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On conformal transformations of Finsler metrics

By

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The purpose of the present paper is to establish the conformal theory of Finsler metrics based on the theory of Finsler spaces by M. Matsumoto [17]. In the first three sections we shall derive some transformation formulas and invariants systematically without any artificial techniques. In the following section (§ 4) we shall treat the spaces with some special Finsler metrics and find the conditions that a space be conformal to one of such spaces. The last section (§ 5) is devoted to studying the special conformal transformation named *C-conformal*, which yields a generalization of the concurrent field treated by S. Tachibana [27] and M. Matsumoto and K. Eguchi [24].

In consequence of these considerations, a geometrical meaning of the condition $T_{hijk}=0$ treated by M. Matsumoto [21,23] and H. Kawaguchi [13] shall be clarified in terms of the tensor P_{hijk} (Theorem 4.3). In the theory of E. Cartan [8], there are three kinds of curvature tensors R_{hijk}, P_{hijk} and S_{hijk}, the second of which seems to offer results most interesting as the Finsler geometry, and a few results have been obtained (G. Landsberg [16], L. Berwald [3,4,5], M. H. Akbar-Zadeh [1], M. Matsumoto [18,19,20,22], H. Kawaguchi [13], M. Hashiguchi [11], M. Hashiguchi, S. Hōjō and M. Matsumoto [12] etc.). Our results may contribute a little to the study of P_{hijk}.

The first to treat the conformal theory of Finsler metrics generally was, to the best of the author's knowledge, M. S. Knebelman [14]. He defined two metric functions L and $\bar{L}$ as *conformal* if the length of an arbitrary vector in the one is proportional to the length in the other, that is, if $\bar{g}_{ij}=\phi g_{ij}$. The length of a vector ξ means here

$(g_{ij}(x, \dot{x})\xi^i\xi^j)^{1/2}$, where g_{ij} is the Finsler metric tensor. And, from the fact that ϕg_{ij}, as well as g_{ij}, must be a Finsler metric tensor, he showed that ϕ falls into at most a point function. We shall call this result *Knebelman's theorem*. In the conformal theory of the Finsler metrics, it seems that few good results are obtained. This situation may inherently be due to Knebelman's theorem. For example, the concept that a space be conformal to a Riemannian space is meaningless, because such a space is nothing but Riemannian. In order that we obtain really Finsler-like results, it might be better that we obey other definitions. In the present paper we shall try to give a somewhat different definition, even if it coincides with Knebelman's one.

Throughout the present paper we shall use the terminologies and notations described in Matsumoto's monograph [17]. As to Finsler connections, for convenience we shall give an outline of the theory in § 2. The Finsler connections used as examples are mainly the ones $C\Gamma$, $R\Gamma$ and $B\Gamma$ given by E. Cartan [8], H. Rund [25] and L. Berwald [2], but we shall also refer to another special connection $H\Gamma$.

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§ 1. Preliminaries.

1. 1. Throughout the present paper, x^i denotes a point of a base manifold M and y^i a supporting element $\dot{x}^i$, and the subscript in parenthesis (i) means $\partial/\partial y^i$. Given a Finsler metric function L, the Finsler metric G is defined by $g_{ij} := (L^2/2)_{(i)(j)}$, and we put $y_i := g_{ir}y^r$, $l^i := y^i/L$, $l_i := L_{(i)} = g_{ir}l^r = y_i/L$ and $h_{ij} := g_{ij} - l_i l_j$.

Definition. Let two Finsler metrics G and $\bar{G}$ be defined by Finsler metric functions L and $\bar{L}$ over a differentiable manifold M of dimension n. The Finsler metrics or the Finsler spaces equipped with them are called *conformal* if the angles between any vector v and *a* supporting element y are equal, that is, if

$$(1.1)\qquad \frac{g_{ij}(x, y)\, y^i v^j}{L(x, y)\,(g_{ij}(x, y)\, v^i v^j)^{1/2}} = \frac{\bar{g}_{ij}(x, y)\, y^i v^j}{\bar{L}(x, y)\,(\bar{g}_{ij}(x, y)\, v^i v^j)^{1/2}}$$

holds for any v^i.

As a formal generalization of conformality in Riemannian geometry, it might be natural that we call G and $\bar{G}$ conformal if

$$\frac{g_{ij}(x,y)u^i v^j}{(g_{ij}(x,y)u^i u^j)^{1/2}(g_{ij}(x,y)v^i v^j)^{1/2}} = \frac{\bar{g}_{ij}(x,y)u^i v^j}{(\bar{g}_{ij}(x,y)u^i u^j)^{1/2}(\bar{g}_{ij}(x,y)v^i v^j)^{1/2}} \tag{1.2}$$

holds for any u^i and v^j. Suggested by the definition of spaces of scalar curvature (L. Berwald [6]), however, we have adapted the weaker and Finsler-like condition (1.1).

Let (1.1) hold for any v^i. Then we have

$$\bar{L}^2(y_i y_j \bar{h}_{kl} + y_i y_k \bar{h}_{jl} + y_i y_l \bar{h}_{jk} + y_j y_l \bar{h}_{ik} + y_j y_k \bar{h}_{il} + y_k y_l \bar{h}_{ij}) = L^2(\bar{y}_i \bar{y}_j h_{kl} + \bar{y}_i \bar{y}_k h_{jl} + \bar{y}_i \bar{y}_l h_{jk} + \bar{y}_j \bar{y}_l h_{ik} + \bar{y}_j \bar{y}_k h_{il} + \bar{y}_k \bar{y}_l h_{ij}). \tag{1.3}$$

If we contract (1.3) by y^l, we obtain owing to $h_{kl}y^l=0$

$$y_i \bar{h}_{jk} + y_j \bar{h}_{ik} + y_k \bar{h}_{ij} = \bar{y}_i h_{jk} + \bar{y}_j h_{ik} + \bar{y}_k h_{ij}\,. \tag{1.4}$$

The contraction of (1.4) by y^k gives

$$\bar{h}_{ij}/\bar{L}^2 = h_{ij}/L^2\,. \tag{1.5}$$

Then, putting $z_i := \bar{y}_i - (\bar{L}/L)^2 y_i$, (1.4) is rewritten in the form

$$z_i h_{jk} + z_j h_{ik} + z_k h_{ij} = 0\,. \tag{1.6}$$

Contracting (1.6) by g^{jk} and paying attention to $z_j h_i^j = z_i$, we obtain $z_i=0$. Thus we have

$$\bar{y}_i/\bar{L}^2 = y_i/L^2\,. \tag{1.7}$$

From (1.5), (1.7) we have immediately

$$\bar{g}_{ij} = \phi g_{ij}\,, \tag{1.8}$$

where $\phi=(\bar{L}/L)^2$, and arrive at Knebelman's definition. The converse holds clearly and we have

Theorem 1.1. *Two Finsler metrics are conformal if and only if the corresponding metric tensors are proportional to each other.*

By Knebelman's theorem, the factor of proportionality depends at most on the point x^i of the base manifold M. For convenience we shall write

$$\bar{g}_{ij}=e^{2\alpha}g_{ij}, \tag{1.9}$$

where $\alpha=\alpha(x)$, and call the transformation of G to $\bar{G}$ the *conformal transformation* α.

