shall prepare a lemma which is a key in our proof.

Lemma 5. *Let G be a connected linear algebraic group defined over k and let X be a variety defined over k on which G acts regularly and let Z be a k-rational cycle on X. Then, for every element g of G, $g\cdot Z$ is rationally equivalent to Z.*

Proof. We may assume that Z is a prime cycle. Let g be a generic point of G over k and let z be a generic point of Z over $k(g)$. Let W be the closure of $(g, g\cdot z)$ in $G\times X$. It is enough to prove that $g_0\cdot Z$ is rationally equivalent to Z for any k-rational point g_0

of G. Since $k(g)$ and $k(z)$ is linearly disjoint over k, if z_0 is a specialization of z over k, then (g_0, z_0) is a specialization of (g, z) over k. Furthermore, $(g_0, g_0{\cdot}z_0)$ is a specialization of $(g, g{\cdot}z)$ over k. Hence, $g_0 \times g_0 Z$ is contained in $(g_0 \times X) \cap W$. On the other hand, if (g_0, z') is a specialization of (g, gz) over k, then $z' = g_0{\cdot}z''$ for some element z'' of Z, because $z = g^{-1}(gz)$. Thus, $g_0 \times g_0 Z$ is the only one component of $(g_0 \times X) \cap W$. Next we shall prove that the multiplicity is equal to one, i.e. $(g_0 \times X) \cdot W = g_0 \times g_0 Z$. In order to prove this, we may assume that $g_0 =$ unit element of G. Let $\varphi: G \times X \ni (g, x) \rightarrow g^{-1} \cdot x \in X$ be the operation map and let $p_1: G \times X \rightarrow G$, $p_2: G \times X \rightarrow X$ be the projections. Put $\varphi^*: k(X) \rightarrow k(G \times X)$ (respectively $p_1^*: k(G) \rightarrow k(G \times X)$, $p_2^*: k(X) \rightarrow k(G \times X)$) be the map induced from φ (respectively p_1 and p_2). Furthermore, let n be the maximal ideal of $O_{Z,X}$ and let m be the maximal ideal of $O_{e,G}$. Then, $(e \times X) \cap \mathrm{Spec}(O_{e \times Z, G \times X})$ is defined by $p_1^*(m) 0_{e \times Z, G \times X}$ and $(e \times Z) \cap \mathrm{Spec}(O_{e \times Z, G \times X})$ is defined by $(p_1^*(m) 0_{e \times Z, G \times X}) + (p_2^*(n) 0_{e \times Z, G \times X})$. Let $I(W)$ be the defining ideal of $W \cap \mathrm{Spec}(O_{e \times Z, G \times X})$. Then $I(W) + (p_1^*(m) 0_{e \times Z, G \times X}) = (\varphi^*(n) 0_{e \times Z, G \times X}) + (p_1^*(m) 0_{e \times Z, G \times X})$. In fact, if $f \in n$, then $\varphi^*(f)(g, g{\cdot}z) = f(z) = 0$ for any $z \in Z$. Let f be an element of $I(W)$. $f(g, x) = f(e, g^{-1} \cdot x) + (f(g, x) - f(e, g^{-1} \cdot x))$ and $f(e, g^{-1} \cdot x) \in \varphi^*(n) O_{e \times Z, G \times X}$, $f(g, x) - f(e, g^{-1} \cdot x) \in p_1^*(m) 0_{e \times Z, G \times X}$, thus $I(W) + (p_1^*(m) 0_{e \times Z, G \times X}) = (\varphi^*(n) 0_{e \times Z, G \times X}) + (p_1^*(m) 0_{e \times Z, G \times X})$. Let f be an element of n. Then, $(\varphi^*(f) - p_2^*(f))(g, x) = f(g^{-1}, x) - f(x)$ and therefore $\varphi^*(f) - p_2^*(f) \in p_1^*(m) 0_{e \times Z, G \times X}$. Hence $(\varphi^*(n) 0_{e \times Z, G \times X}) + (p_1^*(m) 0_{e \times Z, G \times X}) = (p_1^*(m) 0_{e \times Z, G \times X}) + (p_2^*(n) 0_{e \times Z, G \times X})$. Therefore, $(e \times X) \cdot W = e \times Z$. Since G is a rational variety, gZ is ratio nally equivalent to Z. q.e.d.

Now we shall prove the main theorem of this section.

Theorem 1. *Let G be a connected linear algebraic group and let X be a normal quasi-projective variety on which G acts regularly. Then there is a projective embedding $\psi: X \rightarrow \mathbf{P}^n$ and a group representation $\rho: G \rightarrow PGL(n)$ such that $\rho(g)\psi(x) = \psi(gx)$ for every $g \in G$ and $x \in X$.*

Proof. Let D be a very ample effective divisor on X such that

$X-D$ is an affine open subset of X and let g be a generic point of G over k. By virtue of Lemma 5, gD is linearly equivalent to D, hence for a rational function φ_g definec over $k(G)$, we have that $gD=D+(\varphi_g)$. Since φ_g is regular on $X-D$, we may assume that φ_g has the following form: $\varphi_g=\sum_{i=1}^{n} a_i(g)x_i$, where $a_i(g)$ $(i=1, 2,\dots, n)$ are elements of $k[G]$, and x_i $(i=1, 2,\dots, n)$ are elements of $k[X-D]$ and are linearly independent over k. Let $U=\{g\in G \mid a_i(g)\neq 0$ for some $i\}$. Then U is a non-empty k-open subset of G and for any element g of U, we have that $gD=D+(\varphi_g)$. In particular, for any independent generic points g, g' of G over k,

$$gg'D=g(D+(\varphi_{g'}))=gD+(\varphi_{g'}^{g})=D+(\varphi_g\cdot\varphi_{g'}^{g}).$$

Therefore, there is an invertible regular function $\delta(g, g')$ on X defined over $k(G\times G)$ such that $\varphi_{gg'}=\delta(g, g')\varphi_g\varphi_{g'}^{g}$. By virtue of Lemma 3, there exist an element $c(g, g')$ of $k(G\times G)$ and an invertible regular function δ on X defined over k such that $\delta(g, g')=c(g, g')\delta$. By taking $\delta\varphi_g$ instead of φ_g and by virtue of Lemma 2, we may assume that $\delta(g, g')$ is an element of $k(G\times G)$ ($\delta(g, g')$ is defined at g, g' whenever g, g' and gg' are contained in U). Let $\{D=D_0, D_1,\dots, D_m\}$ be very ample divisors which are linearly equivalent to D and such that they give a projective embedding $\varphi: X\to \mathbf{P}^m$. Then, there exist rational functions φ_i defined over k such that $D_i=D+(\varphi_i)$ and $\varphi: X\ni x\to (\varphi_0(x);\dots;\varphi_m(x))\in \mathbf{P}^m$ gives an embedding of X. Put $V=\sum_{\substack{0\leq i\leq m\\ g\in G}} \varphi_g\varphi_i^g\,\Omega$ (If $g\notin U$, then $\varphi_g=0$). We shall prove that V is a finite dimensional vector space over Ω. In fact, let g be a generic point of G over k. Then we have that $gD_i=gD+(\varphi_i^g)=D+(\varphi_g\varphi_i^g)$. Since $\varphi_g\varphi_i^g$ is regular on $X-D$, $\varphi_g\varphi_i^g=\sum_g \alpha_{ij}(g)y_{ij}$ for some $\alpha_{ij}(g)\,(\in k(G))$ and $y_{ij}(\in k[X-D])$. Therefore, V is a vector subspace of the vectorspace generated by y_{ij}'s over Ω, hence V is a finite dimensional vector space over Ω. It is easy to see that we can take $\psi_0=\varphi_{g_0}\varphi_{i_0}^{g_n}, \dots, \psi_n=\varphi_{g_n}\varphi_{i_n}^{g_n}$ $(g_i\in U(k))$ as a basis of V. Let g be a generic point of G over k. Then we have that $\varphi_{g^{-1}}\psi_k^{g^{-1}}=\varphi_{g^{-1}}(\varphi_{g_k}\varphi_{i_k}^{g_k})^{g^{-1}}=\varphi_{g^{-1}}\varphi_{g_k}^{g^{-1}}\varphi_{i_k}^{g^{-1}g_k}=\delta(g^{-1}, g_k)^{-1}\varphi_{g^{-1}g_k}\varphi_{i_k}^{g^{-1}g_k}$. Since $\varphi_{g^{-1}g_k}\varphi_{i_k}^{g^{-1}g_k}$ is contained in V, $\varphi_{g^{-1}}\psi_k^{g^{-1}}=\sum_{i=0}^{n}\alpha_{ik}(g)\psi_i$ for some $\alpha_{ik}(g)(\in k(G))$. Furthermore, for any independent generic points g, g' of G over k,

$$\psi_{(gg')^{-1}}\psi_k^{(gg')^{-1}}=\varphi_{g'^{-1}g^{-1}}\psi_k^{g'^{-1}g^{-1}}=$$

$$\delta(g'^{-1}, g^{-1})\varphi_{g'^{-1}}\varphi_{g^{-1}}^{g'^{-1}}(\psi_k^{g^{-1}})^{g'^{-1}}=\delta(g'^{-1}, g^{-1})\varphi_{g'^{-1}}(\varphi_{g^{-1}}\psi_k^{g^{-1}})^{g'^{-1}}$$

$$=\delta(g'^{-1}, g^{-1})\varphi_{g'^{-1}}(\sum_{i=0}^{n}\alpha_{ik}(g)\psi_i)^{g'^{-1}}=\delta(g'^{-1}, g^{-1})\sum_{i=0}^{n}\alpha_{ik}(g)\varphi_{g'^{-1}}\psi_i^{g'^{-1}}$$

$$=\delta(g'^{-1}, g^{-1})\sum_{i=0}^{n}\alpha_{ik}(g)\sum_{j=0}^{n}\alpha_{ji}(g')\psi_j$$

$$=\delta(g'^{-1}, g^{-1})\sum_{j=0}^{n}(\sum_{i=0}^{n}\alpha_{ik}(g)\alpha_{ji}(g'))\psi_j.$$

