J. Math. Kyoto Univ. (JMKYAZ)
15-1 (1975) 231-236

On the pair of functions on C^2 reduced to a pair of rational functions

By

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(Communicated by Prof. Y. Kusunoki, March 26, 1974)

1. Statement of results

For the integral function of one complex variable, the property that a function is a polynomial or not is independent of the choice of the coordinate. But for the function of several variables, this property is not independent of the choice of the coordinate system. Then the question occurs ; what functions are intrinsic polynomials? Theorem 1 gives one criterion for such a question. Theorem 2 is a byproduct obtained in the process of the proof of Theorem 1.

Theorem 1. *Suppose a m-fold covering domain $\mathscr{D}$ over the product of the complex 1-dimensional projective space $P\times P$. And suppose $\mathscr{D}$ satisfies following conditions.*

i) $\mathscr{D}$ is biholomorphically equivalent to 2-dimensional complex plane C^2.

ii) There is no relative boundary over any point of the base space $P\times P$ except a point on a curve in $P\times P$.

iii) The projection of the set of points of the ramifying point is equal to a curve in $P\times P$ except possibly a discrete set of points. Then the projection mapping is a pair of rational functions with respect to an adequate coordinate system.

Theorem 2. *Let $S=\{P(x, y, z)=0\}$ be an algebraic surface in C^3. And suppose S is biholomorphically equivalent to C^2. Then S has a biholomorphic parametrization of polynomials*

$$\begin{cases} x=\varphi_1(u, v) \\ y=\varphi_2(u, v) \\ z=\varphi_3(u, v), \end{cases}$$

where $\varphi_i\,(i=1, 2, 3)$ is a polynomial in the (u, v)-space.

2. Continuation of a covering domain

Lemma 1. *Let M be a 2-dimensional complex manifold. And let $\mathfrak{D}$ be a m-fold covering domain over M satisfying following conditions.*

i) $\mathfrak{D}$ has no relative boundary over any point of M except a point on an analytic set C of M.

ii) The projection of the set of ramifying points R is equal to an analytic set of M sxcept possibly a discrete set of points.

Then $\mathfrak{D}$ can be continued to the covering domain $\tilde{\mathfrak{D}}$ without relative boundary (namely, there is an into holomorphic mapping from $\mathfrak{D}$ to $\tilde{\mathfrak{D}}$ commutative with the projection).

Proof. Set $A=\{p$; the number of elements of $\pi^{-1}(p)$ is less than $m\}$. For any point p of $M-A$, $\pi^{-1}(p)$ is a set of m-points $\{a_1, \ldots, a_m\}$. For every index i, there is a neighborhood U_i of a_i such that $\pi: U_i \longrightarrow \pi(U_i)$ is a homeomorphism. And one can suppose U_i does not intersect U_j for any index $j \neq i$. Then the intersection of $\pi(U_i)$ $(i=1, \ldots, m)$ is contained in $M-A$. Then $M-A$ is an open set. Let ∂A be the set of boundary points of A. Becanse $M-A$ is open, ∂A is contained in A. Then ∂A is contained in $C \cup R$. If A contains an open set, ∂A is a 3-dimensional set. Because $C \cup R$ is 2-dimensional A does not contain any open set. Namely, A is equal to ∂A. Then A is contained in $C \cup R$. And trivially $C \cup R$ is contained in A. Then A is equal to $C \cup R$. Let

p be an arbitrary point of C. Let $U_p=\{(z, y)\ ;\ |z|<R,\ |y|<R\}$ be a coordinate neighborhood of p. Set $\delta(r_0)=\{(z, y)\ ;\ |z|<r_0,\ |y|<r_0\}$ where r_0 is smaller than R. Let $\mathscr{D}\ (r_0)$ be the restriction of $\mathscr{D}$ over $\delta(r_0)$. Let $D^1(r),\ldots, D^h(r)$ be connected components of $\mathscr{D}(r_0)$, where h is not larger than m. If r is smaller than r_0, there is at least one component of $\mathscr{D}(r)$ in every $D^i(r_0)$. Because h is not larger than m, for sufficiently small r_0 there is exact one component of $\mathscr{D}(r)$ in every $D^i(r_0)$. Then $\{D^i(r)\}$ determines a filter. Then h points are determined over the point p by these filters. Attaching these points a topological covering domain $\tilde{\mathscr{D}}$ is constructed, and $\tilde{\mathscr{D}}$ has no relative boundary. The restriction of $\tilde{\mathscr{D}}$ over a neighborhood of any point of the base space is an analytic cover* (it can be easily examined). Then the analytic structure is determined in $\tilde{\mathscr{D}}$ by Grauert-Remmert [1], and this structure is equal to the one of $\mathscr{D}$.