1. 2. Given two conformal Finsler metrics G and $\bar{G}$, we have from (1.9)

$$\bar{L}=e^{\alpha}L, \tag{1.10}$$

$$\bar{l}^i=e^{-\alpha}l^i, \quad \bar{l}_i=e^{\alpha}l_i, \quad \bar{y}_i=e^{2\alpha}y_i, \tag{1.11}$$

$$\bar{g}=e^{2n\alpha}g, \tag{1.12}$$

$$\bar{g}^{ij}=e^{-2\alpha}g^{ij}, \tag{1.13}$$

where $g:=\det(g_{ij})$ and $(g^{ij}):=(g_{ij})^{-1}$.

As to the torsion tensor of E. Cartan, it holds

$$\bar{C}_{ijk}=e^{2\alpha}C_{ijk}, \tag{1.14}$$

$$\bar{C}^i_{jk}=C^i_{jk}, \tag{1.15}$$

$$\bar{C}_k^{\ ij}=e^{-2\alpha}C_k^{\ ij}, \tag{1.16}$$

where $C_{ijk}:=\frac{1}{2}g_{ij(k)}$, $C^i_{jk}:=g^{ir}C_{rjk}$ and $C_k^{\ ij}:=g^{js}C^i_{sk}=-\frac{1}{2}g^{ij}_{(k)}$.

1.3. We have already some conformal invariants:

Proposition 1.1. *The vector* $l_i/L(=y_i/L^2)$, *the tensors* g_{ij}/L^2, h_{ij}/L^2, C^i_{jk} *and the tensor densities* $g^{-(1/n)}g_{ij}$, $(\frac{1}{2}g^{-(1/n)}L^2)_{(i)(j)}$ *are conformally invariant.*

In the two-dimensional case, it holds

$$\bar{m}^i=e^{-\alpha}m^i, \quad \bar{m}_i=e^{\alpha}m_i, \tag{1.17}$$

where m^i is the unit vector orthogonal to the supporting element and $m_i:=g_{ir}m^r$. So we have

Proposition 1.2. *In the two-dimensional Finsler spaces, the vectors Lm^i, m_i/L, the main scalar $I:=LC_{ijk}m^i m^j m^k$ and the Landsberg angle θ defined by $d\theta=(m_i/L)dy^i$ are conformally invariant.*

Some of the conformal invariants listed above may be used as the definition of conformality. For example we have

Proposition 1. 3. *Two Finsler metrics G and $\bar{G}$ are conformal if and only if $l_i/L=\bar{l}_i/\bar{L}$.*

Proof. Let us assume that $l_i/L=\bar{l}_i/\bar{L}$, i.e.,

$$\bar{y}_i=\phi y_i\,. \tag{1.18}$$

Since it holds $g_{ir(j)}y^r=0$ for the Finsler metric, the differentiation by y^j gives

$$\bar{g}_{ij}=\phi_{(j)}y_i+\phi g_{ij}\,. \tag{1.19}$$

If we contract this by y^i, it follows $\phi_{(j)}=0$ from (1.18). Thus (1.19) becomes $\bar{g}_{ij}=\phi g_{ij}$.

1. 4. To find the further conformal invariants, we shall here pay attention to $\bar{L}^2=e^{2\alpha}L^2$, i.e., $\log\bar{L}^2=2\alpha+\log L^2$. The tensor $B_{ij}:=(\log L^2)_{(i)(j)}$ is clearly a conformal invariant, and it is expressed by

$$B_{ij}=(2/L^2)(g_{ij}-2l_i l_j)\,. \tag{1.20}$$

As it is easily verified, (B_{ij}) has the inverse (B^{ij}), where

$$B^{ij}=(L^2/2)(g^{ij}-2l^i l^j)\,, \tag{1·21}$$

which is also conformally invariant.

1. 5. Next, we shall find the transformation formulas of the connection parameters. As to the Christoffel symbols given by

$$\gamma^i_{jk}:=\tfrac{1}{2}g^{ir}(\partial g_{jr}/\partial x^k+\partial g_{rk}/\partial x^j-\partial g_{jk}/\partial x^r)\,, \tag{1.22}$$

we have

$$\bar{\gamma}^i_{jk}=\gamma^i_{jk}-(g_{jk}\alpha^i-\delta_j{}^i\alpha_k-\delta_k{}^i\alpha_j)\,, \tag{1.23}$$

where $\alpha_i := \partial\alpha/\partial x^i$, $\alpha^i := g^{ir}\alpha_r$ and $\delta_j{}^i$ is the Kronecker delta. Putting

$$G^i := \tfrac{1}{2}\gamma^i_{jk}y^j y^k\,, \tag{1.24}$$

we have from (1.23) and (1.21)

$$\overline{G}^i = G^i - B^i\,, \tag{1.25}$$

where $B^i := B^{ir}\alpha_r$.