By the above fact, if we shall define $\rho(g)$=the class of the transposed matrix of $(\alpha_{ik}(g))$ in $PGL(n)$, then $\rho(gg')=\rho(g)\rho(g')$ for any independent generic points g, g' of G over k. Hence $\rho(g)$ is an everywhere defined rational representation of G. Moreover, $\psi: X\ni x\to(\psi_0(x);\dots;\psi_n(x))\in\mathbf{P}^n$ gives an embedding of X, because V contains $\{\varphi_g\varphi_0^g,\dots,\varphi_g\varphi_n^g\}$ $(g\in U(k))$. These ρ and ψ are desired ones. In fact, let g be a generic point of G over k and let x be a generic point of X over $k(G)$. $\psi(gx)=(\psi_0(gx);\dots;\psi_n(gx))=(\varphi_{g^{-1}}\psi_0^{g^{-1}}(x);\dots;\varphi_{g^{-1}}\psi_n^{g^{-1}}(x))=\rho(g)(\psi_0(x);\dots;\psi_n(x))=\rho(g)\psi(x)$. Therefore, for every element g of G and x of X, we have that $\psi(gx)=\rho(g)\psi(x)$, because ρ and ψ are both regular.

q.e.d.

Remark 1.[*)] If X is not normal, Theorem 1 is not necessarily true.

3. Equivariant Chow lemma.

Let G be a connected linear algebraic group and let X be a variety on which G acts regularly. In this section, we shall prove that there are a quasi-projective variety $\tilde{X}$ on which G acts regularly and a G-birational projective surjective morphism $f: \tilde{X}\to X$ and a non-empty G-stable open subset U of X such that $f|f^{-1}(U)\overset{\sim}{\to}U$ is an isomorphism. This is a generalization of Chow's lemma.

The following Lemma 7 is well-known.

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*) This Remark 1 was pointed out to the author by Professor T. Oda.

Lemma 7. *Let X be a normal variety and let D be an effective divisor of X such that,*

1) There are a finite number of effective divisors $D=D_0, D_1, \dots, D_n$ which are linearly equivalent to D.

2) $X-D_i\,(i=0, 1,\dots, n)$ is affine and $\bigcup_{i=0}^{n}(X-D_i)=X$ Then, X is quasi-projective.

Lemma 8. *Let G be a connected linear algebraic group and let X be a normal variety on which G acts regularly. Then, for any point x of X, there is a G-stable quasi-projective open neighbourhood of x.*

Proof. Let D be an effective divisor of X such that $X-D$ is an affine open neighbourhood of x. Put $Y=\bigcap_{g\in G(k)} gD$ and $U=X-Y$. Then U is a G-stable open neighbourhood of x. If $Y=D$, i.e. D is G-stable, then U is a G-stable affine open neighbourhood of x. If $Y\neq D$, then we put $D'=D-Y$. D' is an effective divisor of U and $U=\bigcup_{g\in G(k)}(U-gD')$. Furthermore, for every element g of $G(k)$, gD' is linearly equivalent to D', by virtue of Lemma 5 and $U-gD'$ is affine. By virtue of Lemma 7, U is quasi-projective. q.e.d.

Corollary 1. *Let G be a connected linear algebraic group and let X be a G-homogeneous variety. Then X is quasi-projective.*

Corollary 2.*) *Let T be a torus group and let X be a normal variety on which T acts regulary. Then, for any point x of X, there is a G-stable affine open neighbourhood of x.*

Proof. We may assume that x is a k-rational point. Furthermore, by virtue of Theorem 1 and Lemma 8, we may assume that X is T-stable locally closed subvariety $\mathbf{P}^n$ on which T acts linearly. Let $\overline{X}$ be the closure of X in $\mathbf{P}^n$ and let $\{x_0,\dots, x_n\}$ be a T-semi invariant,

*) This Corollary 2 was pointed out to the author by Professor T. Oda and the author heard from Professor M. Maruyama that Professor D. Mumford conjectured this Corollary 2.

homogeneous coordinate of $\mathbf{P}^n$. If $\bar{X}=X$, then Corollary 2 is obvious. Let $Y=\bar{X}-X$ and let I be the homogeneous ideal defined by Y_{red} and let m be the homogeneous ideal defined by the point x. Then I is $T(k)$-stable and $m \not\supseteq I$. Take a homogeneous polynomial f which is contained in I and is not contained in m. Put $V=\sum_{t\in T(k)} f^t k$, $J=\bigcap_{t\in T(k)} m^t$ and $W=V\cap J$. Then V and W are $T(k)$-stable and finite dimensional vector spaces. Since every representation of T is completely reducible, there is a $T(k)$-stable vector subspace Z of V such that $V=W\oplus Z$. Therefore, there is a T-semi invariant homogeneous polynomial $F(\neq 0)$ which is contained in Z. Let S be the T-stable hypersurface of $\mathbf{P}^n$ defined by F and let $U=X-S$. Then U is the derired T-stable affine open neighbourhood of x. In fact, we have only to prove that $x\notin S$. If x is contained in S, then F is contained in W. This is a contradiction. q.e.d.

Corollary 3. *Let T be a torus group and let X be a normal variety on which T acts regularly. If the action is closed, i.e. every orbit is closed in X, then there exists a universal geometric quotient Y of X and Y is a normal pre-variety. Furthermore, if the action is separated (cf.* [3]), *then Y is a normal variety.*

Proof. Use Corollary 2 and see Amplication 1.3 and Lemma 0.6 in [3]. q.e.d.

Remark 2. Let G be a connected linear algebraic group (resp. a torus group). Then, Theorem 1 and Lemma 8 (resp. Corollary 2) show that every regular action on a normal algebraic variety is obtained by patching linear actions of G on normal quasi-projective (resp. affine) varieties.

Theorem 2. (Equivariant Chow lemma) *Let X be an algebraic variety on which a connected linear algebraic group G acts regularly. Then, there exist a quasi-projective variety $\tilde{X}$ on which G acts regularly such that.*

1) *There is a G-birational projective surjective morphism $f\colon \tilde{X}\to X$,*
2) *There is a non-empty G-stable open subset U of X such that*

$f|f^{-1}(U) \rightrightarrows U$ is an isomorphism.

Proof. We may assume that X is normal. The almost part of the proof of Theorem 2 is nothing but the one of Theorem 5.6.1. [1]. However, for the completeness of the proof, we shall prove it here. By virtue of Lemma 8, there is a G-stable quasi-projective open covering $\mathfrak{U}=(U_k)_{1\le k\le n}$ of X. By virtue of Theorem 1, for each $k(1\le k\le n)$, there are a projective variety P_k on which G acts reqularly and an open immersion φ_k: $U_k\to P_k$ such that $\varphi_k(gx)=g\varphi_k(x)$ for every element x of U_k and g of G. Put $U=\bigcap_{k=1}^{n} U_k$. Then U is a G-stable open subset of X. Let φ: $U\ni x\to(\varphi_1(x),\dots, \varphi_n(x))\in P_1\times\cdots\times P_n=P$ (φ being a G-open immersion) and let $\psi=(j, \varphi)$: $U\to X\times P$ where j is the inclusion map of U. ψ is a G-immersion. Put $X'=\overline{\psi(U)}$. Then X' is a G-stable closed subset and ψ is factored through X',

