J. Math. Kyoto Univ. (JMKYAZ)
15-3 (1975) 573-605

Equivariant completion II

By

Hideyasu SUMIHIRO

(Communicated by Prof. Nagata, Sept. 5, 1974)

0. Introduction.

In this paper, we shall generalize the results obtained in [13]. Let S be a scheme and let G be a surjective smooth affine group scheme over S with connected fibres and let X be a normal noetherian S-scheme on which G acts regularly. We shall prove following three results;

a) For any line bundle L on X, there is a positive integer m such that $L^m (=L^{\otimes m})$ is G-linearizable (cf. Theorem 1.6). Moreover, if S is noetherian and if X is normal and quasi-projective over S, then there is a coherent $\mathcal{O}_S$-Module E (cf. Theorem 2.5) such that

(1) There is an immersion $\varphi: X \to P(E)$,

(2) There is a representation $\rho: G \to \mathrm{Aut}_S(P(E))$ and

(3) The following diagram is commutative.

$$\begin{array}{ccc} G \underset{S}{\times} X & \xrightarrow{\sigma} & X \\ \Big\downarrow{\scriptstyle \rho \times \varphi} & & \Big\downarrow \\ \mathrm{Aut}_S(P(E)) \underset{S}{\times} P(E) & \xrightarrow{\sigma'} & P(E) \end{array}$$

where $\sigma: G \underset{S}{\times} X \to X$ is the regular action of G on X and σ': $\mathrm{Aut}_S(P(E)) \times P(E) \to P(E)$ is the canonical action of $\mathrm{Aut}_S(P(E))$ on $P(E)$. Therefore, the regular action G on X is linear.

b) If S is normal, noetherian and if X is an S-scheme satisfying the property (N) (cf. Definition 3.5) on which G acts regularly, then X is covered by G-stable open subschemes $(U_i)_{1 \leq i \leq n}$ which are quasi-projective over S (cf. Theorem 3.8). Therefore, combining these results, every regular action of G on X is obtained by patching the quasi-projective S-scheme $(X_i)_{1\leq i\leq n}$ on which G acts linearly (cf. Theorem 4.9). Moreover, if G is of locally multiplicative type, then X is covered by G-stable open subschemes $(U_i)_{1\leq i\leq n}$ which are affine over S (cf. Corollary 3.11).

c) If S is normal, noetherian and if X is an S-scheme satisfying the property (N) on which G acts regulary, then there exists an equivariant completion $\bar{X}$ (cf. Theorem 4.13), i.e., $\bar{X}$ is an S-sheme on which G acts regularly such that

(1) $\bar{X}$ is proper over S,

(2) $\bar{X}$ contains X as a G-stable open dense subscheme and

(3) The action of G on $\bar{X}$ is the extension of the action of G on X.

In the proof of this main theorem, the author owes the most part to the results and arguments of P. Deligne [2].

Notation and convention. Let S be scheme and let G be an S-group scheme. We denote the multiplication of G by μ and the unit section of G over S by e. Let X be an S-scheme. The regular action of G on X is denoted by $\sigma: G\underset{S}{\times}X\to X$ and $p_2: G\underset{S}{\times}X\to X$ is the second projection. Moreover, for every point s of S, X_s is the fibre of X over s.

The author wishes to express his sincere thanks Professor M. Nagata for his many valuable discussions and encouragement and to Professor P. Deligne for his kind letters.

1. Preliminary results.

In this section, we shall prepare several results. Most of them are generalized ones of the results used in [13].

Lemma 1.1. *Let S be a normal noetherian scheme and let G be a surjective smooth affine group over S with connected fibres. Then $\underline{\mathrm{Pic}}_{G/S}(S)=\mathrm{Pic}(G)/\mathrm{Pic}(S)$ is a torsion group.*

Proof. It is well-known that if $S=\mathrm{Spec}\,(k)$ (k being a field), the $\mathrm{Pic}\,(G)$ is a torsion group. Hence, by virtue of [3] Err. IV. 21. 4. 13, Lemma 1. 1 is easily proved. q.e.d.

Using Lemma 1. 1, we shall prove the following.

Lemma 1. 2. *Let G be a surjective smooth affine group scheme over S (S being a scheme) with connecred fibres and let X be a normal noetherian S-scheme on which G acts regularly and let L be an invertible sheaf on X. Then there is a positive integer m such that $\sigma^*(L^m)\xrightarrow{\sim}p_2^*(L^m)$.*

Proof. $G\underset{S}{\times}X$ is a surjective smooth affine group scheme over X. By virtue of Lemma 1. 1, there is a positive integer m such that $\sigma^*(L^m)\otimes p_2^*(L^m)^{-1}\xrightarrow{\sim}p_2^*(M)$ where M is an invertible sheaf on X. Let $e: S\rightarrow G$ be the unit section of G. Restricting these invertible sheaves on the closed subscheme $e\underset{S}{\times}X$ of $G\underset{S}{\times}X$, we get that M is isomorphic to $\mathcal{O}_X$. Hence $\sigma^*(L^m)\xrightarrow{\sim}p_2^*(L^m)$. q.e.d.

For a while, we assume that S, G, X and L are under the situation of Lemma 1. 2 and let $\phi: \sigma^*(L^m)\xrightarrow{\sim}p_2^*(L^m)$ be an isomorphism whose existence has been shown in Lemma 1. 2. Let us modify the ϕ so nicely that it provides a G-linearlization of L. In order to do this, let us consider the following diagram;

(1)
$$\begin{tikzcd}
G\underset{S}{\times}G\underset{S}{\times}X \arrow[r, "1_G\times\sigma"] \arrow[d, "\mu\times 1_X"] & G\underset{S}{\times}X \arrow[d, "\sigma"] \\
G\underset{S}{\times}X \arrow[r, "\sigma"] & X
\end{tikzcd}$$

(2)
$$G\underset{S}{\times}G\underset{S}{\times}X \xrightarrow{\ p_{23}\ } G\underset{S}{\times}X$$

where p_{23} is the projection to the second and the third factors. From these we have the following diagram which is not necessarily commutative. Our present aim is to find ϕ which makes the diagram commutative.

$$\begin{tikzcd}
{[\sigma\circ(1_G\times\sigma)]^*(L^m)} \arrow[r, "(1_G\times\sigma)^*(\phi)"] \arrow[ddd, equal] & {[p_2\circ(1_G\times\sigma)]^*(L^m)} \arrow[d, equal] \\
& {[\sigma\circ p_{23}]^*(L^m)} \arrow[d, "p_{23}^*(\phi)"] \\
& {[p_2\circ p_{23}]^*(L^m)} \arrow[d, equal] \\
{[\sigma\circ(\mu\times 1_X)]^*(L^m)} \arrow[r] & {[p_2\circ(\mu\times 1_X)]^*(L^m)}.
\end{tikzcd}$$

Hence, the obstruction $(\mu\times 1_X)^*(\phi)\circ[p_{23}^*(\phi)\circ(1_G\times\sigma)^*(\phi)]^{-1}$ for the commutativity of the above diagram is in $\mathrm{Isom}([p_2\circ(\mu\times 1_X)]^*(L^m))$ $=H^0(G\underset{S}{\times}G\underset{S}{\times}X, \mathcal{O}^*_{G\underset{S}{\times}G\underset{S}{\times}X})$. Before computing the obstruction, let us recall some lemmas due to M. Raynaud ([9] Cor. VII. 1, 2 and Prop. VII. 1. 3).

Lemma 1. 3. *Let S be a reduced scheme and let G be a flat, locally finite presentation group scheme over S with smooth, connected maximal fibres and let f be an element of $H^0(G, \mathcal{O}_G{}^*)$ which takes 1 on the unit section of G. Then f is a character of G.*

Lemma 1. 4. *Let S be a normal noetherian scheme and let G be a flat, finite presentation S-group scheme such that G_s is smooth at every point $s(s\in S)$ of codimension 1. Then we have that*

$$(*)\qquad \mathrm{Hom}_{gr}(G, G_m)\xrightarrow{\sim}\prod_{\eta}\mathrm{Hom}_{gr}(G_\eta, G_{m,\eta})$$

where η ranges all the maximal points of S and the map $(*)$ is the restriction map.

In addition to the above, we need

Lemma 1. 5. *Let G be a geometrically integral k-group scheme (k being a field) and let K be a regular extension field of k. Then, if f is a character defined over K, f is defined over k.*

Proof. If k is algebraically closed, then lemma 1. 5 is obvious. Since $\bar{k}(G)$ and $K(G)$ are linearly disjoint over $k(G)$, $\bar{k}(G)\cap K(G)$

$=k(G)$ where $\bar{k}$ is an algebraic closure of k. q.e.d.

Now we shall come back to the situation before Lemma 1.3. Let f be the element of $H^0(G \underset{S}{\times} G \underset{S}{\times} X, \mathcal{O}_{G \underset{S}{\times} G \underset{S}{\times} X})$ represented by $(\mu \times 1_X)^*(\phi)\,[p_{23}^*(\phi)\circ(1_G \times \sigma)^*(\phi)]^{-1}$ and let $e\colon S \to G$ be the unit section of G. If we put $f' = f|e \underset{S}{\times} G \underset{S}{\times} X$, then f' can be regarded as an element of $H^0(G \times X, \mathcal{O}_{G\times X}^*)$ and $\lambda_1 = f/p_{23}^*(f')$ is a character of $G \underset{S}{\times} G \underset{S}{\times} X$ by virtue of Lemma 1.3. Since $G \underset{S}{\times} X$ is a smooth X-scheme with connected fibres, every maximal point z of $G \underset{S}{\times} X$ lies over a maximal point x of X and $k(z)$ is a regular extension field of $k(x)$. Hence, by virtue of Lemma 1.4 and Lemma 1.5, $\lambda_1(g_1, g_2, x) = \lambda_1(g_1, e, x)$ for every $(g_1, g_2, x) \in G \underset{S}{\times} G \underset{S}{\times} X$. By the same arguement $p_{23}^*(f') = p_{23}^*(\lambda_2)$ $p_3{}^*(\delta)$ where λ_2 is a character of $G \underset{S}{\times} X$ and δ is an element of $H^0(X, \mathcal{O}_X{}^*)$ and $p_3\colon G \underset{S}{\times} G \underset{S}{\times} X \to X$ is the projection with respect to the third factar. Therefore, $f = \lambda_1 p_{23}^*(\lambda_2) p_3{}^*(\delta)$ and we have that $(\mu \times 1_X)^*(\phi) = \lambda_1 p_{23}^*(\lambda_2) p_3{}^*(\delta)[p_{23}^*(\phi)\circ(1_G \times \sigma)^*(\phi)]$. On the other hand, $(\mu \times 1_X)^*(p_2{}^*(\delta)\phi) = p_3{}^*(\delta)(\mu \times 1_X)^*(\phi)$, $p_{23}^*(p_2{}^*(\delta)\phi) = p_3{}^*(\delta) p_{23}^*(\phi)$ and $(1_G \times \sigma)^*(p_2{}^*(\delta)\phi) = p_{23}^*(\sigma^*(\delta))(1_G \times \sigma)^*(\phi)$. Moreover, $p_{23}^*(\sigma^*(\delta)) = p_{23}^*(\xi)$ $p_3{}^*(\delta)$ where ξ is a character of $G \underset{S}{\times} X$ by virtue of Lemma 1.3 and Lemma 1.4. Hence, if we replace ϕ by $p_2{}^*(\delta)\phi$, then we have an isomorphism $\phi\colon \sigma^*(L^m) \xrightarrow{\sim} p_2{}^*(L^m)$ such that $(\mu \times 1_X)^*(\phi) = \lambda_1 p_{23}(\lambda_2)$ $[p_{23}^*(\phi)\circ(1_G \times \sigma)^*(\phi)]$ with some character λ_1 (or λ_2) of $G \underset{S}{\times} G \underset{S}{\times} X$ (or $G \underset{S}{\times} X$ resp.). Restricting these isomorphisms on the closed subschemes $e \underset{S}{\times} G \underset{S}{\times} X$ (or $G \underset{S}{\times} e \underset{S}{\times} X$) of $G \underset{S}{\times} G \underset{S}{\times} X$ and using the fact that $\lambda_1(g_1, g_2, x) = \lambda_1(g_1, e, x)$ for every $(g_1, g_2, x) \in G \underset{S}{\times} G \underset{S}{\times} X$, we can easily see that $\lambda_2 = 1$ (resp. $\lambda_1 = 1$). Hence we have the desired equaliy $(\mu \times 1_X)^*(\phi) = p_{23}^*(\phi)\,(1_G \times \sigma)^*(\phi)$.