Let us assume that the added subscript means the differentiation by the supporting element except for α_i, e.g., $G_j{}^i := G^i_{(j)}$, $G^i_{jk} := G^i_{j(k)}$, $G^i_{jkl} := G^i_{jk(l)}$. By the successive differentiations of (1.25), it follows

$$\overline{G}_j{}^i = G_j{}^i - B_j{}^i\,, \tag{1.26}$$

$$\overline{G}^i_{jk} = G^i_{jk} - B^i_{jk}, \tag{1.27}$$

$$\overline{G}^i_{jkl} = G^i_{jkl} - B^i_{jkl}. \tag{1.28}$$

By direct calculations we have

$$B_j{}^{ir} := B^{ir}_{(j)} = y_j g^{ir} - \delta_j{}^i y^r - \delta_j{}^r y^i - L^2 C_j{}^{ir}, \tag{1.29}$$

$$\begin{aligned} B^{ir}_{jk} &:= B^{ir}_{j(k)} \\ &= g_{jk}g^{ir} - \delta_j{}^i\delta_k{}^r - \delta_j{}^r\delta_k{}^i - 2C_j{}^{ir}y_k - 2C_k{}^{ir}y_j - L^2 C^{ir}_{j(k)}, \end{aligned} \tag{1.30}$$

$$B_j{}^i := B^i_{(j)} = B_j{}^{ir}\alpha_r = y_j\alpha^i - \delta_j{}^i\alpha_0 - y^i\alpha_j - L^2 C_j{}^i, \tag{1.31}$$

$$\begin{aligned} B^i_{jk} &:= B^i_{j(k)} = B^{ir}_{jk}\alpha_r \\ &= g_{jk}\alpha^i - \delta_j{}^i\alpha_k - \delta_k{}^i\alpha_j - 2C_j{}^i y_k - 2C_k{}^i y_j - L^2 C^i_{j(k)}, \end{aligned} \tag{1.32}$$

where $\alpha_0 := \alpha_r y^r$, $C_j{}^i := C_j{}^{ir}\alpha_r$.

1.6. If we see (1.25) the differential equation of the unknown function α, it is solved with respect to α_i, and α_i is expressed by

$$\alpha_i = B_{ij}(G^j - \overline{G}^j). \tag{1.33}$$

On the other hand, from (1.31) we have

$$B_j{}^i y_i = -L^2\alpha_j\,, \tag{1.34}$$

so it follows, from the formula (1.26) representing the transformation of the non-linear connection parameter by E. Cartan, that

$$\alpha_j = (y_i/L^2)(\bar{G}_j{}^i - G_j{}^i). \tag{1.35}$$

Hence in order that two metrics be conformal it is necessary that the vectors appearing as the right-hand members of (1.33) and (1.35) are gradient vectors over the base manifold M.

1.7. Let Γ^{*i}_{jk} be the connection parameter by E. Cartan, i.e.,

$$\Gamma^{*i}_{jk} = \gamma^i_{jk} - C^i_{jm} G_k{}^m - C^i_{km} G_j{}^m + g^{in} C_{jkm} G_n{}^m. \tag{1.36}$$

The transformation formula of Γ^{*i}_{jk} has the following form:

$$\bar{\Gamma}^{*i}_{jk} = \Gamma^{*i}_{jk} - U^i_{jk}, \tag{1.37}$$

and we have

$$\begin{aligned} U^i_{jk} &= g_{jk}\alpha^i - \delta_j{}^i\alpha_k - \delta_k{}^i\alpha_j - C^i_{jm}B_k{}^m - C^i_{km}B_j{}^m + g^{in}C_{jkm}B_n{}^m \\ &= g_{jk}\alpha^i - \delta_j{}^i\alpha_k - \delta_k{}^i\alpha_j - C_j{}^i y_k - C_k{}^i y_j + C_{jk} y^i + \alpha_0 C^i_{jk} \\ &\quad + L^2(C_j{}^m C^i_{mk} + C_j{}^{im} C_{mk} - C_m{}^i C^m_{jk}), \end{aligned} \tag{1.38}$$

where $C_{jk} := C^r_{jk}\alpha_r$. If we put

$$\begin{aligned} U^{ir}_{jk} &:= g_{jk}g^{ir} - \delta_j{}^i\delta_k{}^r - \delta_j{}^r\delta_k{}^i - C^i_{jm}B_k{}^{mr} - C^i_{km}B_j{}^{mr} + g^{in}C_{jkm}B_n{}^{mr} \\ &= g_{jk}g^{ir} - \delta_j{}^i\delta_k{}^r - \delta_j{}^r\delta_k{}^i - C_j{}^{ir}y_k - C_k{}^{ir}y_j + C^i_{jk}y^r + C^r_{jk}y^i \\ &\quad + L^2(C_j{}^{mr}C^i_{mk} + C_j{}^{im}C^r_{mk} - C_m{}^{ir}C^m_{jk}), \end{aligned} \tag{1.39}$$

it holds $U^i_{jk} = U^{ir}_{jk}\alpha_r$.

1.8. For later use, we shall define the tensors V^{ir}_{jk}, A^{ir}_{jk} and H^{ir}_{jk} as follows:

$$\begin{aligned} V^{ir}_{jk} &:= U^{ir}_{jk} - B^{ir}_{jk} \\ &= L^2 C_j{}^{ir}|_k + C_j{}^{ir}y_k + C_k{}^{ir}y_j + C^i_{jk}y^r + C^r_{jk}y^i, \end{aligned} \tag{1.40}$$

where the long solidus | means the v-covariant differentiation with respect to the connection $C\Gamma$ by E. Cartan,

$$\begin{aligned} A^{ir}_{jk} &:= U^{ir}_{jk} + C^i_{jm}B_k{}^{mr} \\ &= g_{jk}g^{ir} - \delta_j{}^i\delta_k{}^r - \delta_j{}^r\delta_k{}^i - C^i_{km}B_j{}^{mr} + g^{in}C_{jkm}B_n{}^{mr} \\ &= g_{jk}g^{ir} - \delta_j{}^i\delta_k{}^r - \delta_j{}^r\delta_k{}^i - C_k{}^{ir}y_j + C^r_{jk}y^i + L^2 g^{rl}S^i_{jkl}, \end{aligned} \tag{1.41}$$

where

$$(1.42)\quad S^i_{jkl}:=C^m_{jl}C^i_{mk}-C^m_{jk}C^i_{ml},$$

and

$$(1.43)\quad \begin{aligned} H^{ir}_{jk}:&=B^{ir}_{jk}+C^i_{jm}B_k{}^{mr}\\ &=g_{jk}g^{ir}-\delta_j{}^i\delta_k{}^r-\delta_j{}^r\delta_k{}^i-C_j{}^{ir}y_k-2C_k{}^{ir}y_j-C^i_{jk}y^r\\ &\quad -L^2(C^{ir}_{j(k)}+C^i_{jm}C_k{}^{mr}).\end{aligned}$$