$$\psi: U\xrightarrow{\psi'}X'\xrightarrow{h}X\times P$$

where ψ': $U\to X'$ is a G-open immersion and h: $X'\to X\times P$ is a G-closed immersion. Let q_1: $X\times P\to X$ be the first projection and let q_2: $X\times P\to P$ be the second projection and let $f=q_1\circ h$; $X'\xrightarrow{h}X\times P\xrightarrow{q_1}X$. Then X', f and U are the desired ones. First of all, we shall prove that f is a G-projective surjective morphism and that $f|f^{-1}(U)\rightrightarrows U$ is an isomorphism. Since q_1 is a projecitve morphism, f is a G-projective morphism. Furthermore, $f(X')$ contains U, because $f\circ\psi'=q_1\circ h\circ\psi'=j$. Hence f is surjective. We shall put $U'=f^{-1}(U)$. Then $U'=(q_1\circ h)^{-1}(U)=X'\cap q_1^{-1}(U)$ and U' is a closure of Γ_φ(Γ_φ being a graph of φ) in $q_1^{-1}(U)$. However, since φ is a morphism from U to $U\times P=q_1^{-1}(U)$, Γ_φ is a closed subset of $U\times P$. Hence, $U'=\Gamma_\varphi=\psi'(U)$ and $f|U'\simeq U$ is an isomorphism. Next we shall prove that X' is quasi-projective. For the purpose it is enough to prove that g: $X'\xrightarrow{h}X\times P\xrightarrow{q_2}P$ is an immersion. For each $k(1\le k\le n)$, we shall put $V_k=\varphi_k(U_k)$ (G-open subset of P_k), $W_k=p_k^{-1}(V_k)$ (G-open subset of P where p_k: $P\to P_k$ is the k-th projection), $U'_k=f^{-1}(U_k)$ (G-stable open subset of X') and $U''_k=g^{-1}(W_k)$ (G-stable open subset of X'). Then, $\mathfrak{U}'=(U'_k)_{1\le k\le n}$ is a G-open covering of X'. For each $k(1\le$

$k \leq n$), we shall prove that U''_k contains U'_k. Therefore, $\mathfrak{U}'' = (U''_k)_{1 \leq k \leq n}$ is a G-open covering of X, too. For the purpose, we shall prove that the following diagram is commutative.

$$\begin{array}{ccc} U'_k & \xrightarrow{g|U'_k} & P \\ {\scriptstyle f|U'_k}\downarrow & & \downarrow{\scriptstyle p_k} \\ U_k & \xrightarrow[\varphi_k]{} & P_k \end{array}$$

Since all maps are morphisms, it is enough to prove that $p_k \cdot g|U' = \varphi_k \circ f|U'$. However, this equality is true by the definition. Thus, $W = (W_k)_{1 \leq k \leq n}$ is a G-stable open covering of $g(X')$. In order to prove the Theorem 2, we have only to prove that $g\colon U''_k \to W_k (1 \leq k \leq n)$ is an immersion. For any $k (1 \leq k \leq n)$ we shall put

$$u_k\colon W_k \xrightarrow{p_k} V_k \xrightarrow{\varphi_k^{-1}} U_k \longrightarrow X.$$

u_k is a G-morphism. The restriction of q_2 on Γ_{u_k} which is a graph of u_k is a G-isomorphism. Let $v_k\colon U''_k \hookrightarrow X \times W_k$ be a canonical injection and let $w_k = q_2 \cdot v_k$. We have only to prove that $v_k = \Gamma_{u_k} \circ w_k (1 \leq k \leq n)$. Since all maps are morphisms, it is enough to prove that the equality is true on U'. We shall consider all things through the isomorphism $\psi'\colon U \xrightarrow{\sim} U'$. We shall prove that $q_1 \circ v_k = u_k \circ q_2 \circ v_k$, because the second components of v_k and $\Gamma_{u_k} \circ w_k$ are equal. However, since $v_k \circ \psi' | U = \psi | U$, it is enough to prove that $q_1 \circ \psi = u_k \circ q_2 \circ \psi$. Moreover, since $j = q_1 \circ \psi$, $\varphi = q_2 \circ \psi$, it is enough to prove that $j = u_k \circ \varphi$. This is obvious.

q.e.d.

4. G-twisted valuation rings.

Let G be a connected algebraic group (not necessarily linear) and let X be a variety on which G acts regulary and let v be a valuation ring of $k(X)$. We shall define the G-twisted valuation ring $\bar{v}$ of v and study properties of $\bar{v}$. The notion of G-twisted valuation ring plays an important roll in a proof of the existence theorem of G-completion.

Lemma 9. *Let* K_1 *and* K_2 *be extension fields of* k *such that*

1) K_1 *and* K_2 *are linearly disjoint over* k.

2) K_1 *is a regular extension of* k.

Let v *be a valuation ring of* K_2 *and let* m_v *be its maximal ideal. Then* $(K_1 \underset{k}{\otimes} v)_{K_1 \otimes_k m_v}$ *is a valuation ring of the quotient field* $Q(K_1 \underset{k}{\otimes} K_2)$ *of* $K_1 \underset{k}{\otimes} K_2$.

Proof. It is ovbious.

q.e.d.

Remark 3. $(K_1 \underset{k}{\otimes} v)_{K_1 \otimes_k m_v}$ is an extension of v in $Q(K_1 \underset{k}{\otimes} K_2)$.

Definition 1. *Let* X *be a variety on which a connected algebraic group* G *acts regularly and let* $\sigma: G \times X \ni (g, x) \to gx \in X$ *be its operation. Then* σ *induces the injective homomorphism* $\sigma^*: k(X) \to k(G \times X)$. *For any valuation ring* v *of* $k(X)$, *we shall call the induced valuation ring* $\sigma^{*-1}((k(G) \underset{k}{\otimes} v)_{k(G) \otimes_k m_v})$ *in* $k(X)$ *the* G*-twisted valuation ring of* v *and simply denote it by* $\bar{v}$.

Lemma 10. *Let* X *be a variety on which a connected algebraic group* G *acts regularly. Let* v *be a valuation ring of* $k(X)$ *dominating a point* x *of* X, *i.e.* v *dominating* $O_{x,X}$. *Then the* G*-twisted valuation ring* $\bar{v}$ *satisfies the following properties.*

1) $\bar{v}$ *is* $G(k)$*-stable.*

2) $\bar{v}$ *dominates* $\bar{x}$ *where* $\bar{x}$ *is a generic point of the orbit* $G(x)$ *of* x *over* k.

3) *For every element* f *of* $\bar{v}$ *(resp.* $m_{\bar{v}}$*), there is a non-empty open subset* U *of* $G(k)$ *such that* $f^{g^{-1}}$ *belongs to* v *(resp.* $m_{\bar{v}}$*) for every element* g *of* U. *Conversely, if* f *is an element of* $k(X)$ *and if there is a non-empty open subset* U *of* $G(k)$ *such that* $f^{g^{-1}}$ *is an element of* v *(resp.* m_v*) for every element* g *of* U, *then* f *is an element of* $\bar{v}$ *(resp.* $m_{\bar{v}}$*).*

4) $\bigcap_{g \in G(k)} v^g \subseteq \bar{v} \subseteq \bigcup_{g \in G(k)} v^g$, $\bigcap_{g \in G(k)} m_v^g \subseteq m_{\bar{v}} \subseteq \bigcup_{g \in G(k)} m_v^g$.

Proof. 1) and 3). Let f be an element of $\bar{v}$ and let g and x be independent generic points of G and X over k. Then we have that $\sigma^*(f)(g, x)=f^{g^{-1}}(x)=\frac{\sum a_\alpha(g)b_\alpha(x)}{\sum c_\beta(g)d_\beta(x)}$ where $\{a_\alpha, c_\beta\}$ are elements of $k(G)$ and $\{b_\alpha, d_\beta\}$ are elements of v and $\sum c_\beta\otimes d_\beta\notin k(G)\otimes m_v$. Furthermore if f is an element of $m_{\bar{v}}$, then $\sum a_\alpha\otimes b_\alpha\in k(G)\otimes m_v$. At any case, there is a non-empty open subset U of $G(k)$ such that $f^{g^{-1}}$ is an element of v or m_v. If f is an element of $k(X)$ and if f satisfies the last condition of 3), then it is easily seen that f is contained in $\bar{v}$ or $m_{\bar{v}}$ by the same method. Let g' be an element of $G(k)$. $\sigma^*(f^{g'})(g, x)=f^{g'}(g, x)=(f^{g'})^{g^{-1}}(x)=(f^{g^{-1}g'})(x)=\frac{\sum a_\alpha^{g'}(g)b_\alpha(x)}{\sum c_\beta^{g'}(g)d_\beta(x)}$. On the other hand, $k(G)\otimes m_v$ is stable under the ring isomorphism induced by the morphism $L_{g'}$; $G\times X\ni(g, x)\rightarrow(g'^{-1}g, x)\in G\times X$. Therefore, $\bar{v}$ is $G(k)$-stable.