Thus, we have the following which plays an important roll in section 2.

Theorem 1.6. *Let S be a scheme and let G be a surjective smooth affine group scheme over S with connected fibres and let X be a normal noetherian S-scheme on which G acts regularly. Then for any invertible sheaf L on X, there are a positive integer m and an isomorphism $\phi: \sigma^*(L^m) \cong p_2^*(L^m)$ such that $(\mu \times 1_X)^*(\phi) = p_{23}(\phi) \circ (1_G \times \sigma)^*(\phi)$.*

2. Quasi-projective case.

In this section, we shall generalize Theorem 1 in [13] which was a key to prove the existence of equivariant completion.

At first, we shall prepare lemmas on dual actions (cf. [6]). Let S be a scheme and let G be an affine group scheme over S, i.e. $G = \mathrm{Spec}(B)$ where B is an $\mathcal{O}_S$-Algebra. Then we have $\mathcal{O}_S$-Algebra homomorphisms;

$$\hat{\mu}: B \to B \underset{\mathcal{O}_S}{\times} B \quad \text{and} \quad \hat{e}: B \to \mathcal{O}_S$$

which correspond to the multiplication of G and the unit section of G, respectively.

Definition 2.1. (1) Let M be an $\mathcal{O}_S$-Module. If there is an $\mathcal{O}_S$-homomorphism of $\mathcal{O}_S$-Modules; $\sigma: M \to B \underset{\mathcal{O}_S}{\otimes} M$ such that

(a) the following diagram is commutative

$$\begin{tikzcd}
 & B \underset{\mathcal{O}_S}{\otimes} M \arrow[dr, "1_B \otimes \hat{\sigma}"] & \\
M \arrow[ur, "\hat{\sigma}"] \arrow[dr, "\hat{\sigma}"'] & & B \underset{\mathcal{O}_S}{\otimes} B \underset{\mathcal{O}_S}{\otimes} M \\
 & B \underset{\mathcal{O}_S}{\otimes} M \arrow[ur, "\hat{\mu} \otimes 1_M"'] &
\end{tikzcd}$$

and

(b) $M \to B \underset{\mathcal{O}_S}{\otimes} M \xrightarrow{\hat{e} \otimes 1_M} M$ is the identity morphism, then $\hat{\sigma}$ is called a dual action of G on M.

(2) Let $\hat{\sigma}$ be a dual action of G on M and let N be an $\mathcal{O}_S$-submodule of M. Then N is called invariant under the dual action $\hat{\sigma}$, if $\hat{\sigma}(N) \subseteq \mathrm{Im}[B \underset{\mathcal{O}_S}{\otimes} N \to B \underset{\mathcal{O}_S}{\otimes} M]$.

The following is a generalization of a very important lemma due to Cartier ([6]), and though the proof is mostly the same as his, we shall give it here for completeness.

Lemma 2.2. *Let $S = \mathrm{Spec}(A)$ be an affine noetherian scheme and let M be an A-module and let $G = \mathrm{Spec}(B)$ be an affine group scheme over S whose coordinate ring B is a projective A-module. If $\hat{\sigma}: M \to B \underset{A}{\otimes} M$ is a dual action of G on M, and if N is a finitely generated A-submodule of M, then there exists a finitely generated invariant submodule $E(N)$ of M such that*

(1) *$E(E(N)) = E(N)$ and $E(N)$ is the smallest invariant submodule of M which contains N.*

(2) *Let $S' = \mathrm{Spec}(A')$ be a noetherian S-scheme and let $B' = B \underset{A}{\otimes} A'$, $M' = M \underset{A}{\otimes} A'$, $N' = \mathrm{Im}[N \underset{A}{\otimes} A' \to M \underset{A}{\otimes} A']$ and let $\hat{\sigma}': M' \to B' \underset{A'}{\otimes} M'$ be the induced dual action of $G' = \mathrm{Spec}(B')$ on M'. Then $E(N') = \mathrm{Im}[E(N) \underset{A}{\otimes} A' \to M']$. In particular, if A' is A-flat, then $E(N \underset{A}{\otimes} A') = E(N) \underset{A}{\otimes} A'$.*

Proof. Let $B^* = \mathrm{Hom}_A(B, A)$ be the dual module of B. We shall define an A-endmorphism γ_{b^*} of M for any element b^* of B^* as follows;

$$\gamma_{b^*}: M \ni n \to \gamma_{b^*}(n) = \sum_i b^*(b_i) n_i \in M$$

where $\hat{\sigma}(n) = \sum_i b_i \otimes n_i$.

Then $\gamma_{b_1^*} \circ \gamma_{b_2^*} = \gamma_{b_1^* \cdot b_2^*}$ for all $b_1^*, b_2 \in B^*$ (B^* is canonically considered to be an A-algebra and $b_1^* \cdot b_2^*$ is the product of b_1^* and b_2^*). We shall put $E(N) = \sum_{b^* \in B^*} \gamma_{b^*}(N)$. Let us show that $E(N)$ is the desired invariant submodule of M.

(i) It is easily seen that $N \subseteq E(N)$ by Definition 2.1 (b).

(ii) Let $\{n_i\}$ be a generator of N let $\hat{\sigma}(n_i) = \sum_j b_{ij} \otimes n_{ij}$ where

$b_{ij} \in B$ and $n_{ij} \in M$. Then $\gamma_{b^*}(n_i) = \sum_j b^*(b_{ij}) n_{ij}$ for any $b^* \in B^*$, hence $E(N)$ is a submodule of $\sum_{i,j} n_{ij} A$. Since A is noetherian, $E(N)$ is a finitely generated A-submodule of M.

(iii) Since $\gamma_{b_1^* \cdot b_2^*} = \gamma_{b_1^*} \circ \gamma_{b_2^*}$, $\gamma_{b^*}(E(N)) \subseteq E(N)$ for any $b^* \in B^*$. Let m be an element of $E(N)$ and let $\hat{\sigma}(m) = \sum b_i \otimes m_i$. We shall shall show that we can take the m_i's in $E(N)$. Since B is A-flat, it is enough to prove the following; Let N' be any A-module and let $\sum b_i \otimes n_i$ be an element of $B \underset{A}{\otimes} N'$ such that $\sum b^*(b_i) \otimes n_i = 0$ for any $b^* \in B^*$. Then $\sum b_i \otimes n_i = 0$. Since B is A-projective, $B \oplus C = A^{\otimes I}$ for some A-module C. Let $\{e_\lambda\}$ be the free basis of $A^{\oplus I}$ and assume that $b_i = \sum_\lambda a_{i\lambda} e_\lambda \ (a_{i\lambda} \in A)$ for every i. Then $\sum b_i \otimes n_i = \sum_\lambda e_\lambda \otimes \sum_i a_{i\lambda} n_i$ and $\sum_i a_{i\lambda} n_i = 0$ for every λ by our assumption, hence $\sum b_i \otimes n_i = 0$.

The above (i)-(iii) show that our $E(N)$ has the properties (1), (2). The property (1) is obvious. Let $\{n_i\}$ be a generator of N and let $\hat{\sigma}(n_i) = \sum_j b_{ij} \otimes n_{ij}$. Furthermore, let $\{e_\lambda\}$ be a basis of $B \oplus C = A^{\oplus I}$. Then $E(N)$ is generated by $\{\sum a_{ij\lambda} n_{ij}\}$ where $b_{ij} = \sum_\lambda a_{ij\lambda} e_\lambda$. Thus we can see easily the property (2). q.e.d.

Corollary 2.3. *Let S be a noetherian scheme and let $G = \mathrm{Spec}(B)$ be a smooth affine group scheme over S with connected fibres and let $\hat{\sigma}: M \to B \underset{\mathcal{O}_S}{\otimes} M$ be a dual action of G on M where M is an $\mathcal{O}_S$-Module. Then for any coherent $\mathcal{O}_S$-submodule N of M, there is a coherent invariant $\mathcal{O}_S$-submodule $E(N)$ of M which contains N.*

Proof. Let $\mathcal{S} = (S_\alpha)_{\alpha \in A}$ be an affine open covering of S. Then for any α, $H^0(G \underset{S}{\times} S_\alpha, \mathcal{O}_{G \underset{S}{\times} S_\alpha})$ is an $H^0(S_\alpha, \mathcal{O}_S)$-projective module by virtue of [10] Prop. 3.3.1. Hence we can construct a coherent invariant $\mathcal{O}_S$-submodule $E(N)$ of M which contains N by virtue of Lemma 2.2. q.e.d.

Corollary 2.4. *Let S be a noetherian scheme and let $G = \mathrm{Spec}(B)$ be a smooth affine group scheme over S with connected fibres and let H be a closed subgroup scheme of G which is smooth over S and has connected fibres. Then there exist a coherent $\mathcal{O}_S$-Module*

M one which there is a dual action of G and a quotient $\mathcal{O}_S$-Module N of M such that $H(T)$ ($H(T)$ being the set of T-valued points of H) is equal to $\{g\in G(T)\mid \mathrm{Spec}\,(S^{\bullet}(N_T)$ is invariant under $g\}$ ($S^{\bullet}(N_T)$ being the symmetric $\mathcal{O}_T$-Algebra of N_T) for any S-cheme T. Moreover, if S is a noetherian regular scheme of dimension 1, then M and N are locally free sheaves.