The contractions of the above formulas with α_r give us

$$(1.44)\quad \begin{aligned} V^i_{jk}:&=V^{ir}_{jk}\alpha_r=U^i_{jk}-B^i_{jk}\\ &=L^2(C_j{}^i|_k+C_j{}^{ir}C_{rk})+C_j{}^iy_k+C_k{}^iy_j+C_{jk}y^i+\alpha_0C^i_{jk},\end{aligned}$$

$$(1.45)\quad \begin{aligned} A^i_{jk}:&=A^{ir}_{jk}\alpha_r=U^i_{jk}+C^i_{jm}B_k{}^m\\ &=g_{jk}\alpha^i-\delta_j{}^i\alpha_k-\delta_k{}^i\alpha_j-C^i_{km}B_j{}^m+g^{in}C_{jkm}B_n{}^m\\ &=g_{jk}\alpha^i-\delta_j{}^i\alpha_k-\delta_k{}^i\alpha_j-C_k{}^iy_j+C_{jk}y^i+L^2\alpha^lS^i_{jkl},\end{aligned}$$

$$(1.46)\quad \begin{aligned} H^i_{jk}:&=H^{ir}_{jk}\alpha_r=B^i_{jk}+C^i_{jm}B_k{}^m\\ &=g_{jk}\alpha^i-\delta_j{}^i\alpha_k-\delta_k{}^i\alpha_j-2C_k{}^iy_j-\alpha_0C^i_{jk}-C_j{}^iy_k\\ &\quad -L^2(C^i_{j(k)}+C^i_{jm}C_k{}^m).\end{aligned}$$

Here, we have also many conformal invariants as follows:

Proposition 1.4. *The tensors B_{ij}, B^{ir}, $B_j{}^{ir}$, B^{ir}_{jk}, U^{ir}_{jk}, V^{ir}_{jk}, A^{ir}_{jk} and H^{ir}_{jk} are conformally invariant. And, these are symmetric in i, r and j, k except latter two.*

§ 2. Differences of Finsler connections.

2.1. Given a differentiable manifold M of dimension n, we denote by $L(M)$ $(M, \pi, GL(n, R))$ the bundle of linear frames and by $T(M)$ $(M, \tau, V, GL(n, R))$ the tangent bundle, where the standard fibre V is a vector space of dimension n with a fixed base $\{e_a\}$. The induced bundle $\tau^{-1}L(M)=\{(y, z)\in T(M)\times L(M)\,|\,\tau(y)=\pi(z)\}$ is called the *Finsler bundle* of M and denoted by $F(M)(T(M), \pi_1, GL(n, R))$. The Lie algebra of the structural group $GL(n, R)$ of $L(M)$ and $F(M)$ is denoted by $L(n, R)$ and the canonical base by $\{L_a{}^b\}$.

The *Finsler connection* $F\Gamma$ on M is by the third definition of M. Matsumoto a triad (Γ_V, N, Γ^v) of a V-connection Γ_V in $L(M)$, a non-linear connection N in $T(M)$ and a vertical connection Γ^v in $F(M)$.

2. 2. In $F(M)$ the *fundamental vector field* $Z(A)$ $(A \in L(n, R))$ is defined. If a Finsler connection $F\Gamma$ is given, the h- and the v-basic vector fields $B^h(v)$, $B^v(v)$ $(v \in V)$ are defined in $F(M)$, and these three fields span the tangent space of $F(M)$ at each point. In terms of a canonical coordinate system $(x^i, y^i, z_a{}^i)$ of $F(M)$, they are expressed by

$$Z(A) = A_b{}^a z_a{}^i (\partial/\partial z_b{}^i), \tag{2.1}$$

$$B^h(v) = v^a z_a{}^i (\partial/\partial x^i - \boldsymbol{N}_i{}^j \partial/\partial y^j - z_b{}^j \boldsymbol{F}^k_{ji} \partial/\partial z_b{}^k), \tag{2.2}$$

$$B^v(v) = v^a z_a{}^i (\partial/\partial y^i - z_b{}^j \boldsymbol{C}^k_{ji} \partial/\partial z_b{}^k), \tag{2.3}$$

where $A = A_b{}^a \boldsymbol{L}_a{}^b \in L(n, R)$ and $v = v^a \boldsymbol{e}_a \in V$, and $\boldsymbol{F}^i_{jk}$, $\boldsymbol{N}_j{}^i$, $\boldsymbol{C}^i_{jk}$ are called the *connection parameters* of $F\Gamma$. As a trivial vertical connection in $F(M)$, there is the *vertical flat connection,* with respect to which (2. 3) becomes

$$Y(v) = v^a z_a{}^i \partial/\partial y^i . \tag{2.4}$$

2. 3. Let K be a Finsler tensor field. The h- and the v- *covariant derivatives* of K are defined by $\Delta^h K(v) := B^h(v) K$ and $\Delta^v K(v) := B^v(v) K$ respectively. In particular, the v-covariant derivative with respect to the vertical flat connection is called the 0-*covariant derivative* and denoted by $\Delta^\circ K$.

In terms of a canonical coordinate system $(x^i, y^i, z_a{}^i)$, the components of $\Delta^h K$ and $\Delta^v K$ are denoted by $K_j{}^i{}_{|k}$ and $K_j{}^i|_k$ respectively, if K is assumed, for instance, to be of type (1,1), i.e., $K = z^{-1}{}_i{}^a z_b{}^j K_j{}^i \boldsymbol{e}_a \otimes \boldsymbol{e}^b$, and they are expressed as follows:

$$K_j{}^i{}_{|k} = \delta K_j{}^i / \partial x^k + K_j{}^m \boldsymbol{F}^i_{mk} - K_m{}^i \boldsymbol{F}^m_{jk}, \tag{2.5}$$

$$K_j{}^i|_k = K^i_{j(k)} + K_j{}^m \boldsymbol{C}^i_{mk} - K_m{}^i \boldsymbol{C}^m_{jk}, \tag{2.6}$$

where $\delta/\partial x^k := \partial/\partial x^k - \boldsymbol{N}_k{}^m \partial/\partial y^m$. The components of $\Delta^\circ K$ are $K^i_{j(k)}$.

2.4. If we consider the Lie products [,] of the basic vector fields, we have the following *structure equations*:

$$[B^h(1), B^h(2)] = B^h(T(1,2)) + B^v(R^1(1,2)) + Z(R^2(1,2)), \tag{2.7}$$

$$[B^h(1), B^v(2)] = B^h(C(1,2)) + B^v(P^1(1,2)) + Z(P^2(1,2)), \tag{2.8}$$

$$[B^v(1), B^v(2)] = B^v(S^1(1,2)) + Z(S^2(1,2)), \tag{2.9}$$

where we put $i := v_i$ $(i=1,2)$ for brevity, and from which we have five kinds of torsion tensors T, C, R^1, P^1, S^1 and three kinds of curvature tensors R^2, P^2, S^2.