3. Proof of the theorems

For the covering domain $\mathscr{D}$, let $\tilde{\mathscr{D}}$ be the continuation of $\mathscr{D}$ constructed in 2. Let $\mathscr{D}_0$ be a nonsingular model of $\tilde{\mathscr{D}}$. $\mathscr{D}_0-\mathscr{D}$ is contained in an analytic set of $\mathscr{D}_0$. If $\mathscr{D}_0-\mathscr{D}$ is not an analytic set of $\mathscr{D}_0$, there is an irreducible analytic set A of $\mathscr{D}_0$ intersects $\mathscr{D}$ and $\mathscr{D}_0-\mathscr{D}$ simultaneously. Let $\{p_n\}$ be a sequence of points of $\mathscr{D}$, and suppose $\{p_n\}$ tends to the point of $A-\mathscr{D}$. There is a holomorphic function f on $\mathscr{D}$ such that $\{|f(p_n)|\}$ tends to infinity because $\mathscr{D}$ is a Stein manifold. On the other hand f has a holomorphic continuation to the union of $\mathscr{D}$ and A by the theorem of continuity of Thullen-Stein-Remmert [2] [3]. This is a contradiction. Then $\mathscr{D}_0-\mathscr{D}$ is an analytic set in $\mathscr{D}_0$. By blowing up and blowing down on $\mathscr{D}_0-C^2$, $\mathscr{D}_0$ is reduced to the manifold satisfying following conditions.

i) $\mathscr{D}_0-C^2=\bigcup_{i=1}^{N} Y_i$, where Y_i is an irreducible nonsingular curve in $\mathscr{D}_0$.

ii) Either $Y_i\cap Y_j$ is a null set or Y_i intersects Y_j normally at one point.

iii) $Y_i \cap Y_j \cap Y_k$ is a null set for any distinct triple (i, j, k).

iv) If the self intersection number $c^2(Y_i) = -1$, then Y_i has at least three intersection points.

Such a compactification of C^2 is called *minimal normal* in terms of Morrow [4]. And it is easily shown that every Y_i is a projective line P.

Lemma 2. *Let M be a compactification of C^2, and suppose $M - C^2$ is a curve in M. Then there is a birational correspondence Φ between M and P^2 such that the restriction of Φ on C^2-part of M is a into holomorphic mapping.*

Proof. It may be supposed that M is minimal normal. Then by Morrow [4] M is classified to 7 types. We proceed the proof using the classifying table of Morrow.

In case of the type (f).

(P_1). Blow up the intersection point of the lines of weight n and weight 0, and blow down the proper transformation of the line with weight 0. Repeat this operation n times.

(p_2) Blow down the proper transformation of the line with weight $-n-1$, then the line linked to this line becomes to be an exceptional curve of the first kind. Blow down this line, then the line linked to that becomes to be an exceptional curve of the first kind. Repeat this operation $k_1 - 1$ times.

(p_3) Then we get a graph of the type (g). And after the same oparation for this graph as for the original one, we get a graph of the type (f) with $p' = p - 1$.

Repeat this total operation p times, then we get a graph of the type (d).

In case of the type (g).

By the same operation as for the graph of the type (f), we get a graph of the type (e).

In case of the type (d) or the type (e). By the operation (p_1) and (p_2), we get a graph of the type (b).

In case of the type (b).

By the operation (p_1), we get a graph $\underset{0}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\circ$. This graph is correspondent to $P \times P$. Thus the Lemma is proved.

Proof of the theorem 1. Let Φ be the birational mapping from $\mathscr{D}_0$ to P^2 mentioned in Lemma 2. Then Φ induces an isomorphism of the meromorphic function field of P^2 to the one of $\mathscr{D}_0$. Let $\pi = (F, G)$ be the projection mapping from $\mathscr{D}_0$ to $P \times P$. And let f and g be the restrictions of F and G on $\mathscr{D}$, respectively, then $f \circ \Phi^{-1}$ and $g \circ \Phi^{-1}$ are rational functions on $X = \Phi(\mathscr{D})$. Let (u, v) be an afine coordinate in X. Then $(u \circ \Phi, v \circ \Phi)$ is an afine coordinate in $\mathscr{D}$. Then f and g are rational functions with respect to this coordinate system.

Proof of the theorem 2. S is a covering domain over the (x, y)-space, then S has a continuation $\tilde{S}$ over $P \times P$ by Lemma 1. Then S has a biholomorphic parametrization of polynomials by Lemma 2.

Note

*An analytic cover is a triple (X, π, U) such that

i) X is a locally compact Hausdorff space,

ii) U is a domain in C^n,

iii) π is a proper, fibre discrete, continuous mapping of X onto U,

iv) there are a negligible set A in U, and an integer λ, such that π is a λ-sheeted covering mapping from $X - \pi^{-1}(A)$ onto $U - A$,

v) $X - \pi^{-1}(A)$ is dese in X.

Where, a set A in U is said to be negligible if following conditions are satisfied.

a) A is a closed subset of U.

b) A is nowhere dense.

c) For every domain U' in U and every function f holomorphic on $U' - A$ and locally bounded on U', f has a unique holomorphic continuation $\tilde{f}$ to all of U'.

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