Proof. Let I be the defining ideal of H and let $\pi: G\to S$ be the structure morphism of G. Let P be a coherent $\mathcal{O}_S$-submodule of B such that the ideal generated by P in B is I. Then $M=E(P)$ and $N=M/M\cap\pi_*(I)$ are desired ones. In order to prove the first part of Corollary 2.4, we may assume that T and S are affine. Now we shall put $\mathrm{Spec}=(A)$, $T=\mathrm{Spec}(A')$, $G=\mathrm{Spec}(B)$ and let $\hat{\sigma}: M\to B\otimes M$ be the dual action of G on M. Since B/I is a projective A-module $\hat{\sigma}(I\cap M)\subseteq B\otimes M\cap(I\otimes B+B\otimes I)=I\otimes M+B\otimes(I\cap M)$. Thus Spec $(S^{\bullet}(N))$ is invariant for any element g of $H(S)$. Moreover Spec $(S^{\bullet}(N'))$ where $N'=N\otimes A'$ is invariant for any element g' of $H(T)$. Let $B'=B\otimes A'$, $M'=M\otimes A'$ and let $\hat{\sigma}': M'\to B'\otimes M'$ be the dual action of $G'=\mathrm{Spec}(B')$ on M' and let $\xi: B'\to A'$ be an A'-homomorphism such that $\hat{\sigma}'(\mathrm{Im}[I\cap M)\otimes A'\to M'])\subseteq\mathrm{Im}[m\otimes M'\to B'\otimes M']+B'\otimes\mathrm{Im}[(I\cap M)\otimes A'\to M']$ where $m=\mathrm{Ker}\,\xi$. In order to prove our assertion, it is enough to show that $m\supseteq I'=\mathrm{Im}[I\otimes A'\to B']$ which is the defining ideal of $H\times_S T$ in $G\times_S T$. Now let us pick a system of generators $\{f_1,\cdots,f_n\}$ of P. Then $\{f_1\otimes 1,\cdots,f_n\otimes 1\}$ generates the ideal I', hence we have only to prove that $f_i\otimes 1$ is in m ($i=1,2,\cdots,n$). From the above assumption on m, we have that $\hat{\sigma}'(f_i\otimes 1)=\sum x_j\otimes y_j+\sum b_k\otimes c_k$ where the x_j (resp. y_j, b_k or c_k) are in m (resp. M', B' or $[(I\cap M)\otimes A'\to M']$). Let $\hat{e}': B'\to A'$ be the unit section of B' and let $\varphi: M'\to B'$ be the canonical homomorphism. Then $f_i\otimes 1=(1\otimes\hat{e}')\circ(1\otimes\varphi)\circ\hat{\sigma}'(f_i\otimes 1)=\sum\hat{e}'(\varphi(y_j)x_j)+\sum\hat{e}'(\varphi(c_k))b_k$. Since $\hat{e}'|I'=0$, $f_i\otimes 1=\sum\hat{e}'(\varphi(y_j))x_j$, hence $f_i\otimes 1\in m$. In order to prove the second part of Corollary 2.4, we may assume that $S=\mathrm{Spec}\,(A)$ and that A is a discrete valuation ring. Since B is a torsion free A-module, M is a free A-module. If $am\in I\cap M$ where $a\in A-\{0\}$ and $m\in M-(I\cap M)$, then $a\in I$ because I is a prime ideal of A. This is a contradiction because B/I is A-projective. Therefore

N is free. q.e.d.

Now we shall generalize Theorem 1 in [13].

Theorem 2.5. *Let S be a noetherian scheme and let G be a surjective smooth affine group scheme over S with connected fibres and let X be a normal noetherian S-cheme which is quasi-projective over S and on which G acts regularly. Then there is a coherent sheaf E on S such that*

(1) *There is an immersion $\varphi: X \to P(E)$,*

(2) *There is a representation $\rho: G \to \mathrm{Aut}_S(P(E))$ and*

(3) *The following diagram is commutative.*

$$\begin{tikzcd} G \underset{S}{\times} X \arrow[r, "\sigma"] \arrow[d, "\rho\times\varphi"] & X \arrow[d, "\varphi"] \\ \mathrm{Aut}_S(P(E)) \underset{S}{\times} P(E) \arrow[r, "\sigma'"] & P(E) \end{tikzcd}$$

where $\sigma'$is the canonical action of $\mathrm{Aut}_S(P(E))$ *on* $P(E)$.

Proof. Since X is quasi-projective over S, there exist a coherent sheaf M on S and an immersion $i; X \to P(M)$. Let $L = \mathcal{O}_{P(M)}(1)|X$ where $\mathcal{O}_{P(M)}(1)$ is the tautological invertible sheaf on $P(M)$. Then, by virtue of Theorem 1.6, we may assume that

(i) there is an isomorphism $\phi: \sigma^*(L) \to p_2{}^*(L)$ and

(ii) $(\mu \times 1_X)^*(\phi) = p_{23}^*(\phi) \circ (1_G \times \sigma)^*(\phi)$.

Let $f: X \to S$ be the structure morphism and let $G = \mathrm{Spec}(B)$. From (i), we have homomorphisms;

$$f_*(L) \xrightarrow{\sigma} f_* p_{2*} \sigma^*(L) \xrightarrow{f_* p_{2*}(\phi)} f_* p_{2*} p_2{}^*(L) \longrightarrow B \underset{\mathcal{O}_S}{\otimes} f_*(L).$$

(The last isomorphism is obtained from the Künneth formula.) We define $\hat{\sigma}$ to be the composition of above morphisms. Then it is easily seen that $\hat{\sigma}$ is a dual action of G on $f_*(L)$ by virtue of (ii). Let $\alpha: M \to f_* f^*(M) \to f_*(L)$ be the canonical map and let N be the image of M in $f_*(L)$. N is a coherent $\mathcal{O}_S$-submodule of $f_*(L)$. If we put E to be the coherent invariant submodule $E(N)$ that is obtained in Corollary 2.3, then E is the desired coherent sheaf. In fact, since $X\to P(M)$ is an immersion, the canonical surjective map $f^*(E)\to L$ gives an immersion $\varphi: X\to P(E)$. On the other hand, the dual action $\hat{\sigma}$ of G on E gives a linear regular action σ' of G on $P(E)$, i.e., $\sigma': G\underset{S}{\times} P(E)\to P(E)$ such that

$$\text{(i}'\text{)}\qquad \begin{array}{ccc} G\underset{S}{\times}G\underset{S}{\times}P(E) & \xrightarrow{1_G\times\sigma'} & G\underset{S}{\times}P(E) \\ \Big\downarrow{\scriptstyle \mu\times 1_{P(E)}} & & \Big\downarrow{\scriptstyle \sigma'} \\ G\underset{S}{\times}P(E) & \xrightarrow{\sigma'} & P(E) \end{array}$$

is commutative and

(ii′) $P(E)\xrightarrow{e\times 1_{P(E)}} G\underset{S}{\times}P(E)\xrightarrow{\sigma'} P(E)$ is the identity morphism.

Furthermore, it is easily seen that

$$\text{(iii}'\text{)}\qquad \begin{array}{ccc} G\underset{S}{\times}X & \xrightarrow{\sigma} & X \\ \Big\downarrow{\scriptstyle 1_G\times\varphi} & & \Big\downarrow{\scriptstyle \varphi} \\ G\underset{S}{\times}P(E) & \xrightarrow{\sigma'} & P(E) \end{array}$$

is commutative. q.e.d.

Corollary 2.6. *Under the situation of Theorem 2.5, X has an equivariant completion.*

Proof. Let $\bar{X}$ be the smallest closed subscheme of $P(E)$ such that $\bar{X}$ dominates X (cf. [3] Prop. 9.5.10). Then $\bar{X}$ is an equivariant completion of X. q.e.d.

3. G-stable quasi-projective open coverings.

In this section, we shall prove that every point of X is contained in a G-stable open subscheme which is quasi-projective over S. This is a generalization of Lemma 8 in [13].

Let X be a noetherian scheme and let $Z^1(X)$ be the group of cycles of pure codimension 1 of X. Here we shall consider only those elements with integral components of $Z^1(X)$, i.e., $D=\sum n_iD_i$ where the D_i are integral closed subschemes of X of codimension 1 and we shall call such elements Weil divisors of X.

At first, we shall recall a relation between effective Cartier divisors and effective Weil divisors on normal noetherian schemes. Let $D=\sum_{i=1}^{r} n_iD_i$ be an effective Weil divisor on a normal noetherian scheme X and let $U=\{U_\alpha\}$ $(A_\alpha=\Gamma(U_\alpha, \mathcal{O}_{U_\alpha}))$ be an affine open covering of X. We shall put $P_{\alpha,i}$ to be prime ideal of A_α associated with the divisor $D_i\cap U_\alpha$ in A_α for all i and α. If $D_i\cap U_\alpha=\phi$, then $P_{\alpha,i}=A_\alpha$. We shall here define an ideal $I_\alpha(D)$ to be $\bigcap_{i=1}^{r} P_{\alpha,i}^{(n_i)}$ where $P_{\alpha,i}^{(n_i)}$ is the symbolic n_i-th power of $P_{\alpha,i}$. It is easily seen that the associated ideal sheaves $\tilde{I}_\alpha(D)$ on the U_α can be patched to each other, hence we can construct an ideal sheaf $\tilde{I}(D)$ on X such that $\tilde{I}(D)|U_\alpha=\tilde{I}_\alpha(D)$ for all α. We shall here call $\tilde{I}(D)$ the ideal sheaf of D. Then D is a Cartier divisor if and only if $\tilde{I}(D)$ is an invertible sheaf (cf. E.G.A. Vol. IV, Prop. 21. 7. 2, Cor. 21. 7. 3).

Next we shall recall a property on symbolic powers. Let A be a local ring and let $\tilde{A}$ be the (strict) henselization of A. We shall refer to [11] on the (strict) henselisation of A. Then, there is a filtered inductive system of $\{A_i\}$ where the A_i are local étale A-algebras and $\tilde{A}$ is the inductive limit of $\{A_i\}$. $\tilde{A}$ is noetherian (resp. reduced or normal) if and only if A is noetherian (resp. reduced or normal). $\tilde{A}$ is faithfully flat over A. Now let $\tilde{A}=\varinjlim A_i$ where the $A_i=(B_i)_{n_i}$, the B_i are étale A-algebras and the n_i are maximal ideals of B_i lying over the maximal ideal n of A. Assume that B is a reduced A-algebra. Then $\tilde{B}=B\otimes_A\tilde{A}$ is reduced. In fact, since B_i is étale over A, $B\otimes_A A_i$ is reduced, hence $\tilde{B}$ is reduced. Therefore, if we assume that A is a noetherian ring and that B is a normal finitely generated A-algebra, then for any prime ideal of p in B of codimension 1 (i.e., $\dim B_p=1$), $p\tilde{B}$ has no embedded prime ideals and $p^{(m)}\tilde{B}=p_1^{(m)}\cap\cdots\cap p_r^{(m)}$ for every positive integer m where the p_i $(i=1, 2, \cdots, r)$ are the prime ideals in $\tilde{B}$ of codimension 1. Note here that $\tilde{B}$ is also normal noetherian.

Let k be a field and let B be an integral, finitely generated k-algebra and let k' be a finite purely inseparable extension field of k. Then Spec $(B \otimes_k k')$ is irreducible. Therefore, if p is a prime ideal in B of codimension 1, then $p \otimes_k k'$ is a primary ideal in $B \otimes k'$. Furthermore, if Spec (B) is geometrically normal, then $p^{(m)} \otimes k' = p'^{(me)}$ for every positive integer m where e is a power of characteristic of k and p' is a prime ideal in $B \otimes k'$ of codimension 1. Therefore, we have the following.

Lemma 3.1. (1) *Let A be a noetherian local ring, B be a normal finitely generated A-algebra and let p be a prime ideal in B of codimension 1. Then $p(B \otimes_A \tilde{A})$ ($\tilde{A}$ being the* (*strict*) *henselization of A) has no embedded prime ideals and $p^{(m)}(B \otimes_A \tilde{A}) = p_1^{(m)} \cap \cdots \cap p_r^{(m)}$ for every positive m where the p_i $(i = 1, \cdots, r)$ are prime ideals in B of codimension 1.*

(2) *Let k be a field, B be a finitely generated k-algebra such that* Spec(B) *is geometrically normal and let p be a prime ideal in B of codimension 1. Then, for any finite purely inseparable extension field k' of k, $p \otimes k'$ is a primary ideal in $B \otimes k'$ and $p^{(m)} \otimes k' = p'^{(me)}$ for every positive integer m where e is a power of characteristic of k and p' is a prime ideal in $B \otimes k'$ of codimension 1.*

We shall add two elementary lemmas on Weil divisors.