They are called the $(h)h$-, the $(h)hv$-, the $(v)h$-, the $(v)hv$- and the $(v)v$-*torsion tensors* and the h-, the hv- and the v-*curvature tensors*, and their components are denoted by $\boldsymbol{T}^i_{jk}$, $\boldsymbol{C}^i_{jk}$, $\boldsymbol{R}^i_{jk}$, $\boldsymbol{P}^i_{jk}$, $\boldsymbol{S}^i_{jk}$, $\boldsymbol{R}_h{}^i{}_{jk}$, $\boldsymbol{P}_h{}^i{}_{jk}$ and $\boldsymbol{S}_h{}^i{}_{jk}$ respectively. If $\mathfrak{S}_{jk}\{\cdots\}$ denotes, for instance,

$$\mathfrak{S}_{jk}\{A_{jk}\} = A_{jk} - A_{kj}, \tag{2.10}$$

they are expressed as follows:

$$\boldsymbol{T}^i_{jk} = \mathfrak{S}_{jk}\{\boldsymbol{F}^i_{jk}\}, \tag{2.11}$$

$$\boldsymbol{C}^i_{jk} = \text{the same as the connection parameter } \boldsymbol{C}^i_{jk}, \tag{2.12}$$

$$\boldsymbol{R}^i_{jk} = \mathfrak{S}_{jk}\{\delta \boldsymbol{N}_j{}^i/\partial x^k\}, \tag{2.13}$$

$$\boldsymbol{P}^i_{jk} = \boldsymbol{N}_j{}^i{}_{(k)} - \boldsymbol{F}^i_{kj}, \tag{2.14}$$

$$\boldsymbol{S}^i_{jk} = \mathfrak{S}_{jk}\{\boldsymbol{C}^i_{jk}\}, \tag{2.15}$$

$$\boldsymbol{R}_h{}^i{}_{jk} = \mathfrak{S}_{jk}\{\delta \boldsymbol{F}_h{}^i{}_j/\partial x^k + \boldsymbol{F}_h{}^m{}_j \boldsymbol{F}_m{}^i{}_k\} + \boldsymbol{C}_h{}^i{}_m \boldsymbol{R}^m_{jk}, \tag{2.16}$$

$$\boldsymbol{P}_h{}^i{}_{jk} = \boldsymbol{F}_h{}^i{}_{j(k)} - \boldsymbol{C}_h{}^i{}_{k|j} + \boldsymbol{C}_h{}^i{}_m \boldsymbol{P}^m_{jk}, \tag{2.17}$$

$$\boldsymbol{S}_h{}^i{}_{jk} = \mathfrak{S}_{jk}\{\boldsymbol{C}_h{}^i{}_{j(k)} + \boldsymbol{C}_h{}^m{}_j \boldsymbol{C}_m{}^i{}_k\}. \tag{2.18}$$

2.5. When a Finsler metric is given, various Finsler connections are defined from the metric. The well-known examples are the ones $C\Gamma$, $R\Gamma$ and $B\Gamma$ defined by E. Cartan, H. Rund and L. Berwald. There are some methods by which one Finsler connection is converted to some others. The author [9] proposed two such methods. If we apply them to $C\Gamma$, we have $R\Gamma$ and $B\Gamma$ respectively. The former is equal to the one called the *C-process* by M. Matsumoto. Besides, M. Matsumoto gave the important one called the *P^1-process*. The C-

and the P^1-processes are characterized by expelling the torsion tensors C and P^1 respectively.

Interestingly, the two processes commute with each other. If we apply to $C\Gamma$ the P^1-process after applying the C-process, we have $B\Gamma$ following $R\Gamma$. On the other hand, if the C-process is applied after the P^1-process, we have also $B\Gamma$ via a new connection, which we shall denote by $H\Gamma$, even if its applications belong to the future.

As a Finsler connection defined by a given Finsler metric, we shall use one of the above four in the following. To denote the h-covariant differentiation, we shall use the short solidus $_{|}$ in the cases of $C\Gamma$, $R\Gamma$, and the semi-colon in the cases of $H\Gamma$, $B\Gamma$ in which P^1 vanishes. As to the v-covariant differentiation, the long solidus $|$ is used in the cases of $C\Gamma$, $H\Gamma$, but since in the cases of $R\Gamma$, $B\Gamma$ in which C vanishes it reduces to the 0-covariant differentiation, the parenthesis ( ) may be used. And for the above four connections we have

Proposition 2. 1. *The connection parameters and the components of the torsion and the curvature tensors are as follows*:

	Γ_V	N	Γ^v	T	C	R^1	P^1	S^1	R^2	P^2	S^2
general	$\boldsymbol{F}^i_{jk}$	$\boldsymbol{N}_j{}^i$	$\boldsymbol{C}^i_{jk}$	$\boldsymbol{T}^i_{jk}$	$\boldsymbol{C}^i_{jk}$	$\boldsymbol{R}^i_{jk}$	$\boldsymbol{P}^i_{jk}$	$\boldsymbol{S}^i_{jk}$	$\boldsymbol{R}^i_{hjk}$	$\boldsymbol{P}^i_{hjk}$	$\boldsymbol{S}^i_{hjk}$
$C\Gamma$	Γ^{*i}_{jk}	$G_j{}^i$	C^i_{jk}	0	C^i_{jk}	R^i_{jk}	P^i_{jk}	0	R^i_{hjk}	P^i_{hjk}	S^i_{hjk}
$R\Gamma$	Γ^{*i}_{jk}	$G_j{}^i$	0	0	0	R^i_{jk}	P^i_{jk}	0	R^{*i}_{hjk}	Γ^{*i}_{hjk}	0
$H\Gamma$	G^i_{jk}	$G_j{}^i$	C^i_{jk}	0	C^i_{jk}	R^i_{jk}	0	0	H^i_{hjk}	Q^i_{hjk}	S^i_{hjk}
$B\Gamma$	G^i_{jk}	$G_j{}^i$	0	0	0	R^i_{jk}	0	0	K^i_{hjk}	G^i_{hjk}	0

(2. 13*) $$R^i_{jk}=\mathfrak{S}_{jk}\{\delta G_j{}^i/\partial x^k\},$$

(2. 14*) $$P^i_{jk}=C^i_{jk|0},$$

(2. 16C) $$R^i_{hjk}=R^{*i}_{hjk}+C^i_{hm}R^m_{jk},$$

(2. 16R) $$R^{*i}_{hjk}=\mathfrak{S}_{jk}\{\delta\Gamma^{*i}_{hj}/\partial x^k+\Gamma^{*m}_{hj}\Gamma^{*i}_{mk}\},$$