4) follows immediately from 3). Let f be an element of $0_{\bar{x},X}$ where $\bar{x}$ is a generic point of the orbit $G(x)$ of x over k and let g be a generic point of G over $k(x)$. Since $f^{g^{-1}}$ is defined at x, $f^{g^{-1}}=\frac{\sum a_\alpha(g)b_\alpha}{\sum c_\beta(g)d_\beta}$ where $\{b_\alpha, d_\beta\}$ are elements of $0_{x,X}$ and $\sum c_\beta(g)\otimes d_\beta\notin k(g)\otimes m_x$. Since v dominates x, i.e. $v\supseteq 0_{x,X}$ and $m_v\cap 0_{x,X}=m_x$, $\sigma^*(f)=\frac{\sum a_\alpha\otimes b_\alpha}{\sum c_\beta\otimes d_\beta}\in(k(G)\otimes v)_{k(G)\otimes m_v}$. Hence f is an element of $\bar{v}$ and $\bar{v}$ contains $0_{\bar{x},X}$. Let f be an elements of $m_{\bar{x},X}$. Then $\sigma^*(f)=\frac{\sum a_\alpha\otimes b_\alpha}{\sum c_\beta\otimes d_\beta}$, where $\{a_\alpha, c_\beta\}$ are elements of $k(G)$ and $\{b_\alpha, d_\beta\}$ are elements of $0_{x,X}$ and where $\sum a_\alpha\otimes b_\alpha\in k(G)\otimes m_x$, $\sum c_\beta\otimes d_\beta\notin k(G)\otimes m_x$. Since $m_v\cap 0_{x,X}=m_x$, $\sigma^*(f)$ is contained in $(k(G)\otimes m_v)_{k(G)\otimes m_v}$. Hence $m_{\bar{v}}\cap 0_{\bar{x},X}=m_{\bar{x}}$. q.e.d.

The G-twisted valuation ring $\bar{v}$ of v is characterized by the property 3) of Lemma 10.

Lemma 11. *Under the situation of Definition* 1, *let* v' *be a valuation ring of* $k(X)$ *satisfying the property* 3) *of Lemma* 10, *i.e. for every element* f *of* v' *(resp.* $m_{v'}$*), there is a non-empty open subset*

U of G(k) such that $f^{g^{-1}}$ is an element of v (resp. m_v) for every element g of U. Then, $\bar{v}=v'$.

Proof. Let f be an element of $\bar{v}$. Then there is a non-empty open subset U of $G(k)$, such that $f^{g^{-1}}$ is an element of v for every element g of U. If $f\notin v'$, then $f^{-1}\in m_{v'}$. Therefore, there is a non-empty open subset V of $G(k)$ such that $(f^{-1})^{g^{-1}}$ is an element of m_v for every element g of V. Since G is connected, $U\cap V\neq\phi$. This is a contradiction. Hence, $\bar{v}\subseteq v'$. The inverse inclusion relation is proved similarly. q.e.d.

Corollary 4. *Let G be a connected algebraic group and let X be a G-homogeneous variety. If v is a valuation ring of k(X) which dominates a point of X. Then we have that* $\bigcup_{g\in G(k)} v^g=k(X)$ *and* $\bigcap_{g\in G(k)} m_v^g=(0)$.

Proof. Since $\bar{v}=k(X)$, $m_{\bar{v}}=(0)$, Corollary 4 is easily seen by Lemma 10. 4). q.e.d.

Corollary 5. *Let X be a variety on which G acts regularly and let R be a G(k)-stable local ring in k(X). Then there is a G(k)-stable valuation ring v of k(X) which dominates R.*

Proof. Let v' be a valuation ring of $k(X)$ which dominates R and let $v=\bar{v}'$ be the G-twisted valuation ring of v'. Then v is the desired one. In fact, $R\subseteq\bigcap_{g\in G(k)} v'^g\subseteq\bar{v}'=v$ by virtue of Lemma 10. Furthermore, m_R (the maximal ideal of $R)=\bigcap_{g\in G(k)} m_R^g=\bigcap_{g\in G(k)}(m_{v'}\cap R)^g$ $=\bigcap_{g\in G(k)}(m_{v'}^g\cap R)=(\bigcap_{g\in G)k)} m_{v'}^g)\cap R\subseteq m_v\cap R$ by virtue of Lemma 10. Hence, $m_R=m_v\cap R$. q.e.d.

Lemma 12. *Let X be a variety on which a connected algebraic group G acts regularly and let v be a valuation ring of k(X) and let $\bar{v}$ be the G-twisted valuation ring of v. Then v dominates a point of X if and only if $\bar{v}$ dominates a point of X.*

Proof. It is enough to prove that if $\bar{v}$ dominates a point of X, then v dominates a point of X. It is easily seen that a valuation ring v dominates a point of X if and only if v contains a coordinate ring of an affine open subset of X. Therefore, there is an affine open subset U of X such that $\bar{v}\supseteq A=k[f_1,\dots,f_n]$ where A is the coordinate ring of U. By virtue of Lemma 10, for every i $(1\leq i\leq n)$, there is an open subset U_i of $G(k)$ such that $f_i^{g^{-1}}$ is an element of v for every element g of U_i. Since G is connected, $U_1\cap\cdots\cap U_n\neq\phi$. Therefore, $A^{g^{-1}}=k[f_1^{g^{-1}},\dots,f_n^{g^{-1}}]$ is contained in v for every element g of $U_1\cap\cdots\cap U_n$. Hence v dominates a point of X.

q.e.d.

We shall next study the rational rank, rank and dimension of v.

Lemma 13. *Let K and $L(K\supset L)$ be extension fields of k and let v be a valuation ring of K and let $v'=v\cap L$ be the restriction of v on L. Then we have that*

1) *rational rank* $v'\leq$ *rational rank* v, *rank* $v'\leq$ *rank* v.
2) $\dim v'\leq\dim v$.
3) *If K and L are algebraic function fields over k, then rational rank* $v'+\dim v'+\mathrm{tr}_L K\geq$ *rational rank* $v+\dim v$.

Proof. We can prove easily Lemma 13 by elementary calculations. See also the appendix 2 of [9]. q.e.d.

Lemma 14. *Let X be a variety on which G acts regularly and let v be a valuation ring of $k(X)$ and let $\bar{v}$ be the G-twisted valuation ring of v. Then we have that*

1) *rational rank* $\bar{v}\leq$ *rational rank* v, *rank* $\bar{v}\leq$ *rank* v.
2) $\dim v\leq\dim\bar{v}\leq\dim v+\dim G$.
3) *rational rank* $\bar{v}+\dim\bar{v}\geq$ *rational rank* $v+\dim v$.

Proof. Let v' be the valuation ring $(k(G)\otimes v)_{k(G)\otimes m_v}$ of $Q(k(G)\underset{k}{\otimes}k(X))$. Then v' is an extension of v in $Q(k(G)\underset{k}{\otimes}k(X))$ and $\bar{v}=\sigma^{*-1}(v'))$. Therefore rational rank $v'=$ rational rank v, rank $v'=$ rank v and $\dim v'=\dim G+\dim v$. Hence 1) and 3) are obvious and $\dim\bar{v}\leq$

$\dim v + \dim G$ by virtue of Lemma 13. Furthermore, $\dim \bar{v} - \dim v \geq$ rational rank $v-$ rational rank $\bar{v} \geq 0$ by 1) and 3). Hence, $\dim \bar{v} \geq \dim v$.

q.e.d.

Remark 4. Let X be a variety on which a connected algebraic group G acts regularly and let $ZR(X)$ be the Zariski-Riemann space of X, i.e. $ZR(X) = \{v|$ valuation ring v of $k(X)$ which dominates a point of $X\}$. Then it is well-known that $ZR(X)$ is a quasi-compact topological space. Lemma 11 implies that a valuation ring v of $k(X)$ is an element of $ZR(X)$ if and only if the G-twisted valuation ring $\bar{v}$ of v is an element of $ZR(X)$. Lemma 10 implies that if v is an element of $ZR(X)$, then $\bar{v}$ is $G(k)$-stable and is contained in the closure of the orbit $G(k)(v)$ of v in $ZR(X)$ and that if v_1 and v_2 are valuation rings of $k(X)$ such that $v_1 = v_2^g$ for an element g of $G(k)$, then $\bar{v}_1 = \bar{v}_2$. Conversely, if $\bar{v}_1 = \bar{v}_2$ for two elements v_1, v_2 of $ZR(X)$, what can we say about v_1 and v_2? Finally we shall make the following remark.

Let $\varphi: G(k) \times ZR(X) \ni (g, v) \rightarrow v^g \in ZR(X)$ be the operation. Is this φ continuous under the product topology on $G(k) \times ZR(X)$? The answer is no. There is a following easy counter example.

Example. Let $G = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \right\}$ be a parabolic subgroup of $GL(3, k)$ and let X be P^2(projective space of 2-dimension) on which G acts canonically. Let $v = k\left[\frac{X}{Z}, \frac{Z}{Y}\right]_{\frac{Z}{Y}k[\frac{Z}{Y}, \frac{Z}{X}]}$ in $k\left(\frac{X}{Z}, \frac{Y}{Z}\right) = k(P^2)$. Then v is an element of $ZR(X)$. We shall assume that φ is continuous at (e, v) where e is the unit element of $G(k)$. Therefore, for any open neighbourhood V of v, there is an open subset U on $G(k)$ such that v^g is contained in V for every element g of U. However, this contradicts with the following.