Lemma 3.2. *Let S be a noetherian scheme and let X be an S-cheme such that*

a) *X is of finite type and flat over S and*

b) *X_t is integral for all points $t\,(t \in S)$ of codimension 1.*

Furthermore let D be a Weil divisor on X such that $D \cap X_t = \phi$ for all maximal points t of S. Then there is a Weil divisor E on S such that D is the inverse image of E.

Proof. We may assume that D has only one integral component. Let x be the maximal point of D and let y be the image of x. Then, since X is flat over S and $x \notin X_t$ for every maximal point t of S, y is the point of codimension 1. Let $E = \overline{\{y\}}$ in S. Then the inverse image of E is a Weil divisor of X by our hypotheses a) and b). It is easily seen that $D=$ inverse image of E. q.e.d.

Let Spec(A) be a normal noetherian scheme. For a regular meromorphic function f on Spec(A), let cyc$(f)=D-E$ be the Weil divisor associated with the Cartier divisor (f) with the positive (or, negative) part D (or E resp.) (cf. [3] Chap. IV). Take a non-zero divisor t of A such that Spec(A/tA) is nomal. Let $D=\sum m_iD_i$ (resp. $E=\sum n_jE_j$) and let p_i (resp. q_j) be the defining ideal of D_i (resp. E_j). Assume that every $\bar{p}_i=p_i+tA/tA$ (resp. $\bar{q}_j=q_j+tA/tA$) is a semi-prime ideal in A/tA of codimension 1 for every i (resp. j). In this case, we shall say that D_i (resp. E_j) meets with Spec(A/tA) at closed integral subschemes in Spec(A) transversally. For every i (resp. j), we shall put $\bar{p}_i=\bar{p}_i{}^1\cap\cdots\cap\bar{p}_i{}^{k_i}$ (resp. $\bar{q}_j=\bar{q}_j{}^1\cap\cdots\cap\bar{q}_j^{e_j}$) where the $\bar{p}_i{}^k$ (resp. $\bar{q}_j{}^e$) are the associated prime ideals of $\bar{p}_i$ (resp. $\bar{q}_j$) and define $\bar{D}=\sum_{i,k}m_i\bar{D}_i{}^k$ (resp. $\bar{E}=\sum_{j,}{}^e$) where the $\bar{D}_i{}^k$ (resp. $\bar{E}_j{}^e$) are the closed integral subschemes of Spec(A/tA) defined by $\bar{p}_i{}^k$ (resp. $\bar{q}_j{}^e$) $(k=1,\cdots,k_i;\,e=1,\cdots,e_j)$. Then we have the following lemma.

Lemma 3.3. *Under the above situation, $\bar{D}$ and $\bar{E}$ are linearly equivalent to each other.*

Proof. In order to prove Lemma 3.3, we may assume that Spec(A) is integral because Spec(A) is normal. Let $\{r_k\}$ be the set of associated prime ideals of tA. Then every r_k is of codimension 1 in A. Moreover, $r_k\neq p_i$ and $r_k\neq q_j$ for all i, j and k by our assumption. Hence we can write $f=a/b$ where $a\in A$, $b\in A$ and $a\notin r_k$, $b\notin r_k$ for all k. Let $\bar{f}=\bar{a}/\bar{b}$ where $\bar{a}$ (resp. $\bar{b}$) is the image of a (resp. b) in A/tA. Then $\bar{f}$ is a regular meromorphic function on Spec(A/tA) and cyc$(\bar{f})=\bar{D}-\bar{E}$ because A_p is regular for every prime ideal p $(\supset tA)$ in A of codimension 2. q.e.d.

Under the above preparations, we shall prove the following Lemma 3.6 which is analogous to Lemma 1.2 in section 1.

Before proving Lemma 3.6, let us introduce the following notion (cf. [9] Lemma IV. 2.4.).

Definition 3.4. Let S be a noethrian scheme. We say that an S-scheme X has the property (N) if the following conditions are satisfied;

(1) X is flat and of finite type over S,

(2) X_t is geometrically normal for all maximal points t of S and

(3) X_t is geometrically integral for all points t $(t \in S)$ of codimension 1.

Remark 3.5. M. Raynaud proved the following (cf. [9] Lemma (v. 2.4.)). Let S be a normal noetherian scheme and let X be an S-scheme such that

a) X is flat and of finite type over S,

b) X_t is geometrically normal for all maximal points t of S and

c) X_t is geometrically reduced for all points t $(t \in S)$ of codimension 1.

Then X is also normal.

Now we shall prove Lemma 3.6.

Lemma 3.6. *Let S be a noetherian scheme, G be a surjective smooth affine group scheme over S with connected fibers and let X be a normal S-scheme which has the property (N) and on which G acts regularly. Assume that D is a Weil divisor of X such that* Supp D *contains no maximal points of fibers of X. Then there are a positive integer m and a regular meromorphic function f on $G \underset{S}{\times} X$ and a Cartier divisor E on S such that*

$$\sigma^*(mD) - p_2{}^*(mD) + p_2{}^*\pi^*(E) = \mathrm{cyc}(f)$$

where $\pi: X \to S$ is a structure morphism. Therefore, if $\mathrm{Pic}(S)$ *is a torsion group, then we have that*

$$\sigma^*(mD) - p_2{}^*(mD) = \mathrm{cyc}(f)$$

for a positive integer m and a regular meromorphic function f.

Proof. Of course, we may assume that D is an integral closed subscheme and that $\pi: X \to S$ is surjective, S and X are integral because X is normal. At first we assume that $S=\mathrm{Spec}(k)$ where k is a field (p=characteristic of k). Let k' be a finite algebraic extension field of k such that G is rational over k' and let k_s' be the separable closure of k in k'. Then $\tilde{I}(D)\otimes k_s'=\tilde{I}(D')$ where $\tilde{I}(D)$ (resp. $\tilde{I}(D')$) is the ideal sheaf of D (resp. D') and D' is a Weil divisor on $X\otimes k_s'$ which is written in the form $D'=D_1'+\cdots+D_r'$ with some integral closed subschemes D_i' of codimension 1 in $X\otimes k_s'$ (cf. Lemma 3.1). Moreover, $\tilde{I}(D)\otimes k'=\tilde{I}(D'')$ where D'' is a Weil divisor on $X\otimes k'$ (cf. Lemma 3.1). Since G is rational over k', we may assume that $\sigma^*(D'')-p_2^*(D'')=(f')$ for some $f'\in k'(G\times X)$ by virtue of Lemma 5 [13] and lemma 1.1. Let $\Gamma=\mathrm{Aut}_k(k')$ and let l be the order of Γ. Then $\sigma^*(D'')-p_2^*(D'')=(f'^{\alpha})$ for all $\alpha\in\Gamma$ because $\tilde{I}(D'')=\tilde{I}(D)\otimes k'$. Therefore, there is a positive integer n such that $\sigma^*(p^n lD'')-p_2^*(p^n lD'')=((\prod_\alpha f'^{\alpha})^{p^n})$. Since $(\prod_\alpha f^{\alpha})^{p^n}\in k(G\times X)$, $\sigma^*(mD)-p_2^*(mD)=(f)$ for a positive integer m and a non-zero meromorphic function $f\in k(G\times X)$. Next we consider the general case. Let η be the maximal point of S. By the above result, there are a positive integer m and a non-zero meromorphic function $f_\eta\in k(\eta)(G_\eta\times X_\eta)$ such that $\sigma_\eta^*(mD_\eta)-p_{2,\eta}^*(mD_\eta)=(f_\eta)$. By virtue of lemma 3.3, we have that $\sigma^*(mD)-p_2^*(mD)-p_2^*\pi^*(E)=(f)$ where E is a Weil divisor on S and $f=f_\eta$ is non-zero meromorphic function on $G\underset{S}{\times}X$. Let $e: S\to G$ be the unit section of G. Since $G\underset{S}{\times}X$ is smooth over X, $X\cong e\underset{S}{\times}X$ is a regular immersion in $G\underset{S}{\times}X$. Therefore, for any point x of X, there is an affine open subscheme $U=\mathrm{Spec}(C)$ ($\ni x$) in $G\underset{S}{\times}X$ such that there is an étale morphism $\varphi: U\to X[t_1,\cdots,t_n]$, $U\cap X=\varphi^{-1}$ (X= the closed subscheme in $X[t_1,\cdots,t_n]$ defined by the ideal $(t_1,\cdots,t_n)$) (cf. [4] Expose II. Theorem 4.10.). Hence, if we denote the element ($\in C$) corresponding to each t_i $(i=1,\cdots,n)$ through φ by t_i also, then $\{t_1,\cdots,t_n\}$ is a C-prime sequence in the sense of Serre. For each i, $\mathrm{Spec}(C/t_1,\cdots,t_i))$ is smooth over X and $\mathrm{Spec}(C/(t_1,\cdots,t_i))$ meets with $p_2^{-1}(D)$ at closed integral subschemes transversally in $\mathrm{Spec}(C/t_1,\cdots,t_{i-1}))$ where D is any closed integral subscheme in X of codimension 1. Since X is normal and $\mathrm{Spec}(C/(t_1,\cdots,t_{i-1}))$ is smooth over X, $\pi^*(E)$ is a Cartier divisor on X by virtue of Lemma 3.3. Moreover, since $\pi: X\to S$ is a faithfully flat morphism, E is a Cartier divisor on S. The second part of Lemma 3.6 is obvious. q.e.d.

Corollary 3.7. *Under the situation of Lemma 3.6 if $g\in G(S)$ and if* Pic(S) *is a torsion group, then there are a positive integer m and a regular meromorphic function f on X such that*

$$g.(mD)-mD=\mathrm{cyc}(f)$$

Proof. The proof is similar to that of Lemma 3.6.

Now we are in position to generalize Lemma 8 in [13].