(2. 16H) $$H^i_{hjk}=K^i_{hjk}+C^i_{hm}R^m_{jk},$$

(2. 16B) $$K^i_{hjk}=\mathfrak{S}_{jk}\{\delta G^i_{hj}/\partial x^k+G^m_{hj}G^i_{mk}\},$$

(2. 17C) $$P^i_{hjk}=\Gamma^{*i}_{hjk}-C^i_{hk|j}+C^i_{hm}P^m_{jk},$$

(2. 17R) $$\Gamma^{*i}_{hjk}=\Gamma^{*i}_{hj(k)},$$

(2.17H) $$Q^i_{hjk}=G^i_{hjk}-C^i_{hk;j},$$

(2.17B) $$G^i_{hjk}=G^i_{hj(k)},$$

(2.18*) $$S^i_{hjk}=\mathfrak{S}_{jk}\{C^m_{hk}C^i_{mj}\},$$

where $\delta/\partial x^k=\partial/\partial x^k-G_k{}^m\partial/\partial y^m$.

P^i_{hjk} is also expressed as follows:

(2.19) $$P^i_{hjk}=g^{ir}\mathfrak{S}_{hr}\{P_{rjk(h)}+C_{rkm}P^m_{hj}\},$$

where $P_{rjk}:=g_{ri}P^i_{jk}$.

2.6. Given two Finsler connections $F\Gamma$ and $\bar{F}\bar{\Gamma}$, the basic vector fields $\bar{B}^h(v)$, $\bar{B}^v(v)$ with respect to $\bar{F}\bar{\Gamma}$ are expressed by the basic vector fields with respect to $F\Gamma$ and the fundamental vector field as follows:

(2.20) $$\bar{B}^h(v)=B^h(v)+B^v(D^h(v))+Z(A^h(v)),$$

(2.21) $$\bar{B}^v(v)=\qquad B^v(v)+Z(A^v(v)),$$

where D^h, A^h and A^v are called the *difference tensors*, and are expressed, in terms of a canonical coordinate system $(x^i, y^i, z_a{}^i)$, by

(2.22) $$(D^h)_j{}^i=\boldsymbol{N}_j{}^i-\bar{\boldsymbol{N}}_j{}^i,$$

(2.23) $$(A^h)^i_{jk}=(\boldsymbol{F}^i_{jk}-\bar{\boldsymbol{F}}^i_{jk})+\boldsymbol{C}^i_{jm}(\boldsymbol{N}_k{}^m-\bar{\boldsymbol{N}}_k{}^m),$$

(2.24) $$(A^v)^i_{jk}=\boldsymbol{C}^i_{jk}-\bar{\boldsymbol{C}}^i_{jk}.$$

2.7. We shall here consider the conformal transformations. From (1.15), (1.26), (1.27), (1.37), (1.45) and (1.46), we have

Theorem 2.1. *Let two Finsler connections $F\Gamma$ and $\bar{F}\bar{\Gamma}$ be defined from the Finsler metrics G and $\bar{G}$ conformally corresponded.*

If we denote by $\boldsymbol{B}_j{}^i$ and $\boldsymbol{A}^i_{jk}$ the components of the difference tensors D^h and A^h respectively, they become as follows:

(2.25) $$\boldsymbol{B}_j{}^i=B_j{}^i,$$

(2.26) $$(\textit{Case of } C\Gamma)\quad \boldsymbol{A}^i_{jk}=A^i_{jk}(=U^i_{jk}+C^i_{jm}B_k{}^m),$$

$$(\textit{Case of } R\Gamma)\quad \boldsymbol{A}^i_{jk}=U^i_{jk},$$

(*Case of* $H\Gamma$) $\quad A^i_{jk}=H^i_{jk}(=B^i_{jk}+C^i_{jm}B_k{}^m)$,

(*Case of* $B\Gamma$) $\quad A^i_{jk}=B^i_{jk}$.

And the difference tensor A^v *vanishes.*

2. 8. There is an important tensor D called the *deflection tensor*, which expresses a relation between the V-connection Γ_V and the non-linear connection N. In terms of a canonical coordinate system $(x^i, y^i, z_a{}^i)$, the components are

$$D_j{}^i=y^kF^i_{kj}-N_j{}^i\,. \tag{2.27}$$

In the typically used Finsler connections, it is imposed that the D vanishes as a natural assumption. For example, it is so in the cases of $C\Gamma$, $R\Gamma$, $H\Gamma$ and $B\Gamma$. From the standpoint that a non-linear connection N may be freely chosen in parallel displacements, the author [10] has treated the Finsler connections with a non-vanishing deflection tensor in order to characterize the affine connection ([15]).

Given a conformal transformation α of a metric G to a metric $\overline{G}$, we can choose the respective Finsler connections $F\Gamma$ and $\overline{F\Gamma}$ defined from the metrics G and $\overline{G}$ such that the difference tensor D^h disappears. We have only to take the non-linear connection $\overline{N}$ of $\overline{F\Gamma}$ such that

$$\overline{N}_j{}^i=\overline{G}_j{}^i+B_j{}^i. \tag{2.28}$$

In this case the deflection tensor $\overline{D}_j{}^i$ of $\overline{F\Gamma}$ becomes $-B_j{}^i$. However, such a modification is not treated in the following.

If $B_j{}^i=0$ for a conformal transformation α, a modified non-linear connection is nothing but the usual one. Owing to (1. 34), such a transformation α satisfies $\alpha_j=0$. The conformal transformation α is called *homothetic* if $\alpha_j=0$. The converse holds clearly and we have

Proposition 2. 2. *A conformal transformation* α *is homothetic if and only if* α *satisfies* $B_j{}^i=0$.

$B_j{}^i=0$ is also equivalent to $\overline{N}=N$. And, all difference tensors vanish for a homothetic transformation, so $\overline{F\Gamma}$ coincides with $F\Gamma$. Thus we have

Theorem 2.2. *A conformal transformation α of two Finsler metrics is homothetic if and only if two non-linear connections resulting from the metrics coincide. In this case two Finsler connections coincide too.*

§ 3. Transformation formulas of the torsion and the curvature tensors.

3.1. Let $F\Gamma$ and $\bar{F}\bar{\Gamma}$ be any two Finsler connections. If we substitute (2.20), (2.21) into the structure equations (2.7), (2.8), (2.9) with respect to $\bar{F}\bar{\Gamma}$, we obtain by direct calculations the following transformation formulas communicated by M. Matsumoto some years ago.