Let $f = \frac{X}{Z}$ and let $V = \{v \in ZR(X) | v \ni f\}$. Then $f^{g^{-1}} =$

$$\frac{a_{11}X + a_{12}X + a_{13}Z}{a_{33}Z} = \frac{a_{11}}{a_{33}} \cdot \frac{X}{Z} + \frac{a_{12}}{a_{33}} \cdot \frac{Y}{Z} + \frac{a_{13}}{a_{33}} \text{ where } g = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}.$$

Hence $f^{g^{-1}}$ is contained in v if and only if $a_{12} = 0$. This contradicts our assumption.

5. Main Theorem.

Let X be a normal algebraic variety on which a linear algraic group G (not necessarily connected) acts regularly. In this section, we shall prove that there is a G-completion (or eqivariant completion) $\bar{X}$ of X, i.e. X is embeded as a G-stable open subset of a complete variety $\bar{X}$ on which G acts regularly. In [4, 5] Nagata proved that there is a completion $\bar{X}$ of X for any variety X. His method is effective in our case, too. Therefore, with above preparations on G-twisted valuation rings, we shall follow it in order to prove our main theorem. We note here that crucial algebraic sets which show up in the process of the proof are $G(k)$-stable.

At first, we shall recall several notations which were used in [4, 5].

Notations.

1) Let $f: X \to X'$ be a birational map. Then the set of points of X at which f is regular is denoted by $D_{X,X'}$ and the set of points at which f is biregular is denoted by $X \cap X'$.

2) Let X be a variety and let $f_i: X_1 \to X_i (i=2,\dots, n)$ be a birational map over X, i.e. for every $X_i (i=1, 2,\dots, n)$, there is a canonical morphism $p_i: X_i \to X$ which satisfies $p_1 = p_i f_i$. Then we shall denote the closure of $\{(x, f_2(x),\dots, f_n(x)) | x \in X_1 \cap X_2 \cap \cdots \cap X_n\}$ in $X_1 \underset{X}{\times} X_2 \times \cdots \underset{X}{\times} X_n$ by $J_X(X_1, X_2,\dots, X_n)$ and we shall call it the join of $\{X_1, X_2, \dots, X_n\}$ over X. If X is a point, then we shall simply denote it by $J(X_1, X_2,\dots, X_n)$ and we shall call it the join of $\{X_1, X_2,\dots, X_n\}$.

3) Let X and X' be birational varieties and let x and x' be points of X and X' respectively. If (x, x') is contained in the join $J(X, X')$ of $\{X, X'\}$, then we shall say that x and x' correspond to each other and we shall denote it by $x \sim x'$. It is easily seen that x and x' correspond to each other if and only if there is a valuation ring v of $k(X)$ such that v dominates x and x', i.e. $v \geq 0_{x,X}$ and $v \geq 0_{x',X'}$. Let f be the birational map between X and X'. If f is regular at every point x of X which corresponds to a point of X', we shall say that X is quasi-dominant over X'.

4) Let X and X' be birational varieties and let Y' be a subset of X'. We shall denote the set of all points x of X such that x corresponds to a point of Y' by $T_{X',X}(Y')$.

From now on, we shall frequently consider varieties on which a connected linear algebraic group G acts regularly. Hence, for simplicity, we shall call such varieties G-varieties.

Lemma 15. *Let G be a connected linear algebraic group and let X and X' be G-birational G-varieties and let v be an element of $ZR(J(X, X'))$. Then there is a G-variety X'' such that*

1) *X'' is G-projective and G-birational over X.*

2) $X \cap X' \subseteq X''$.

3) *If x' and x'' are points of X' and X'' which are dominated by v, then x'' dominates x', i.e. $0_{x'',X''} \geq 0_{x',X'}$.*

Proof. Let $f: X \to X'$ be the G-birational map and let $\bar{v}$ be the G-twisted valuation ring of v and let (x, x') and $(\bar{x}, \bar{x}')$ be the points of $J(X, X')$ which are dominated by v and $\bar{v}$ respectively. If Lemma 15 is true for $\bar{v}$, then Lemma 15 is also true for v. In fact, let X'' be the G-variety which satisfies the conditions 1), 2) and 3) for $\bar{v}$. Let $\bar{f}: X'' \to X$ be the G-projective and G-birational map and let $\bar{x}''$ be the point of X'' which is dominated by $\bar{v}$. Then, by 3) $0_{\bar{x}'',X''} \geq 0_{\bar{x}',X'}$. Therefore, $f \circ \bar{f}$ is a G-birational map from X'' to X' and is regular at $\bar{x}''$. By virtue of Lemma 10, $\bar{x}''$ is a generic point of the orbit $G(x'')$ of x'', hence $\bar{x}'' = g \cdot x''$ where g is a generic point of G over $k(x'')$. Since $f \circ \bar{f}$ is G-birational, $f \circ \bar{f}$ is regular at x'' and x'' dominates x'. Therefore, X'' is a desired one. Since $\bar{v}$ is $G(k)$-stable, we may assume that v is $G(k)$-stable. We shall prove Lemma 15 by induction on rank v. If rank $v=0$, i.e. $v=k(X)$ or x dominate x', then we may take $X''=X$. Hence we may assume that rank $v \geq 1$ and x does not dominate x'. Since v is $G(k)$-stable, every prime ideal of v is also $G(k)$-stable. Let m_v be the maximal ideal of v and let p be the prime ideal of v which is the next prime ideal of m_v with respect to inclusion. By the induction hypothesis, we may assume that $0_{y,X} \geq 0_{y',X'}$ where y and y' are points of

X and X' which are dominated by v_p. Let U be an affine open neighbourhood of x' in X' and let $B=k[b_1,\dots, b_n]$ be the coordinate ring of U. Since $0_{y',X'}$ is a quotient ring of B and $0_{y',X'}\subseteq 0_{y,X}$, every $b_i(1\leqq i\leqq n)$ is contained in $0_{y,X}$. If every $b_i(1\leq i\leq n)$ is contained in $0_{x,X}$, then $0_{x',X'}\leq 0_{x,X}$. Therefore, $b_{i_0}\notin 0_{x,X}$ for some i_0 and there is an element s of m_x such that $s\notin p\cap 0_{x,X}$ and $s\ b_i\in 0_{x,X}(1\leqq {}^{\forall}i\leq n)$. Put $\mathfrak{a}_x=\sum_{g\in G(k)}(s^g v\cap 0_{x,X})$. Then $\mathfrak{a}_x$ is a $G(k)$-stable ideal, because v and $0_{x,X}$ are $G(k)$-stable. Furthermore, $\mathfrak{a}_x$ is m_x-primary ideal. Let $i\colon \mathrm{Spec}(0_{x,X})\to X$ be the embedding and let $\theta\colon 0_X\to i_*(0_{x,X})$ be the canonical sheaf homomorphism. Put $I=\theta^{-1}(i_*(\mathfrak{a}_x))$. Then I is a $G(k)$-stable quasi-coherent ideal of 0_X and the closed subset of X defined by I is contained in $X-(X\cap X')$. Let X'' be the blowing up of X with center I. Then X'' is a G-projective, G-birational over X, and X'' is a desired one. In fact, let $\{a_1, a_2,\dots, a_r\}$ $(a_1=s)$ be a generator of $(sv\cap 0_{x,X})$ as $0_{x,X}$-module. Then, for every element g of $G(k)$, $\{a_1^g,\dots, a_r^g\}$ is a generator of $s^g v\cap 0_{x,X}$. Since $0_{x,X}$ is noetherian, there are finitely many elements $\{g_1,\dots, g_m\}(g_i\in G(k))$ such that $a_i^{g_j}(1\leq i\leq r,\ 1\leq j\leq m)$ is a generator of $\mathfrak{a}_x$ as $0_{x,X}$-module. Without loss of generality, we may assume that $v(s^{g_1})=\min_{1\leq i\leq m}\{v(s^{g_i})\}$. Put $t=s^{g_1}$ and $C=0_{x,X}\left[\frac{a_i^{g_j}}{t}\right](1\leq i\leq r,\ 1\leq j\leq m)$. Then C is contained in v. If we shall put $q=C\cap m_v$, then $v\geq C_q\geq 0_{x,X}$. Since the coordinate ring $k[b_1^{g_1},\dots, b_n^{g_1}]$ of U^{g_1} which is an affine open neighbourhood of x'^{g_1} is contained in C_q, $C_q\geq 0_{x'^{g_1},X'}=0_{x',X'}$. On the other hand, $C_q=0_{x'',X''}$. q.e.d.