Theorem 3.8. *Let S be a normal noetherian scheme and let G be a surjective smooth affine group scheme over S with connected fibres and let X be an S-scheme which satisfies the property (N) and on which G acts regularly. For any dense open subscheme U of X which is affine over S, we shall put $W=p_2(\sigma^{-1}(U))$ and let $\{D_1,\cdots,D_r\}$ be the set of irreducible components (with reduced structures) of $X-U$. Then, for $D=\sum n_iD_i$ with positive integers n_i, there is a positive integer m such that $mD|W$ is a Cartier divisor on W and $\mathcal{O}_W(mD|W)$ is S-very ample. In particular W is a G-stable quasi-projective open subscheme of X over S.*

Proof. In order to prove Theorem 3.8, we may assume that X and S are integral and that $X=W$. At first, we assume $S=\mathrm{Spec}(k)$ where k is a field. Let k' be the separable closure of k. Then, since the set of k'-rational points of G is dense in $G\otimes k'$, there are finitely many k'-rational points $\{g_0,\cdots,g_r\}$ of $G\otimes k'$ ($g_0=$the unit of $G\otimes k'$) such that $X\otimes k'=\bigcup_{i=0}^r g_i(U\otimes k')$. By virtue of lemma 3.2, there is a Weil divisor D' on $X\otimes k'$ such that $\tilde{I}(mD)\otimes k'=\tilde{I}(mD')$ for every positive integer m. On the other hand, there is a positive integer m such that $\sigma^*(mD')=p_2{}^*(mD')+(f)$ where f is a regular meromorphic function on $(G\times X)\otimes k'$ by virtue of Lemma 3.6. Therefore, mD', $m(g_1D'),\cdots,m(g_rD')$ have following properties;

(1) $m(g_iD')$ is linearly equivalent to mD' and $X\otimes k'-m(g_iD')$

is affine for every i $(i=0, 1, \cdots, r)$ and

(2) $X\otimes k' = \bigcup_{i=0}^{r} (X\otimes k' - m(g_i D'))$

Hence, mD' is a very ample divisor and is a Cartier divisor for a sufficiently large m. Thanks to the descent theory, mD is a very ample Cartier divisor on X. We consider next the general case. By virtue of [3] IV. 8. 5. 5 and IV. 8. 10. 5. 2, we may assume that $S=\mathrm{Spec}(A)$ where A is a local normal noetherian ring. We shall prove Theorem 3.8 by an induction on dimension of A. When $\dim A=0$, we have already proved it. Let s be the closed point of S and let $S_0=S-\{s\}$. By the induction hypothesis, we may assume that Theorem 3.8 is true over S_0. Let $\tilde{A}$ be the strict henselization of A and let $\tilde{S}=\mathrm{Spec}(\tilde{A})$, $\tilde{S}_0=\tilde{S}-\tilde{s}$ where $\tilde{s}$ is the closed point of $\tilde{S}$. Since $\tilde{s}$ is the unique closed point of $\tilde{S}$ lying over s, the dimension of $\tilde{S}_0$ is less than that of $\tilde{S}$. By virtue of Lemma 3.1, $\tilde{I}(mD)\otimes\mathcal{O}_{\tilde{S}}=\tilde{I}(m\tilde{D})$ where $\tilde{D}$ is a Weil divisor on $\tilde{X}=X\underset{S}{\times}\tilde{S}$ for every positive integer m. On the other hand, by the induction assumption, there is a positive integer m_1 such that $m_1\tilde{D}|\tilde{\pi}^{-1}(\tilde{S}_0)$ is a Cartier divisor and is $\tilde{S}_0$-very ample where $\tilde{\pi}: \tilde{X}\to\tilde{S}$ is the structure morphism. Since $\tilde{A}$ is the strict henselization of A and $\tilde{G}=G\underset{S}{\times}\tilde{S}$ is smooth over $\tilde{S}$, there are finitely many $\tilde{A}$-rational points $g_0=e, g_1, \cdots, g_r$ of $\tilde{G}$ such that if we put $\tilde{V}=\bigcup_{i=1}^{r} g_i\tilde{U}$ $(\tilde{U}=U\underset{S}{\times}\tilde{S})$, then $\tilde{V}_{\tilde{s}}=\tilde{X}_{\tilde{s}}$. Furthermore, since $\mathrm{Supp}\,\tilde{D}$ contains no maximal points of fibres of $\tilde{X}$ and since $\mathrm{Pic}(\tilde{S})=0$, there is a positive integer m_2 such that $\tilde{g}(m_2\tilde{D})$ is linearly equivalent to $m_2\tilde{D}$ for all $\tilde{g}\ni G(\tilde{S})$ by virtue of Corollary 3.7. Hence, if we restrict $m_2\tilde{D}$ on $\tilde{V}$, then $m_2\tilde{D}|\tilde{V}$ is a Cartier divisor and is very ample for a sufficiently large m_2 by virtue of Corollary 3.7. Thus $m\tilde{D}$ is a Cartier divisor on $\tilde{X}$ and is $\tilde{S}$-very ample for $m=m_1m_2$. By faithfully flat descent theory, mD is a Cartier divisor and is S-very ample.

q.e.d.

By Theorem 2.5 and Theorem 3.8, we have the following theorem.

Theorem 3.9. *Let S be a normal noetherian scheme and let G be a surjective smooth affine group scheme over S and let X be* *an S-cheme having the property (N). Then every regular action of G on X is obtained by patching the linear actions of G on X_i which are noetherian, normal and quasi-projective over S.*

By Theorem 2.5 and Theorem 3.8, we have the following Corollary 3.10 on connected smooth affine k-group schemes. Let G be a connected smooth affine k-group scheme (k being a field) and let X be a geometrically normal and geometrically integral k-algebraic scheme on which G acts regularly. If L is the k-rational functions field of X, then G acts rationally on L. In fact, if we put $f^g = T^*_{g^{-1}}(f)$ for any $f \in L$ where g is the generic point of G and $T_{g^{-1}}: X \ni x \to g^{-1}x \in X$, then $\xi_g: L \in f \to \xi_g(f) = f^g \in Q(k(G) \otimes_K L) =$ the quotient field of $k(G) \otimes_k L$ is a k-homomorphism such that $\xi_{g_1 g_2}(f) = \xi_{g_1}\xi_{g_2}(f))$ for any $g_1, g_2 \in G$ and $\xi_e(f) = f$ (e being the unit element of G). Furthermore, by virtue of Theorem 2.5 and Theorem 3.8, we have a rational 1-cocycle $\delta = \{\delta_g\} \in H^1(G(\bar{k}), \bar{k}(L))$ ($\bar{k}$ being the algebraic closure of k) and finitely many elements $f_1, \cdots, f_n$ of L such that $L = k\ (f_1, \cdots, f_n)$ and $\delta_g f_i^g \in \sum_i f_i \bar{k}$ for any $g \in G(\bar{k})$ and i $(1 \leq i \leq n)$. Conversely, if G acts rationally on a regular extension field L over k and if there are a rational 1-cocycle $\delta = \{\delta_g\} \in H^1(G(\bar{k}), \bar{k}(L))$ and finitely many elements $f_1, \cdots, f_n$ of L satisfying the above conditions, then it is easily seen that there is a geometrically normal and geometrically integral k-algebraic scheme X on which G acts regularly such that $L = k(X)$ and the action of G on X induces the rational action of G on L. Therefore, we have the following corollary.

Corollary 3.10. *Let G be a connected smooth affine algebraic k-group scheme (k being a field) and let L be a regular extension field of k on which G acts rationally. Then the following two conditions are equivalent;*

(1) *There are a rational 1-cocycle* $\delta = \{\delta_g\} \in H^1(G(\bar{k}), \bar{k}(L))$ *($\bar{k}$ being the algebraic closure of k) and finitely many elements* $f_1, \cdots, f_n$ *of L such that* $L = k(f_1, \cdots, f_n)$ *and* $\delta_g f_i^g \in \sum_i f_i \bar{k}$ *for any* $g \in G(\bar{k})$ *and i* $(1 \leq i \leq n)$

(2) *There is a geometrically normal and geometrically integral k-algebraic scheme X on which G acts regularly such that* $L = k(X)$

and the action of G on X induces the rational action of G on L.

We shall now generalize Corollary 2 in [13] which has many applications (cf. [7]).

Corollary 3.11. *Let S be a normal noetherial scheme and let G be a smooth locally diagonalisable group scheme over S wtih connected fibres (cf. [5] Exposé VIII) and let X be an S-cheme which satisfies the property (N) and on which G acts regularly. Then X is covered by G-stable open subschemes which are affine over S.*

Proof. By virtue of Theorem 3.8, we may assume that X is quasi-projective over S. Furthermore, we may assume that $S=\mathrm{Spec}(A)$ is affine and $G=\mathrm{Spec}(A[M])$ where M is a finitely generated free abelian group. By virtue of Theorem 2.5, there exist a finitely generated A-module F and a dual action $\hat{\sigma}$ of G such X is embedded into $P(F)$ equivariantly. Let x be a closed point of X. Then we show that there is a G-stable affine open subscheme U of X which contains x. Let P be the homogeneous defining ideal of $p_2(\sigma^{-1}(x))_{\mathrm{red}}$ in $S^{\bullet}(F)$ ($S^{\bullet}(F)$ being the symmetric algebra of F) and let $\bar{X}$ be the smallest closed subscheme in $P(F)$ which contains X. If $\bar{X}=X$, then Corollary 3.11 is obvious. In fact, if $F=\bigoplus_{m\in M}F_m$ be the decomposition of F associated with the action of G on F, then some $f_m\,(\in F_m)$ is not contained in P. Therefore, $\mathrm{Spec}(S^{\bullet}(F)_{(f_m)})$ is the desired open subscheme. Hence, we assume that $\bar{X}\neq X$ and that I is the homogeneous defining ideal of $(\bar{X}-X)_{\mathrm{red}}$ in $S^{\bullet}(F)$. Since $P\not\subseteq I$, there is an element $f\ (\in I)$ which is not contained in P. Let $V=E(A\cdot f)$ (cf. Lemma 2.2), i.e., V is the smallest invariant submodule of $S^{\bullet}(F)$ under the dual action $\hat{\sigma}$ which contains f. Since $V\not\subseteq P$, there is an element $f_m\ (\in V_m)$ which is not contained in P. It is clear that $\mathrm{Spec}(S^{\bullet}(F)_{(f_m)})$ is the desired G-stable affine open subscheme. q.e.d.

Remark 3.12. If G is multiplicative type over S (cf. [5] Exposé IX), then Corollary 3.11 is not necessarily true. We can construct a counter-example easily.

4. Main theorem.

In this section, we shall prove our main theorem, i.e., the existence of equivariant completion which is a generalization of Theorem 3 in [13]. In [2], P. Deligne proved M. Nagata's result [8] on the existence of completion, without using valuation rings. It is not too difficult to see that his elegant proof is effective in our case, however, we shall give a proof, along his line, of our main theorem for completeness. Roughly speaking, our problem is to show that crucial algebraic schemes appearing in the process of Deligne's proof are G-stable. At first, we shall prepare several lemmas on blowings-up.

Let A be a commutative noetherian ring and let $\mathfrak{a}$ be an ideal of A and let $X=\mathrm{Spec}(A)$, $Y=V(\mathfrak{a})$ (Y being the closed subscheme of X defined by $\mathfrak{a}$). Then the blowing-up $\tilde{X}$ of X with center $\mathfrak{a}$ is covered by open subschemes $\{D(x)=\mathrm{Spec}(A[\mathfrak{a}/x])\}_{x\in\mathfrak{a}}$. Let $Z=V(\mathfrak{b})$ be a closed subscheme of X and let Z' be the pure transform of Z in $\tilde{X}$. Then $Z'\cap D(x)$ is the closed subscheme of $D(x)$ defined by $\mathfrak{b}A[1/x]\cap A[\mathfrak{a}/x]$ for every $x\in\mathfrak{a}$. Hence, if $x\in\mathfrak{a}\cap\mathfrak{b}$, then $Z'\cap D(x)=\phi$. Therefore we have the following.