Proposition 3.1. *The torsion and the curvature tensors of $\bar{F}\bar{\Gamma}$ are expressed by the ones of $F\Gamma$ as follows:*

$$\bar{T}(1,2)=T(1,2)-\mathfrak{S}_{12}\{A^h(1,2)-C(1,D^h(2))\}, \tag{3.1}$$

$$\bar{C}(1,2)=C(1,2)-A^v(1,2), \tag{3.2}$$

$$\begin{aligned}\bar{R}^1(1,2)=&R^1(1,2)-D^h(T(1,2))+S^1(D^h(1),D^h(2))\\&-\mathfrak{S}_{12}\{\Delta^hD^h(1,2)+\Delta^vD^h(1,D^h(2))+D^h(C(1,D^h(2)))\\&-P^1(1,D^h(2))\},\end{aligned} \tag{3.3}$$

$$\begin{aligned}\bar{P}^1(1,2)=&P^1(1,2)+S^1(D^h(1),2)\\&+A^h(2,1)-D^h(C(1,2))-\Delta^vD^h(1,2),\end{aligned} \tag{3.4}$$

$$\bar{S}^1(1,2)=S^1(1,2)-\mathfrak{S}_{12}\{A^v(1,2)\}, \tag{3.5}$$

$$\begin{aligned}\bar{R}^2(1,2)=&R^2(1,2)+S^2(D^h(1),D^h(2))\\&-A^h(T(1,2))-A^v(R^1(1,2))+A^v(D^h(T(1,2)))\\&-A^v(S^1(D^h(1),D^h(2)))-\mathfrak{S}_{12}\{\Delta^hA^h(1,2)\\&+\Delta^vA^h(1,D^h(2))-A^h(A^h(1),2)+A^h(C(1,D^h(2)))\\&+A^v(P^1(1,D^h(2)))-A^v(D^h(C(1,D^h(2))))\\&-A^v(\Delta^hD^h(1,2))-A^v(\Delta^vD^h(1,D^h(2)))-P^2(1,D^h(2))\},\end{aligned} \tag{3.6}$$

$$\bar{P}^2(1,2)=P^2(1,2)+S^2(D^h(1),2)-\Delta^vA^h(1,2)-A^h(C(1,2)) \tag{3.7}$$

$$-A^v(P^1(1,2))-A^v(S^1(D^h(1),2))+A^v(D^h(C(1,2)))$$

$$-A^h(A^v(2),1)+A^v(A^h(1),2)+\Delta^h A^v(2,1)$$

$$+A^v(\Delta^v D^h(1,2))+\Delta^v A^v(2,D^h(1)),$$

$$\bar{S}^2(1,2)=S^2(1,2)-A^v(S^1(1,2)) \tag{3.8}$$

$$-\mathfrak{S}_{12}\{\Delta^v A^v(1,2)-A^v(A^v(1),2)\}.$$

Since the used connections are general ones, we have to notice that usual symmetries do not necessarily hold. For example, $S^1=0$ if and only if

$$C(1,2)=C(2,1). \tag{3.9}$$

3.2. From Proposition 3.1 we have as a particular case

Proposition 3.2. *If $F\Gamma$ and $\bar{F}\bar{\Gamma}$ satisfy the conditions $T=\bar{T}=0$, $S^1=\bar{S}^1=0$, $A^v=0$, the formulas in Proposition* 3.1 *are reduced to the following forms*:

$$\bar{C}(1,2)=C(1,2), \tag{3.2'}$$

$$\bar{R}^1(1,2)=R^1(1,2) \tag{3.3'}$$

$$-\mathfrak{S}_{12}\{\Delta^h D^h(1,2)+\Delta^v D^h(1,D^h(2))+D^h(C(1,D^h(2)))$$

$$-P^1(1,D^h(2))\},$$

$$\bar{P}^1(1,2)=P^1(1,2)+A^h(2,1)-D^h(C(1,2))-\Delta^v D^h(1,2), \tag{3.4'}$$

$$\bar{R}^2(1,2)=R^2(1,2)+S^2(D^h(1),D^h(2)) \tag{3.6'}$$

$$-\mathfrak{S}_{12}\{\Delta^h A^h(1,2)+\Delta^v A^h(1,D^h(2))-A^h(A^h(1),2)$$

$$+A^h(C(1,D^h(2)))-P^2(1,D^h(2))\},$$

$$\bar{P}^2(1,2)=P^2(1,2)+S^2(D^h(1),2) \tag{3.7'}$$

$$-\Delta^v A^h(1,2)-A^h(C(1,2)),$$

$$\bar{S}^2(1,2)=S^2(1,2), \tag{3.8'}$$

and it holds

$$A^h(1,2)-A^h(2,1)=C(1,D^h(2))-C(2,D^h(1)). \tag{3.1'}$$

3. 3. For later use, we shall replace Δ^v in the formulas in Proposition 3.2 by Δ°. Δ^v and Δ° are related such that

$$(3.10)\quad \Delta^v D(1,2)=\Delta^\circ D(1,2)+C(D(1),2)-D(C(1,2))$$

for any tensor D of type $(1,1)$, and

$$(3.11)\quad \Delta^v A(1,2)=\Delta^\circ A(1,2)+C(A(1),2)-A(C(2),1) \\ -A(C(1,2))$$

for any tensor A of type $(1,2)$. If we recall (3.9), we have easily

Theorem 3.1. *If two Finsler connections $F\Gamma$ and $\bar{F}\bar{\Gamma}$ satisfy the conditions $T=\bar{T}=0$, $S^1=\bar{S}^1=0$, $A^v=0$, the torsion and the curvature tensors of $\bar{F}\bar{\Gamma}$ are expressed by the ones of $F\Gamma$ as follows:*

$$(3.2')\quad \bar{C}(1,2)=C(1,2),$$

$$(3.3'')\quad \bar{R}^1(1,2)=R^1(1,2) \\ -\mathfrak{S}_{12}\{\Delta^h D^h(1,2)+\Delta^\circ D^h(1,D^h(2))-P^1(1,D^h(2))\},$$

$$(3.4'')\quad \bar{P}^1(1,2)=P^1(1,2)+A^h(2,1)-C(D^h(1),2)-\Delta^\circ D^h(1,2),$$