Lemma 16. *Let X and X' be G-birational G-varieties. Then there is a G-variety X'' such that*

1) *X'' is G-projective and G-birational over X.*
2) *$X\cap X'\subseteq X''$*
3) *X'' is quasi-dominant over X'.*

Proof. For every element v of $ZR(J(X, X'))$, there is a G-variety X_v which satisfies the conditions 1), 2) and 3) of Lemma 15. Let $f_{J(X_v,X'),X_v}$ be the restriction on $J(X_v, X')$ of the first projection map $(X_v \times X') \to X_v$. Then $f_{J(X_v, X'), X_v}$ is a G-birational morphism between $J(X_v, X')$ and X_v. Put $U_v = \{v' \in ZR(J(X, X')) | v'$ dominates a point of $f^{-1}_{J(X_v, X'), X_v}(D_{X_v, X'})\}$. Then U_v is an open neighbourhood of v in $ZR(J(X, X'))$. Since $ZR(J(X, X'))$ is quasi-compact, there are finitely many elements $\{v_1, \dots, v_n\}$ of $ZR(J(X, X'))$ such that $ZR(J(X, X')) = \bigcup_{i=1}^{n} U_{v_i}$. Here, we shall put $X'' = J_X(X_{v_1}, \dots, X_{v_n})$. Then X'' is a G-variety which is G-projective and G-birational over X and $X \cap X' \subseteq X''$. X'' is a desired one. In fact, let (x'', x') be a point of $J(X'', X')$ and let v be an element of $ZR(J(X'', X'))$ such that v dominates (x'', x'). If x is the point of X which is the image of x'' by the G-projective morphism from X'' to X, then $v \geq 0_{x,X}$ and v is contained in $ZR(J(X, X'))$. Therefore there is some open subset U_{v_i} which contains v. If we shall denote the point by x_i which is dominated by v in X_{v_i}, then $0_{x_i, X_{v_i}} \geq 0_{x', X'}$. On the other hand, $0_{x'', X''} \geq 0_{x_i, X_{v_i}}$. Hence $0_{x'', X''} \geq 0_{x', X'}$. q.e.d.

The next Lemma 17 is one of key lemmas to prove the existence of G-completion.

Lemma 17. *Let X be a normal G-variety and let v be a valuation ring of $k(X)$. Then there is a G-variety X' such that*

1) *X is a G-stable open subset of X'.*
2) *v dominates a point of X'.*

Proof. Let $\bar{v}$ be the G-twisted valuation ring of v. If Lemma 17 is true for $\bar{v}$, then Lemma 17 is true for v by virtue of Lemma 11. Hence we may assume that v is $G(k)$-stable. We shall prove Lemma 17 by induction on rank v. If rank $v=0$ or v dominates a point of X, then we may take $X'=X$. Therefore, we may assume that rank $v \geq 1$ and that v does not dominate any point of X. Let p be the next prime ideal of m_v with respect to inclusion. By the induction hypothesis, we may assume that v_p dominates a point x of X. By virtue of Lemma 8, there is a G-stable quasi-projective open neighbourhood U of x and by virtue of Theorem 1, there is a G-completion $\bar{U}$ of U. Applying Lemma 16 to $\bar{U}$ and X, we have a complete G-variety X^* such that X^* is G-birational to X, $\bar{U} \cap X \subseteq X^*$ and X^*

is quasi-dominant over X. Since X^* is complete, v dominates a point x^* of X^*. Let $p: J(X, X^*)\to X^*$ be the second projection and let $Z^*=p(J(X, X^*))-X\cap X^*$ and let $\bar{Z}^*$ be the closure of Z^* in X^*. Then $\bar{Z}^*$ is a G-stable closed subset of X^*.

Case 1. $x^*\notin\bar{Z}^*$. $X^*-\bar{Z}^*$ is a G-stable open subset of X^* which containes $X\cap X^*$ and $X^*-\bar{Z}\ni x^*$. Let X' be the G pre-variety which is obtained by patching X and $X^*-\bar{Z}^*$ along $X\cap X^*$. Then X' is a desired one. It is enough to prove that X' is separated. Let v' be a valuation ring of $k(X)$ such that v' dominates points y and y^* of X and $X^*-\bar{Z}^*$ respectively. We shall prove that $0_{y,X'}=0_{y^*,X'}$. Since $y\sim y^*$ and $y^*\in X^*-\bar{Z}^*$, y^* is contained in $X\cap X^*$. Hence $0_{y,X'}=0_{y^*,X'}$ because $X\cap X^*$ is separated.

Case 2. $x^*\in\bar{Z}^*$. Since x is contained in $X\cap X^*$, $\bar{Z}^*$ does not contain x. Therefore, there is an element s of $0_{x^*,X^*}$ such that $x\notin p\cap 0_{x^*,X^*}$ and s is contained in the ideal defined by $\bar{Z}^*\cap\operatorname{Spec}(0_{x^*,X^*})$ in $\operatorname{Spec}(0_{x^*,X^*})$. Put $\mathfrak{a}_{x^*}=\sum_{g\in G(k)}(s^g v\cap 0_{x^*,X^*})$. Then $\mathfrak{a}_{x^*}$ is a $G(k)$-stable ideal and an m_{x^*,X^*}-primary ideal. Let $i:\operatorname{Spec}(0_{x^*,X^*})\to X^*$ be the injection map and let $\theta: 0_{X^*}\to i_*(0_{x^*,X^*})$ be the canonical sheaf homomorphism. Put $I=\theta^{-1}(i_*(\mathfrak{a}_{x^*}))$. Then I is a $G(k)$-stable quasi-coherent ideal of 0_{X^*}. Let X^{**} be the blowing up of X^* with center I. X^{**} is a G-variety which is G-projective and G-birational over X^*. Let $\overline{\{x^*\}}$ be the closure of x^* in X^*. Then $Z^*\cap\overline{\{x^*\}}=\phi$. In fact, if y^* is an element of $Z^*\cap\overline{\{x^*\}}$, then there are a point y of X and a valuation ring v' of $k(X)$ such that $v'\geq 0_{y,X}$ and $v'\geq 0_{y^*,X^*}$. Since X^* is quasi-dominant over X, $0_{y^*,X^*}$ dominates $0_{y,X}$. Let q be the prime ideal defined by $\overline{\{x^*\}}$ in $\operatorname{Spec}(0_{y^*,X^*})$ and let $r=q\cap 0_{y,X}$. Then we have that $0_{x^*,X^*}=(0_{y^*,X^*})_q\geq(0_{y,X})_r=0_{x',X}$ where x' is a point of X. Since v dominates $0_{x^*,X^*}$, v dominates $0_{x',X}$. This contradicts with our first assumption that v does not dominate any point of X. Thus, $Z^*\cap\overline{\{x^*\}}=\phi$. Therefore, the blowing up of X^* with center I does not have any effect upon Z^* and $X\cap X^*=X\cap X^{**}$. Let Z^{**} be the subset of X^{**} which is obtained by the same method as the construction of Z^*. Then $Z^*=Z^{**}$ because X^* is quasi-dominant over X. Let x^{**}

be the point of X^{**} which is dominated by v. Then x^{**} is not contained in $\bar{Z}^{**}$. In fact, if x^{**} is contained in $\bar{Z}^{**}$, then is there a generalization z^{**} of x^{**} in Z^{**}. If z^* is the image of z^{**}, then z^* is a generalization of x^*. Let q be the ideal defined by $\bar{Z}^* \cap \mathrm{Spec}(0_{x^*,X^*})$ and let $\{a_1=s, a_2,\dots, a_r\}$ be a generator of $sv \cap 0_{x^*,X^*}$. Since $0_{x^*,X^*}$ is noetherian, there are finite number of elements $\{g_1, \dots, g_m\}(g_i \in G(k))$ such that $a_i^{g_j}(1 \leq i \leq r,\ 1 \leq j \leq m)$ is a generator of $\mathfrak{a}_{x^*}$. Without loss of generality, we may assume that $v(a_1^{g_1}) = \min_{1 \leq j \leq m} \{v(a_1^{g_j})\}$ We shall put $t = a_1^{g_1}$. Then t is contained in q, because q is $G(k)$-stable. Since $\mathfrak{a}_{x^*}$ is an m_{x^*,X^*}-primary ideal, there is some element $a_{i_0}^{g_{j_0}}$ which is not contained in q. On the other hand, $0_{x^{**},X^{**}} = 0_{x^*,X^*}\left[\frac{a_i^{g_j}}{t}\right]_w$, where $w = 0_{x^*,X^*}\left[\frac{a_i^{g_j}}{t}\right] \cap m_v (1 \leq i \leq r,\ 1 \leq j \leq m)$. Therefore, for every i and j, $a_i^{g_j}/t \in 0_{x^{**},X^{**}} \subseteq 0_{z^{**},X^{**}} = 0_{z^*,X^*} = (0_{x^*,X^*})_q$. In particular, $a_{i_0}^{g_{j_0}}/t = c/u$ for some $u \notin q$ and $c \in 0_{x^*,X^*}$. $ua_{i_0}^{g_{j_0}} = tc$ gives a contra diction, because $ua_{i_0}^{g_{j_0}}$ is not contained in q and tc is contained in q. Therefore, the situation is reduced to case 1. q.e.d.