Lemma 4.1. *Let $X=\mathrm{Spec}(A)$ and let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of A and let $\tilde{X}$ be the blowing-up of X with the center $\mathfrak{a}+\mathfrak{b}$. Then the pure transform of $V(\mathfrak{a})$ is contained in $\bigcup_{x\in\mathfrak{b}} D(x)$.*

Lemma 4.2. *Let the following be a commutative diagram of noetherian schemes;*

$$\begin{tikzcd} Y \arrow[rr, "k"] \arrow[d] & & Z \arrow[d, "p"] \\ U \arrow[r, "i"] & V \arrow[r, "j"] & X \end{tikzcd}$$

where

(1) *p is of finite type,*

(2) *i and j are open immersions and*

(3) *k is an immersion and Y is closed in $p^{-1}(V)$.*

Let $\mathfrak{a}$ be an ideal of X such that $V(\mathfrak{a})=F=\overline{V-U}$ ($\overline{V-U}$ being *the closure of* $V-U$ *in* X), $F_2=V(\mathfrak{a}^2)$ *and let* $\mathfrak{b}$ *be an ideal of* X *such that* $G=V(\mathfrak{b})$ *is contained in* $F-V$ *set-theoretically and that* $p^{-1}(G)$ *contains* $p^{-1}(F_2)\cap\overline{Y}$ *scheme-theoretically* ($\overline{Y}$ *being the closure of* Y *in* Z *with reduced structure*). *Then, after the base change* $g:\tilde{X}\to X$ (g *being a morphism defined by the blowing-up of* X *with the center* $\mathfrak{a}+\mathfrak{b}$), *the new closure* $\overline{Y}$ *of* Y *is disjoint with the new closure* $\overline{V-U}$ *of* $V-U$.

$$\begin{tikzcd} Z \arrow[r, "g'"] \arrow[d, "p"'] & \tilde{Z}=Z\times_X\tilde{X} \arrow[d, "p'"] \\ X \arrow[r, "g"] & \tilde{X} \end{tikzcd}$$

Proof. It is enough to prove $p'(\overline{Y})\cap\overline{V-U}=\phi$ in $\tilde{X}$. Thus we may assume $X=\mathrm{Spec}(A)$. Let b be an element of $\mathfrak{b}$. Then $b=0$ on $\overline{Y}\cap p^{-1}(F_2)$, hence $b=\sum\alpha_{ij}a_ia_j$ on Y locally where the $a_i, a_j\in\mathfrak{a}$, and the α_{ij} are local regular functions. Since a_i/b is a regular function on $D(b)$ in $\tilde{X}$, $1=\sum\alpha_{ij}(a_i/b)a_j$ on $\overline{Y}$ locally. Therefore, $p'^{-1}(q^{-1}(F))\cap p'^{-1}(D(b))\cap\overline{Y}=\phi$. Since $\bigcup_{b\in\mathfrak{b}}D(b)$ contains the new closure $\overline{V-U}$ of $V-U$ in $\tilde{X}$ by virtue of Lemma 4.1, $\phi=q^{-1}(F)\cap p'(\overline{Y})\supseteq(\overline{V-U})\cap p'(\overline{Y})$. q.e.d.

Lemma 4.3. *Let the following be a commutative diagram of noetherian schemes;*

$$\begin{tikzcd} & X' \arrow[r] \arrow[d, "p"] & \overline{X}' \arrow[d, "q"] \\ U \arrow[ur, hook] \arrow[r, hook] & X \arrow[r] & \overline{X} \end{tikzcd}$$

where

(1) U *is dense, open in* $\overline{X}$ *and* $\overline{X}'$,

(2) p *and* q *are of finite type and*

(3) $X'=q^{-1}(X)$.

Let F *be a closed subscheme of* X *and let* E *be a closed subscheme of* X' *which contains* $p^{-1}(F)$. *Then there is an ideal* $\mathfrak{a}$ *of* X *such that*

(1) $V(\mathfrak{a})\subseteq\overline{F}-F$ *and*

(2) *after the base change* r; $\tilde{\bar{X}}\to\bar{X}$ *defined by the blowing-up of* $\bar{X}$ *with the center* $\mathfrak{a}$ *and after replacing* $\bar{X}'$ *by the closure* $\bar{U}$ *of in* $\tilde{\bar{X}}\underset{\bar{X}}{\times}\bar{X}'$, *we have* $q^{-1}(\bar{F})\subseteq E$.

Proof. Let $q^*(\bar{F})=E\cup q^{-1}(\bar{F})=E\cup q^{-1}(\bar{F}-F)$. Then, applying Lemma 4.2 to the following commutative diagram of schemes;

$$\begin{tikzcd}
\bar{X}'-q^*(\bar{F}) \arrow[rr] \arrow[d] & & \bar{X}'-E \arrow[d] \\
\bar{X}-\bar{F} \arrow[r] & \bar{X}-(\bar{F}-F) \arrow[r] & \bar{X},
\end{tikzcd}$$

we find an ideal $\mathfrak{a}$ of $\bar{X}$ such $\operatorname{Supp}\mathfrak{a}\subseteq\bar{F}-F$ and that after the base change $\tilde{\bar{X}}\to\bar{X}$ defined by the blowing-up X with the center $\mathfrak{a}$, the new closure $\bar{F}$ of F is disjoint with the image of the closure of $\bar{X}'-q^*(\bar{F})$ in $\bar{X}'-E$. In particular, replacing $\bar{X}'$ by $\bar{U}$, this new closure $\bar{F}$ is disjoint with the image of $\bar{X}'-E$. q.e.d.

Lemma 4.4. *Let S be a noetherian scheme and let the following be a commutative diagram of separated, finite type S-chemes;*

$$(*)\qquad\begin{tikzcd}
 & X_i \arrow[r] & \bar{X}_i \\
U \arrow[ur, hook] \arrow[r, hook] & X &
\end{tikzcd}\qquad (1\leq i\leq n)$$

where

(1) *U is dense, open in X and X_i $(1\leq^{\forall} i\leq n)$ and*

(2) *$\bar{X}_i$ is proper over S,*

and let $X^*=$ the closure of $\{(u,\cdots,u)\,|\,u\in U\}$ in $\bar{X}_1\underset{S}{\times}\cdots\underset{S}{\times}\bar{X}_n$. Moreover assume that F_i is closed in X_i $(1\leq^{\forall} i\leq n)$ and $\bigcap_{i=1}^{n}p_i^{-1}(F_i)=\phi$ where $p_i\colon X^*\to\bar{X}_i$ is the i-th projection. Then, after replacing $\bar{X}_i$ by a blowing-up of $\bar{X}_i$ with a suitable center contained in $\bar{F}_i-F_i$ for each i $(1\leq i\leq n)$, we can obtain the same diagram as $(*)$ satisfying the conditions (1) and (2) such that the intersection of the inverse images of $\bar{F}_i$ by p_i is empty.

Proof. Let $\mathfrak{a}_i$ be the ideal of X^* such that $V(\mathfrak{a}_i)=\overline{p_i^{-1}(F_i)}$ (with reduced structure) for each i $(1\leq i\leq n)$ and let X^{**} be the blowing-up of X^* with the center $\mathfrak{a}=\mathfrak{a}_1+\cdots\mathfrak{a}_n$ and let $q_i: X^{**}\to X^*\xrightarrow{p_i}X_i$ for each i $(1\leq i\leq n)$. Then $\bigcap_{i=1}^n \overline{q_i^{-1}(F_i)}=\phi$ by virtue of Lemma 4. 1. Applying Lemma 4. 3 to the following diagram;

$$\begin{tikzcd}
 & q_i^{-1}(X_i) \arrow[r] \arrow[d] & X^{**} \arrow[d, "q_i"] \\
U \arrow[ur, hook] \arrow[r, hook] & X_i \arrow[r] & \overline{X}_i
\end{tikzcd}$$

and $F_i\subset X_i$, $E_i=\overline{q_i^{-1}(F_i)}$, we find an ideal $\mathfrak{a}_i'$ of $\overline{X}_i$ such that

(1) $V(\mathfrak{a}_i')\subseteq\overline{F}_i-F_i$ and

(2) after the base change $r_i: \tilde{\overline{X}}_i\to\overline{X}_i$ defined by the blowing-up of $\overline{X}_i$ with the center $\mathfrak{a}_i'$ and after replacing X^{**} by the closure $\overline{U}$ of U in $X^{**}\underset{\overline{X}_i}{\times}\tilde{\overline{X}}_i$, we have $q_i^{-1}(\overline{F}_i)\subseteq E_i$. In the new situation, we have that $\bigcap_{i=1}^n q_i^{-1}(\overline{F}_i)\subseteq\bigcap_{i=1}^n E_i=\phi$ on $\overline{U}$ ($\overline{U}$ being the closure of U in X^{**}). Since $\overline{U}\to X^*$ is surjective, $\bigcap_{i=1}^n q_i^{-1}(\overline{F}_i)=\phi$. q.e.d.

Remark 4. 5. The above lemmas are the same ones due to P. Deligne [2] and if a group scheme G acts on the algebraic schemes which appear in the above lemmas, 4.3 and 4.4 then we can find a G-stable ideal satisfying the same property in Lemma 4. 3 and the equivariant blowing-up of the X_i's having the same property in the Lemma 4. 4.

Let us recall the notion of (proper) quasi-dominations of rational maps.

Definition 4. 6. Let X and Y be S-schemes. A quasi-domination $f: X\to Y$ is a couple (U, f) formed by a dense open subscheme U of X and an S-morphism $f: U\to Y$ such that the $\Gamma_f=\{(x, f(x))\,|\,x\in U\}$ is closed in $X\underset{S}{\times}Y$. In this case, we also say that X is quasi-dominant over Y or that $f: U\to Y$ is a quasi-domination of X over Y. In particular, if $f: U\to Y$ is proper, then we say that $f: X\to Y$ is a proper quasi-domination.

Next Theorem 4. 7 is one of key facts to prove the existence of equivariant completion.

Theorem 4.7. *Let S be a noetherian scheme and let G be an affine group scheme over S and let X, Y be finite type S-schemes on which G acts regularly. Furthermore, assume that X is integral, and Y is covered by G-stable open subschemes $(Y_i)_{1\leq i\leq n}$ such that every Y_i is equivariantly embedded into $P(E_i)$ where E_i is a coherent $\mathcal{O}_S$-Module on which there is a dual action of G. Let U, $V (U\subset V \subset X)$ be G-stable open subschemes of X. If a G-morphism $f: U\to Y$ is a quasi-domination of V over Y, then there is a G-stable ideal $\mathfrak{a}$ of X such that* Supp $\mathfrak{a}\subseteq X-V$ *and that $f^*: X^*\xrightarrow{f} Y$ is quasi-dominant where X^* is the blowing-up of X with the center $\mathfrak{a}$.*

Proof. The proof of Theorem 4.7 consists of several steps.

a) Let G act on $X\underset{S}{\times} Y$ diagonally. Replacing Y by $X\underset{S}{\times} Y$. We may assume that there is a G-morphism $p: Y\to X$ such that f is a section of p on U and $f(U)$ is a G-stable closed subscheme of $p^{-1}(V)$. Moreover, we may assume that Y has a G-stable open covering $Y=(Y_i)_{1\leq i\leq n}$ such that every Y_i is equivariantly embedded in $P(E_i)$ where E_i is a coherent $\mathcal{O}_X$-Module on which there is a dual action of G.

b) Reduction to the quasi-projective case. Let $U_i=f^{-1}(Y_i)$, $p_i=p|Y_i\to X$ and $f_i=f|U_i\to Y_i$ for every i $(1\leq i\leq n)$. If $U_i=\phi$ for some i, we may omit such U_i to prove Theorem 4.7. If Theorem 4.7 is true for every $f_i: U_i\to Y_i$, then there is a G-stable ideal $\mathfrak{a}_i$ of X such that Supp $\mathfrak{a}_i\subseteq X-V$ and $X_i\to X\xrightarrow{f_i} Y_i$ is quasi-dominant where X_i is the blowing-up X with the center $\mathfrak{a}_i$. Then, let $\mathfrak{a}=\mathfrak{a}_1\cdots\mathfrak{a}_n$ be the product of all the ideals $\mathfrak{a}_i$ and let $\tilde{X}$ be the blowing-up of X with the center $\mathfrak{a}$. Since there is a G-morphism $\tilde{X}\to X_i$ for every i, $\tilde{X}\to X\xrightarrow{f_i} Y_i$ is quasi-dominant for every i. Therefore, there is a G-stable open subscheme U_i' $(U_i'\supseteq U_i)$ of $\tilde{X}$ such that the graph $\Gamma_{f_i}=\{(x,f_i(x))\,|\,x\in U_i'\}$ is closed in $\tilde{X}\underset{X}{\times} Y_i$ for every i. Hence $\tilde{X}\to X \xrightarrow{f} Y$ is quasi-dominant because X is integral.

c) Reduction to the case $U=V$. Consider the following commutative diagram of schemes.

$$\begin{tikzcd} f(U) \arrow[r] \arrow[d] & p^{-1}(V) \arrow[r] \arrow[d] & Y \arrow[d, "p"] \\ U \arrow[r] & V \arrow[r] & X \end{tikzcd}$$

Let C be the defining ideal of $(\overline{V-U})_{\mathrm{red}}$ ($\overline{V-U}$ being the closure of $V-U$ in X) in X and let I be the defining ideal of $(\overline{f(U)})_{\mathrm{red}}$ ($\overline{f(U)}$ being the closure of $f(U)$ in Y) in Y. Then, if we put $\mathfrak{D}$ to be the kernel of the morphism;

$$O_X \to p_*(O_Y) \to p_*(O_Y/\mathrm{Im}[p^*(C^2) \to O_Y] + I)$$

$\mathfrak{D}$ is G-stable, $\mathrm{Supp}(\mathfrak{D}) \subseteq X-V$ and $\mathfrak{D}$ satisfies the condition of Lemma 4.2. Lemma 4. 2 shows that we may assume $U=V$ after blowing up X with the center $\mathfrak{D}$. Moreover, after blowing up X with the center $X-U$, we may assume that $X-U$ is a Cartier divisor of X.

d) Quasi-projective case. By step (b), we may assume that there is a coherent $\mathcal{O}_X$-module E with a dual action of G and that Y is embedded equivariantly into $P(E)$. For every point x of X, we shall define an ideal $\mathfrak{a}_x$ of $\mathcal{O}_{x,X}$ in the following way. Let $E_x = H^0(\mathrm{Spec}(\mathcal{O}_{x,X}), E)$, $U_x = U \cap \mathrm{Spec}(\mathcal{O}_{x,X})$, $Y_x = p^{-1}(\mathrm{Spec}(\mathcal{O}_{x,X}))$ (p being the structure morphism: $Y \to X$), $f_x = f | U_x \to Y_x$ and let $\overline{f_x(U_x)}$ (with reduced structure) be the closure of $f_x(U_x)$ in $P(E_x)$.

$$\begin{tikzcd} P(E_x) \arrow[r] & P(E) \\ Y_x \arrow[u, hook] \arrow[d, shift left, "p_x"] & Y \arrow[u, hook] \arrow[d, shift left, "p"] \\ \mathrm{Spec}(\mathcal{O}_{x,X}) \arrow[u, shift left, "f_x"] & X \arrow[u, shift left, "f"] \\ U_x \arrow[u, hook] \arrow[r] & U \arrow[u, hook] \end{tikzcd}$$

Then, by the above assumption, $\overline{f_x(U_x)} = \mathrm{Proj}(S^{\cdot}((F_x))$ for a coherent $\mathcal{O}_{x,X}$-module F_x on which there is a dual action of $G \underset{S}{\times} \mathrm{Spec}(\mathcal{O}_{s,S})$ (s being the image of x in S). Let $\{u_1, \cdots, u_n\}$ be a minimal generator of F_x. Then $u_1/u_i, \cdots, u_n/u_i$ can be regarded as rational functions on $\mathrm{Spec}(\mathcal{O}_{x,X})$ through f_x for every i $(1 \leq i \leq n)$. Here we shall define the ideal $\mathfrak{a}_x$ of $\mathcal{O}_{x,X}$ to be the ideal generated by the set $A_x=\{h\cdot u_1/u_i, \cdots, h\cdot u_n/u_i | h\in\mathcal{O}_{x,X}, h\cdot u_j/u_i\in\mathcal{O}_{x,X}$ for some i $(1\leq i\leq n)$ and every j $(1\leq j\leq n)\}$. Then $\mathfrak{a}_x$ is independent of the choice of a minimal generator of F_x. In fact, let $\{u_1', \cdots, u_n'\}$ be an another minimal generator of F_x. Then there is a non-singular $(n\times n)$-matrix (a_{ij}) $(a_{ij}\in\mathcal{O}_{x,X})$ such that $u_i'=\sum_j a_{ji}u_j$ for every i. Assume that $h\cdot u_j/u_i$ is an element of A_x. For every α, β $(1\leq\alpha\leq n, 1\leq\beta\leq n)$, we have $(\sum_j a_{j\alpha}(h\cdot u_j/u_i))u_\beta'/u_\alpha'=\sum_j a_{j\beta}(h\cdot u_j/u_i)\in\mathcal{O}_{x,X}$. Hence, if we put $h_\alpha=\sum_j a_{j\alpha}(h\cdot u_j/u_i)$, then $h_\alpha\in A_x'$ where A_x' is the set defined by $\{u_1', \cdots, u_n'\}$ similarly. Thus $\mathfrak{a}_x$ is independent of the choice of a minimal generator of F_x because (a_{ij}) is a non-singular matrix. We shall put $\mathfrak{a}=(\mathfrak{a}_x)_{x\in X}$. This is a desired G-stable ideal of X. In fact, it is easily seen that $\mathfrak{a}$ is G-stable by the same method in Lemma 18 in [13] and that Supp $\mathfrak{a}\subseteq X-U$, $\tilde{X}\to X\xrightarrow{f}Y$ is quasi-dominant where $\tilde{X}$ is the blowing-up X with the $\mathfrak{a}$. q.e.d.

Now we shall prove an equivariant Chow's lemma. This is a generalization of Theorem 2 in [13]. Before proving it, we note here the following.

Remark 4.8. If X has the property (N) over a noetherian normal scheme S, then Theorem 2.5 and Theorem 3.8 shows that X is covered by G-stable open subschemes $(X_i)_{1\leq i\leq n}$ such that every X_i is equivariantly embedded into $P(E_i)$ where E_i is a coherent $\mathcal{O}_S$-Module on which there is a dual action of G. On the other hand, if X has such an open covering and if $\tilde{X}$ is a blowing-up X with a G-stable ideal of X, then $\tilde{X}$ has also such an open covering. Moreover, if X and Y has such an open covering and if Z is the scheme obtained by patching X and Y along a G-stable open subscheme in X and Y, then Z has also such an open covering. From now on, we shall call such an open covering, a G-stable, quasi-projective open covering, for the simplicity.

Theorem 4.9. (*Equivariant Chow's lemma*) *Let S be a normal noetherian scheme and let G be a surjective smooth affine group scheme over S with connected fibres and let X be an S-scheme having* *the property* (*N*) *on which* G *acts regularly. Assume that* U *is a* G*-stable dense open subscheme of* X *which is quasi-projective over* S. *Then we have a diagram of* S*-schemes;*

$$\begin{tikzcd} X \arrow[dr] & X' \arrow[l, "q"'] \arrow[r, hook] \arrow[d, dash] & \overline{X} \arrow[dl] \\ & S & \end{tikzcd}$$

where

(1) $q: X' \to X$ *is a blowing-up* X *with a center* $\mathfrak{a}$ *such that* $\mathfrak{a}$ *is a* G*-stable ideal of* X, $\operatorname{Supp} \mathfrak{a} \subseteq X - U$ *and*

(2) $\overline{X}$ *is an* S*-projective scheme on which* G *acts regularly and contains* X' *as a* G*-stable dense open subscheme.*

Proof. We may assume that X and S are integral because X and S are normal. Since U is quasi-projective, there is a projective integral S-scheme U^* on which G acts regularly by virtue of Corollary 2.6 and U^* contains U as a G-stable open subscheme. Applying theorem 4.7 to U^* and X, we may assume that U^* is quasi-dominant over X. Thus, there is a G-stable open subscheme V of U^* and a G-morphism $\varphi: V \to X$ such that there is the following diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d, equal] & V \arrow[r] \arrow[d, "\varphi\text{: morphism}"] & U^* \arrow[d] \\ U \arrow[r] & X \arrow[r] & S \end{tikzcd}$$

and the graph $\Gamma_\varphi = \{(x, \varphi(x)) \mid x \in V\}$ is closed in $U^* \underset{S}{\times} X$. Since $U^* \to S$ is projective, φ is proper. Hence $U^* \to X$ is a proper quasi-domination.

We can apply the next lemma to this situation.

Lemma 4.10. *Let* S *be a noetherian scheme and let* U, V, X *and* Y *be finite type* S*-schemes such that there is the following diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d, equal] & V \arrow[r] \arrow[d, "\varphi"] & Y \\ U \arrow[r] & X & \end{tikzcd}$$

where U is a dense open subscheme in V, X and Y and $\varphi\colon V\to X$ is a quasi-domination of Y over X. Moreover assume that $\mathfrak{a}$ is an ideal of X such that $\operatorname{Supp}\mathfrak{a}\subseteq X-U$, *$\tilde{X}\to X\to V$ is a quasi-domination where $\tilde{X}$ is the blowing-up of X with the center $\mathfrak{a}$ and that $\Psi\colon W\to V$ is a quansi-domination of $\tilde{X}$ over V. Then we have that*

(1) $W=\operatorname{Proj}(\sum_{n\geq 0}\varphi^*(\mathfrak{a})^n)$ *and*

(2) *the graph Γ_Ψ of Ψ is closed in* $Y\underset{S}{\times}\tilde{X}$.

Proof. Let $\tilde{V}=\operatorname{Proj}(\sum_{n\geq 0}\varphi^*(\mathfrak{a})^n)$. Then $\tilde{V}=V\underset{X}{\times}\tilde{X}$. Let $\alpha\colon \tilde{V}\to\tilde{X}$ be the second projection and let $\beta\colon W\to\tilde{V}$ be the morphism induced by Ψ.

$$\begin{tikzcd}
\tilde{V} \arrow[r, "\alpha"] \arrow[dd] & \tilde{X} \arrow[dd] & \\
 & & W \arrow[ull, "\beta"'] \arrow[ul, hook] \arrow[dll, "\psi"] \\
V \arrow[r, "\varphi"] & X &
\end{tikzcd}$$

We prove that $\alpha\circ\beta=1_W$ and $\beta\circ\alpha=1_{\tilde{V}}$. Then we can prove $W=\operatorname{Proj}(\sum_{n\geq 0}(\varphi^*(\mathfrak{a})^n)$. First of all, we prove that $\Psi\colon W\to V$ is proper. Let Γ_φ be the graph of φ and let $\pi\colon\tilde{X}\to X$ be the structure morphism. Then $V=(1_V\underset{S}{\times}\pi)^{-1}(\Gamma_\varphi)$ and fhe projection map $\tilde{V}\to V$ is proper. Hence $\Gamma_\psi\to V$ is also proper because Γ_Ψ is closed in $\tilde{V}=V\underset{X}{\times}\tilde{X}$. Therefore, $\Psi\colon W\to V$ is proper. On the other hand, $\beta=(\Psi,\text{ injection})$, $\beta\colon W\to\tilde{V}$ is proper. Since U is contained in W and $\tilde{V}$ as dense open subschemes, $\beta(W)=\Gamma_\Psi=\tilde{V}$, i.e., $\alpha\circ\beta=1_W$ and $\beta\circ\alpha=1_{\tilde{V}}$. Since Γ_φ is closed in $Y\underset{S}{\times}X$ by our assumption, $(1_V\times\pi)^{-1}(\Gamma_\varphi)=\tilde{V}=\Gamma_\Psi$ is closed in $Y\underset{S}{\times}\tilde{X}$. q.e.d.