Let X be a G-variety and let X^* be a G-projective variety and let $f\colon X \to X^*$ be a generically surjective G-rational map from X to X^*. We assume that the action of G on X^* is linear, i.e. if $B = k[t_0, \dots, t_n]$ is the homogeneous coordinate ring of X^*, then $V = \sum_{i=0}^{n} t_i \Omega$ is a rational projective G-module. For every point x of X, we shall define the ideal $\mathfrak{a}_x$ of $0_{x,X}$ in the following way. Let A_x be the subset of all elements f of $k(X)$ such that for some $i(0 \leq i \leq n)$ and every $j(0 \leq j \leq n)$, $f \cdot t_j/t_i$ is contained in $0_{x,X}$ and let $\mathfrak{a}_x$ be the ideal of $0_{x,X}$ which is generated by $f \cdot t_j/t_i(0 \leq {}^\forall j \leq n,\ {}^\forall f \in A_x)$. We shall define $\mathfrak{a} = (\mathfrak{a}_x)_{x \in X}$. Then $\mathfrak{a}$ is a quasi-coherent ideal of 0_X.

Lemma 18. *$\mathfrak{a}$ is $G(k)$-stable.*

Proof. Let f be an element of A_x, i.e. for some $i(0 \leq i \leq n)$ and every $j(0 \leq j \leq n)$, $f \cdot t_j/t_i$ is contained in $0_{x,X}$. We have only to prove that for every element g of $G(k)$, $f^g \cdot t_j^g/t_i^g$ is contained in $\mathfrak{a}_{gx}$. By the assumption, $f^g \cdot t_j^g/t_i^g(0 \leq j \leq n)$ is contained in $0_{gx,X}$. Let α

and β be any integer $(0\leq\alpha\leq n, 0\leq\beta\leq n)$. Then $f^g\cdot t_\beta/t_\alpha=f^g\cdot\sum_\gamma a_{\beta\gamma}(g)t_\gamma^g/\sum_\gamma a_{\alpha\gamma}(g)t_\gamma^g=f^g\cdot\sum a_{\beta\gamma}(g)\cdot\frac{t_\gamma^g}{t_i^g}/\sum a_{\alpha\gamma}(g)\cdot\frac{t_\gamma^g}{t_i^g}$ where $(a_{\alpha\beta}(g))$ $(0\leq\alpha\leq n, 0\leq\gamma\leq n)$ is a regular matrix. Therefore, $(\sum_\gamma a_{\alpha\gamma}(g)f^g\cdot\frac{t_\gamma^g}{t_i^g}t_\beta/t_\alpha$ is contained in 0_{gx}. Put $h_\alpha=\sum_\gamma a_{\alpha\gamma}(g)f^g\cdot t_\gamma^g/t_i^g$ for every $\alpha(0\leq\alpha\leq n)$. Then h_α is an element of $\mathfrak{a}_{gx}$. Since $(a_{\alpha\gamma}(g))$ $(0\leq\alpha\leq n, 0\leq\gamma\leq n)$ is a regular matrix, $f^g\cdot t_\gamma^g/t_i^g(0\leq{}^\forall\gamma\leq n)$ is contained in $\mathfrak{a}_{gx}$. q.e.d.

Let Y be the closed subset of X defined by $\mathfrak{a}$ and let X' be the blowing up of X with center $\mathfrak{a}$. It is easily seen that Y is the G-stable set of all points x of X at which X does not dominate any point of X^* and that every point of X' dominates a point of X^*.

The next complicated Lemma 19 is another one of key lemmas to prove the existence of G-completion.

Lemma 19. *Let X_1 and X_2 be G-varieties such that X_1 is G-birational to X_2 and let $X=X_1\cap X_2$. Assume that X_1-X is contained as a G-stable subvariety (not necessarily closed) in a G-projective variety X^* which is G-birational to X_1 and that the action of G on X^* is linear. Then there is a G-variety X_3 such that*

1) *X_3 contains X as G-stable open subset*
2) *$ZR(X_3)=ZR(X_1)\cup ZR(X_2)$.*

Proof. We may assume that X_2 is quasi-dominant over X_1 by virtue of Lemma 16. Let $Y=X-(X^*\cap X)$ and let $Y^*=X^*-(X^*\cap X_1)$. Then Y and Y^* are G-stable closed subset in X and X^* respectively.

a) $Y=T_{X^*,X}(Y^*)$. In fact, let x be a point of $T_{X^*,X}(Y^*)$. Then there is a point x^* of Y^* such that $x\sim x^*$. If x is contained in $X^*\cap X$, then $0_{x,X^*}=0_{x^*,X^*}$ because x and x^* are points of X^*. Since $Y^*\cap(X\cap X^*)=\phi$, this does not occur. Hence $T_{X^*,X}(Y^*)\subseteq Y$. Conversely, let x be a point of Y. Since X^* is complete, there is a point x^* of X^* such that $x\sim x^*$. If x^* is contained in $X^*\cap X_1$, then $0_{x,X_1}=0_{x^*,X_1}$. This is a contradiction because $Y\cap(X^*\cap X_1)=\phi$. Hence $Y\subseteq T_{X^*,X}(Y^*)$.

b) If we shall define $Z=X_2-T_{X_1,X_2}(X_1)$, i.e. the set of all points of X_2 which does not correspond to any point of X_1. Then $Z\cup Y=T_{X^*,X_2}(Y^*)$. By a) $Y\subseteq T_{X^*,X_2}(Y^*)$. Let x be a point of Z. Since X^* is complete, there is a point x^* of X^* such that $x\sim x^*$. By the definition of Z, x^* is contained in Y^*. Hence $Z\subseteq T_{X^*,X_2}(Y^*)$. Therefore, $Z\ \cup Y\subseteq T_{X^*,X_2}(Y^*)$. Conversely, let x be a point of $T_{X^*,X_2}(Y^*)$ and let x^* be the point of Y^* such that $x\sim x^*$. If x is contained in X, then x is a point of Y by a). Therefore we assume that x is not contained in X. If x is not contained in Z, then there is a point x_1 of X_1 such that $x\sim x_1$. Since X_2 is quasi-dominant over X_1, $x_1\sim x^*$. Hence $0_{x_1,X^*}=0_{x^*,X^*}$ because $X_1-X\subseteq X^*$. This is a contradiction because $Y^*\cap X_1=\phi$. Therefore $T_{X^*,X_2}(Y^*)=Z\cup Y$.

c) Let $p_1: J(X^*, X_2)\to X^*$ be the first projection and let $p_2: J(X^*, X_2)\to X_2$ be the second projection. Then $T_{X^*,X_2}(Y^*)=p_2(p_1^{-1}(Y^*))$ and it is a closed subset of X_2 because p_2 is a proper morphism. Hence $Z\cup Y$ is a G-stable closed subset of X_2. Let Y_2 be the closure of Y in X_2. Then Y_2 is a G-stable closed subset of X_2 and we have that $(+) Y_2-Y\subseteq Z$.

d) Let $W_2=T_{X^*,X_2}(X_1-X)$ and let $W_1=T_{X_2,X_1}(W_2)$, i.e. the set of all points x of X_1 such that x corresponds to a point of X_2 and x does not dominate the point. Now we change the situation. We may assume that X_1 is quasi-dominant over X_2. In fact, let J be a $G(k)$-stable quasi-coherent ideal of 0_{X_1} whose support is contained in the closure $\overline{W}_1$ of W_1 in X_1 and let $J^*=\theta^{-1}(i_*(J|X_1\cap X^*))$ where $(i, \theta): X_1\cap X^*\to X^*$ is the injection. Then J^* is a $G(k)$-stable quasi-coherent ideal of 0_{X^*}. Let X'_1 (resp. $X^{*\prime}$) be the blowing up of X_1 (resp. X^*) with the center J (resp. J^*) and let π (resp. π^*) be the cannoical projective morphism from X'_1 to X_1 (resp. from $X^{*\prime}$ to X^*). If we shall define similarly X', Y', $Y^{*\prime}$ and Z' with respect to X'_1, X_2 and $X^{*\prime}$ as X, Y, Y^* and Z, then $X'\supseteq X$, $T_{X^{*\prime},X'}(Y^{*\prime})=Y'=Y$, $Z'=Z$ and the relation $(+)$ is held. Furthermore, X'_1-X' is embedded in $X^{*\prime}$ as a G-stable subvariety and the action on $X^{*\prime}$ of G is linear. Therefore, iterating the above blowing up, we may assume that X_1 is quasi-dominant over X_2 by Lemma 15 and Lemma

16. Next we may assume that X^* is quasi-dominant over X_2. In fact, let J be a $G(k)$-stable quasi-coherent ideal of 0_{X^*} whose support is contained in $Y^* \cap (X^* - (X^* \cap X_2))$ and let $X^{*\prime}$ be the blowing up of X^* with the center J. If we shall define X', Y', $Y^{*\prime}$ and Z' with respect to X_1, X_2 and $X^{*\prime}$ similarly, then $X' = X, Z' = Z$ $Y' \subseteq Y$ and $\overline{Y}' - Y' \subseteq Z$ because Y' is closed in X, i.e. the relation $(+)$ is held. Therefore, we may assume that X^* is quasi-dominant over X_2.

e) Let $\overline{W}_2$ be the closure of W_2 in X_2. Then $Y_2 \cap \overline{W}_2 \subseteq Z$ because $Y \cap \overline{W}_2 = \phi$ and the relation $(+)$. Hence Y_2 and $\overline{W}_2$ do not have the same irreducible components.