Let us come back to the proof of Theorem 4.9; By virtue of Theorem 4.7 (cf. step (*d*) Theorem 4.7 and Remark 4.8), there is a G-stable ideal $\mathfrak{a}$ of X such that $\operatorname{Supp}\mathfrak{a}\subseteq X-U$ and $\tilde{X}\to X\to V$ is quasi-dominant where $\tilde{X}$ is the blowing-up of X with the center $\mathfrak{a}$. Applying Lemma 4.10 to this situation, we see that $W=\operatorname{Proj}(\sum_{n\geq 0}\varphi^*(\mathfrak{a})^n)$ where $\Psi\colon W\to V$ is a quasi-domination of $\tilde{X}$ over V. However, φ is a proper morphism, hence $W=\tilde{X}$. Now let $\mathfrak{b}$ be the extension of $\varphi^*(\mathfrak{a})$ to U^* and let $\bar{X}$ be the blowing-up U^* with the center $\mathfrak{b}$. The $X'=\tilde{X}$ and $\bar{X}$ are desired ones. q.e.d.

We shall prepare one more lemma to prove the existence of equivariant completion.

Lemma 4.11. *Let S be a noetherian scheme and let G be an affine group scheme over S and let X_1 and X_2 be integral, finite type S-schemes on which G acts regularly. Moreover assume that each X_i $(i=1,2)$ has G-stable, quasi-projective open coverings and that U is a G-stable open subscheme in both X_1 and X_2. Then there exists a finite type S-scheme X on which G acts regularly such that*

(1) *U is a G-stable open subscheme in X and*

(2) *there are G-rational maps $p_i\colon X\to X_i$ $(i=1,2)$ which are proper quasi-dominations.*

Proof. By virtue of Theorem 4.7, we may assume that X_1 is quasi-dominant over X_2. Therefore we have the following diagram of schemes;

$$\begin{tikzcd} & V \arrow[r] \arrow[d, "\varphi"] & X_1 \\ U \arrow[ur, hook] \arrow[r, hook] & X_2 & \end{tikzcd}$$

($\varphi\colon V\to X_2$ is a morphism and φ is a quasi-domination of X_1 over X_2). By virtue of Theorem 4.7 again, there is a G-stable ideal $\mathfrak{a}$ of X_2 such that $\operatorname{Supp}\mathfrak{a}\subseteq X_2-U$ and $\tilde{X}_2\to X_2\to V$ is quasi-dominant. Lemma 4.10 shows that $W=\operatorname{Proj}(\sum_{n\geq 0}\varphi^*(\mathfrak{a})^n)$ and Γ_Ψ is closed in $X_2\underset{S}{\times} X_1$ where $\Psi\colon W\to V$ is a quasi-domination of X_2 over V. Let $\mathfrak{b}$ be the extension of $\varphi^*(\mathfrak{a})$ to X_1 and let X_1' be the blowing-up of X_1 with the center $\mathfrak{b}$ and let $q\colon X_1'\to X_1$ be the structure morphism. Then W is isomorphic to $q^{-1}(V)$. We shall denote the isomorphism by α. The graph Γ_α of α is closed in $\tilde{X}_2\underset{S}{\times} X_1'$. In fact, since $\Psi=q\circ\alpha$ and Γ_Ψ is closed in $\tilde{X}_2\underset{S}{\times} X_1$, $(1\times q)^{-1}(\Gamma_\Psi)=\Gamma_\alpha$ is closed in $\tilde{X}_2\underset{S}{\times} X_1'$. Let

X be the S-scheme obtained by patching X_1' and X_2 along W and $q_i: X \to X_i$ be the G-rational map $(i=1,2)$. Then X and p_i $(i=1,2)$ are desired ones. q.e.d.

Remark 4.12. The above proof shows that we can take such $p_i: U_i \to X_i$ ($U \subseteq X, p_i: U_i \to X_i$ being a proper morphism) to be a morphism obtained by a blowing-up with a G-stable ideal $\mathfrak{a}_i$ of X_i with Supp $\mathfrak{a}_i \subseteq X_i - U$. This remark is used in the proof of Theorem 4.13.

Now we are in position to prove the existence of equivariant completions.

Theorem 4.13. *Let S be a noetherian normal scheme and let G be a surjective smooth affine group scheme over S with connected fibres and let X be an S-scheme having the property (N) on which G acts regularly. Then there exists a finite type S-scheme $\bar{X}$ on which G acts regularly such that*

(1) *$\bar{X}$ is proper over S,*

(2) *$\bar{X}$ contains X as a G-stable open subscheme and*

(3) *The action of G on $\bar{X}$ is the extension of the action G on X.*

Proof. We may assume that X and S are integral. By virtue of Theorem 2.5 and and Theorem 3.8, X has a G-stable, quasi-projective open covering $(U_i)_{1 \leq i \leq n}$. For every U_i $(1 \leq i \leq n)$, construct the following diagram of integral schemes on which G acts regularly by using Theorem 4.9;

$$\begin{tikzcd} & X_i \arrow[r] \arrow[d, "q_i"] & \bar{X}_i \\ U_i \arrow[ur, hook] \arrow[r, hook] & X & \end{tikzcd}$$

where

(1) $q_i: X_i \to X$ is a G-projective, surjective and birational morphisms,

(2) U_i is G-stable open in both X_i and X and

(3) $\bar{X}_i$ is proper over S and X_i is G-stable open in $\bar{X}_i$.

Let $F_i = X - U_i$ and $F_i' = q_i^{-1}(F_i)$ for every $i (1 \leq i \leq n)$ and let $X^* =$ the closure of $\{(u, \cdots, u) | u \in U\}$ in $\tilde{X}_1 \underset{S}{\times} \cdots \underset{S}{\times} \tilde{X}_n$ where $U = \bigcap_{i=1}^n U_i$ and let $p_i : X^* \to \bar{X}_i$ be the i-th projection for every i $(1 \leq i \leq n)$. Then $\bigcap_{i=1}^n p_i^{-1}(F_i') = \phi$. Moreover, by virtue of Lemma 4. 4, we may assume that $\bigcap_{i=1}^n p_i^{-1}(\bar{F}_i') = \phi$ where $\bar{F}_i'$ is the closure of F_i' in $\bar{X}_i$. Let $\Gamma_i = \{(u, u) | u \in U_i\} \subseteq X \times (\bar{X}_i - \bar{F}_i')$ for i $(1 \leq i \leq n)$. Then Γ_i is closed in $X \underset{S}{\times} (\bar{X}_i - \bar{F}_i')$ because $q_i : X_i \to X$ is proper. Now let M_i be the S-scheme obtained by patching X and $\bar{X}_i - \bar{F}_i'$ along U_i for every i $(1 \leq i \leq n)$. Applying Lemma 4. 11 to X and each M_i, we can construct a finite type S-scheme $\bar{X}$ on which G acts regularly such that

(1) X is a G-stable open subscheme of $\bar{X}$,

(2) $\bar{X}$ quasi-dominates M_i properly $(i = 1, 2, \cdots, n)$ and

(3) The action of G on $\bar{X}$ is an extension of the action on X.

Then, $\bar{X}$ is the desired one. Indeed, $\bar{X}$ is proper over S. Let φ_i be the composite morphism of $X^* - p_i^{-1}(\bar{F}_i') \xrightarrow{p_i} \bar{X}_i - \bar{F}_i'$ and $\bar{X}_i - \bar{F}_i' \xrightarrow{\text{open imm.}} M_i$ for every i $(1 \leq i \leq n)$. Then $\varphi_i | U = \varphi_j | U$ for every i and j. Moreover, let $p_i : V_i \to M_i$ be the proper quasi-domination of $\bar{X}$ over M_i for every i $(1 \leq i \leq n)$. Then $V = \mathrm{Proj}\,(\sum_{m \geq 0} \mathfrak{a}_i^m)$ for some G-stable ideal $\mathfrak{a}_i$ of M_i with $\mathrm{Supp}\,\mathfrak{a}_i \subseteq M_i - X$ as we pointed out in Remark 4. 12. Now assume that $\mathfrak{a}_i'$ is the extended sheaf of $\varphi_i^*(\mathfrak{a}_i)$ to X^* $(i = 1, 2, \cdots, n)$ and that $X^{**} = \mathrm{Proj}\,(\sum_{m \geq 0} (\mathfrak{a}_1' \cdots \mathfrak{a}_n')^m)$. Then there is a G-morphism $\Psi_i : X^{**} - t_i^{-1}(F_i') \to V_i \hookrightarrow \bar{X}$ $(i = 1, 2, \cdots, n)$ (t_i being the composite morphism of $X^{**} \to X^*$ and $p_i : X^* \to X_i$) and $\Psi_i | \bar{U} = \Psi_j | \bar{U}$ for every i and j ($\bar{U}$ being the closure of U in X^{**}) because X^{**} is separeted over S, $\Psi_i | U = \Psi_j | U$ for every i and j and $\{X^{**} - t_i^{-1}(\bar{F}_i')\}_{1 \leq i \leq n}$ is an open covering of X^{**}. Therefore, there is a G-morphism $\Psi : U \to X$ such that $\Psi | U - t_i^{-1}(F_i') = \Psi_i$ $(i = 1, 2, \cdots, n)$ and Ψ is surjective because U is open dense in both $\bar{U}$ and $\bar{X}$. Since $\bar{U}$ is proper over S, so is $\bar{X}$. q.e.d.

DEPARTMENT OF APPLIED MATHEMATICS
KŌNAN UNIVERSITY

References

[1] C. Chevalley, Classification des groupes de Lie algébriques, Seminâire C. Chevalley, 1956-1958.
[2] P. Deligne, Le théorème de plongement de Nagata (unpublished).
[3] A. Grothendieck et J. Dieudonné, Éléments de Géométrie Algébrique, I.H.E.S. Publications mathématiques.
[4] A. Grothendieck, Seminâire de Géométrie Algébrique, Revetements étales et groupe fondamental, Fas. 1, I.H.E.S., 1960-1961.
[5] A. Grothendieck et M. Demazure, Seminâire de Géométrie Algébrique, Schemas en groupes, Fas. 1-3, I.H.E.S., 1963-1964.
[6] D. Mumford, Geometric invariant theory, Springer-Verlag, Ergebnisse der Math. Band 34, 1965.
[7] G. Kempf, F. Knudsen, D. Mumford and B. Sait-Donat, Toroidal Embeddings I, Springer-Verlag, Lecture notes in math. 339, 1973.
[8] M. Nagata, A generalization of the imbedding problem of an abstract variety in a complete variety, Jour. of Math. Kyoto Univ. Vol. 3, 1963, 89-102.
[9] M. Raynaud, Faisceaux amples sur les schemas en groupes et les espaces homogeneous, Springer-Verlag, Lecture notes in math. 19, 1966.
[10] M. Raynaud et L. Gruson, Critéres de platitude et de projectivité, Inventiones math. 13, 1971, 1-89.
[11] M. Raynaud, Anneaux Locaux Henséliens, Springer-Verlag, Lecture notes in math. 169, 1970.
[12] M. Rosenlicht, Some rationality questions on algebraic groups, Annali di matematica pure ed applicata. Vol. 43, 1957, 25-50.
[13] H. Sumihiro, Equivariant completion, Jour. of Math. Kyoto Univ. Vol. 14, 1974, 1-28.