If $Y_2 \cap \overline{W}_2 \neq \phi$, then we shall denote the ideals defined by Y_2 and $\overline{W}_2$ by I and I' respectively. Then, I and I' are $G(k)$-stable ideals of 0_{X_2}. Let X'_2 be the blowing up of X_2 with the center $I + I'$ and let $\varphi: X'_2 \to X_2$ be the canonical projective morphism. Then the proper transform of Y_2, $\overline{\varphi^{-1}(Y_2 - Y_2 \cap \overline{W}_2)}$ and the proper transform of $\overline{W}_2$, $\overline{\varphi^{-1}(\overline{W}_2 - Y_2 \cap \overline{W}_2)}$ do not meet with each other. Therefore, we may assume that $Y_2 \cap \overline{W}_2 = \phi$. Let $\mathfrak{a}$ be the ideal defined in Lemma 18 for X_2 and X^* and let P be the G-stable closed subset defined by $\mathfrak{a}$. For every point x of X_2, let $\mathfrak{a}_x = q_1 \cap \cdots \cap q_m$ be the irredundant decomposition of $\mathfrak{a}_x$ by primary ideals and let $\{q_{i_k}\}$ $(1 \leq k \leq s)$ be all those of primary ideals $q_i (1 \leq i \leq m)$ such that the closed subset $V(q_i) (\subset \operatorname{Spec}(O_{x,X_2}))$ is contained in $Z - Z \cap Y_2 - Z \cap \overline{W}_2$. Then we shall define $\mathfrak{b}_x$ by $\mathfrak{b}_x = q_{i_1} \cap \cdots \cap q_{i_s}$ and $\mathfrak{b} = (\mathfrak{b}_x)_{x \in X_2}$. $\mathfrak{b}$ is a $G(k)$-stable quasi-coherent ideal of 0_{X_2} because $\mathfrak{a}$ and $Z - Z \cap Y_2 - Z \cap \overline{W}_2$ are $G(k)$-stable. $\mathfrak{b}$ has the following properties;

1) The closed subset Q defined $\mathfrak{b}$ is G-stable and is contained in Z.

2) $\mathfrak{b}_x = \mathfrak{a}_x$ for every point x of $Z - Z \cap Y_2 - Z \cap \overline{W}_2$.

3) $Q = P \cap \overline{(Z - Z \cap Y_2 - Z \cap \overline{W}_2)}$.

f) Let X_2^* be the blowing up of X_2 with the center $\mathfrak{b}$ and let $f: X_2^* \to X_2$ be the canonical G-projective morphism. Let $X_3 = X_1 \cup J(X_2^* - Y_2^*, X^*) \cup (X_2^* - \overline{W}_2^*)$ where $Y_2^* = f^{-1}(Y_2)$ and $\overline{W}_2^* = f^{-1}(\overline{W}_2)$. Then X_3

is a desired one. Since X_3 is a G-prevariety satisfying the condition 1) of Lemma 19, it is enough to prove that $ZR(X_3)=ZR(X_1)\cup ZR(X_2)$. In order to prove this, it is sufficient to show that every v in $ZR(X_1)\cup ZR(X_2)$ has one and only one center on X_3. Let x_1, x^*, x_3 and x_2 be the centers of v on X_1, $J(X_2^*-Y_2^*, X^*)$, $X_2^*-W_2^*$ and X_2 respectively if they exist.

Case 1. When x_1 exists. If x_1 is contained in X, then x_1 is contained in $X_2^*-W_2^*$ and therefore $0_{x_1,X_3}=0_{x_2,X_3}=0_{x_3,X_3}$ since X_2^* is separated. If x^* exists, then x_2 is not contained in Y. Hence, $0_{x_1,X_3}=0_{x^*,X_3}=0_{x_3,X_3}$. If x_1 is contained in X_1-X, then x_3 does not exist in this case. If x^* exists, then x_2 is contained in W_2-Y_2. Since $W_2\cap Q=\phi$ and X^* is quasi-dominant over X_2, $0_{x_1,X_3}=0_{x^*,X_3}$.

Case 2. When x_1 does not exist. In this case, x_2 exists and x_2 is contained in Z. Since $Y_2^*\cap W_2^*=\phi$, $X_2^*=(X_2^*-Y_2^*)\cup(X_2^*-W_2^*)$. Therefore, either x^* or x_3 exists. Thus is it sufficient to prove that $0_{x^*,X_3}=0_{x_3,X_3}$ if both x^* and x_3 exist. Since x_2 is contained in $X_2-(Y_2\cup \overline{W}_2)$, x_3 dominates a point of X^*. Hence $x_3\in J(X_2^*-Y_2^*, X^*)$ and $0_{x^*,X_3}=0_{x_3,X_3}$. q.e.d.

Theorem 3. *Let X be a normal variety on which a linear algebraic group G (not necessarily connected) acts regularly. Then there exists a G-completion (or equivariant completion) $\overline{X}$ of X.*

Proof. At first, we shall assume that G is connected. Let X^* be a projective model of X. For every v in $ZR(X^*)$, there is a normal G-variety X_v such that X is a G-stable open subset of X_v and v has a center x_v in X_v. Let U_v be the G-stable quasi-projective open neighbourhood of x_v. Then $X\cup U_v$ is a G-variety and plays the same roll as X_v. Hence, considering $X\cup U_v$ instead of X_v if necessary, we may assume that X_v-X is contained in a quasi-projective variety on which G acts linearly. Since $ZR(X_v)$ is open in $ZR(X^*)$, there are finitely many G-varieties $\{X_1,\dots, X_n\}$ such that

1) $ZR(X^*)=\bigcup_{i=1}^{n} ZR(X_i)$

2) For every $i(1\leq i\leq n)$, X is a G-stable open subset of X_i and

$X_i - X$ is a G-subvariety of a quasi-projective variety on which G acts linearly.

We prove the Theorem 3 by induction on n. If $n=1$, then we have nothing to prove. If $n>1$, then, applying Lemma 19 to $\{X_{n-1}, X_n\}$, we see that there is a G-variety X^*_{n-1} such that $X^*_{n-1} \supseteq X_{n-1} \cap X_n \supseteq X$ and $ZR(X^*_{n-1}) = ZR(X_{n-1}) \cup ZR(X_n)$. Therefore we complete the proof by our induction assumption. We shall now consider the general case. Let G_0 be the connected component of G which contains the unit element e of G and let $G = \sigma_1 G_0 + \sigma_2 G_0 + \cdots + \sigma_n G_0$ where $\sigma_1 = e$, $\sigma_i \in G(k)(1 \leq i \leq n)$. By the above argument in the connected case, there is a G_0-completion of X and we shall denote it by X'. Let $\bar{X}$ be the closure of the set $\{(\sigma_1 x, \dots, \sigma_n(x) | x \in X\}$ in $X' \times \cdots \times X'$. Then $\bar{X}$ is a desired G-completion of X. In fact, let $\varphi: X \ni x \to (\sigma_1 x, \dots, \sigma_n x) \in \bar{X}$. Then φ is an open immersion from X to $\bar{X}$. Here we shall define an action of G on $\bar{X}$ in following way. Let $\sigma_i \sigma_j = \sigma_{l(i,j)} g_{(i,j)}$ $(1 \leq i, j \leq n)$ where $l(i, j)$ is an integer $(1 \leq l(i, j) \leq n)$ and $g(i, j)$ is an element of $G_0(k)$. Then, for every point $(y_1, \dots, y_n)$ of $\bar{X}$, every $\sigma_j (1 \leq j \leq n)$ and every element g of G_0, we define,

$$\sigma_j g(y_1, \dots, y_n) = (z_1, \dots, z_n)$$

where $z_i = \sigma_{l(i,j)} g_{(i,j)} g \sigma_{l(i,j)}{}^{-1} y_{l(i,j)}$ $(1 \leq {}^\forall i \leq n)$. We can see easily that this is a regular action of G on $\bar{X}$ and $\varphi(\sigma_j g x) = \sigma_j g \varphi(x)$ for every σ_j $(1 \leq j \leq n)$, $g(\in G_0)$ and $x(\in X)$. Therefore, $\bar{X}$ is a G-completion of X. q.e.d.

Problem. *Is Theorem* 3 *true, without assuming X is not normal*?

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