$$(3.6'')\quad \bar{R}^2(1,2)=R^2(1,2)+S^2(D^h(1),D^h(2)) \\ -\mathfrak{S}_{12}\{\Delta^h A^h(1,2)+\Delta^\circ A^h(1,D^h(2))+A^h(A^h(2),1) \\ -C(A^h(2),D^h(1))-A^h(C(D^h(2)),1)-P^2(1,D^h(2))\},$$

$$(3.7'')\quad \bar{P}^2(1,2)=P^2(1,2)+S^2(D^h(1),2) \\ -\Delta^\circ A^h(1,2)-C(\mathrm{A}^h(1),2)+A^h(C(2),1),$$

$$(3.8')\quad \bar{S}^2(1,2)=S^2(1,2),$$

and it holds

$$(3.1')\quad A^h(1,2)-A^h(2,1)=C(1,D^h(2))-C(2,D^h(1)).$$

In terms of a canonical coordinate system $(x^i, y^i, z_a{}^i)$, these are expressed by

$$(3.2^*)\quad \bar{\boldsymbol{C}}^i_{jk}=\boldsymbol{C}^i_{jk},$$

$$(3.3^*)\quad \bar{\boldsymbol{R}}^i_{jk}=\boldsymbol{R}^i_{jk}-\mathfrak{S}_{jk}\{\boldsymbol{B}^i_{j|k}+(\boldsymbol{B}^i_{j(n)}-\boldsymbol{P}^i_{jn})\boldsymbol{B}_k{}^n\},$$

(3.4*) $\bar{\boldsymbol{P}}^i_{jk}=\boldsymbol{P}^i_{jk}+(\boldsymbol{A}^i_{kj}-\boldsymbol{C}^i_{km}\boldsymbol{B}_j{}^m)-\boldsymbol{B}^i_{j(k)}$,

(3.6*) $\bar{\boldsymbol{R}}^i_{hjk}=\boldsymbol{R}^i_{hjk}+\boldsymbol{S}^i_{hmn}\boldsymbol{B}_j{}^m\boldsymbol{B}_k{}^n$

$$-\mathfrak{S}_{jk}\{\boldsymbol{A}^i_{hj|k}+\boldsymbol{A}^i_{hj(n)}\boldsymbol{B}_k{}^n+(\boldsymbol{A}^i_{nj}-\boldsymbol{C}^i_{nm}\boldsymbol{B}_j{}^m)\boldsymbol{A}^n_{hk} -\boldsymbol{A}^i_{mj}\boldsymbol{C}^m_{hn}\boldsymbol{B}_k{}^n-\boldsymbol{P}^i_{hjn}\boldsymbol{B}_k{}^n\},$$

(3.7*) $\bar{\boldsymbol{P}}^i_{hjk}=\boldsymbol{P}^i_{hjk}+\boldsymbol{S}^i_{hmk}\boldsymbol{B}_j{}^m-\boldsymbol{A}^i_{hj(k)}-\boldsymbol{C}^i_{mk}\boldsymbol{A}^m_{hj}+\boldsymbol{A}^i_{nj}\boldsymbol{C}^n_{hk}$,

(3.8*) $\bar{\boldsymbol{S}}^i_{hjk}=\boldsymbol{S}^i_{hjk}$,

and

(3.1*) $\boldsymbol{A}^i_{jk}-\boldsymbol{A}^i_{kj}=\boldsymbol{C}^i_{jn}\boldsymbol{B}_k{}^n-\boldsymbol{C}^i_{kn}\boldsymbol{B}_j{}^n$.

3.4. We shall here investigate how the torsion and the curvature tensors change by a conformal transformation of Finsler metrics. In the cases of $C\Gamma$, $R\Gamma$, $H\Gamma$ and $B\Gamma$ the assumptions in Theorem 3.1 are satisfied. Hence, the following transformation formulas are obtained.

Theorem 3.2. *By a conformal transformation of two Finsler metrics, the torsion and the curvature tensors are changed as follows:*

(i) *Case of* $C\Gamma$

(3.2C) $\bar{C}^i_{jk}=C^i_{jk}$,

(3.3C) $\bar{R}^i_{jk}=R^i_{jk}-\mathfrak{S}_{jk}\{B^i_{j|k}+(B^i_{jn}-P^i_{jn})B_k{}^n\}$,

(3.4C) $\bar{P}^i_{jk}=P^i_{jk}+V^i_{jk}$,

(3.6C) $\bar{R}^i_{hjk}=R^i_{hjk}+2S^i_{hmn}B_j{}^mB_k{}^n$

$$-\mathfrak{S}_{jk}\{A^i_{hj|k}+A^i_{hj(n)}B_k{}^n+U^i_{jn}U^n_{hk}-P^i_{hjn}B_k{}^n\},$$

(3.7C) $\bar{P}^i_{hjk}=P^i_{hjk}+2S^i_{hmk}B_j{}^m-A^i_{hj(k)}-C^i_{mk}U^m_{hj}+U^i_{jn}C^n_{hk}$,

(3.8C) $\bar{S}^i_{hjk}=S^i_{hjk}$,

where $A^i_{hj}=U^i_{hj}+C^i_{hm}B_j{}^m$.

(ii) *Case of* $R\Gamma$

(3.3R) $\bar{R}^i_{jk}=R^i_{jk}-\mathfrak{S}_{jk}\{B^i_{j|k}+(B^i_{jn}-P^i_{jn})B_k{}^n\}$,

(3. 4R) $\bar{P}^i_{jk} = P^i_{jk} + V^i_{jk}$,

(3. 6R) $\bar{R}^{*i}_{hjk} = R^{*i}_{hjk} - \mathfrak{S}_{jk}\{U^i_{hj|k} + U^i_{hjn}B_k{}^n + U^i_{jn}U^n_{hk} - \Gamma^{*i}_{hjn}B_k{}^n\}$,

(3. 7R) $\bar{\Gamma}^{*i}_{hjk} = \Gamma^{*i}_{hjk} - U^i_{hjk}$,

where $U^i_{hjk} := U^i_{hj(k)}$.

(iii) *Case of* $H\Gamma$

(3. 2H) $\bar{C}^i_{jk} = C^i_{jk}$,

(3. 3H) $\bar{R}^i_{jk} = R^i_{jk} - \mathfrak{S}_{jk}\{B^i_{j;k} + B^i_{jn}B_k{}^n\}$,

(3. 6H) $\bar{H}^i_{hjk} = H^i_{hjk} + 2S^i_{hmn}B_j{}^m B_k{}^n$

$$- \mathfrak{S}_{jk}\{H^i_{hj;k} + H^i_{hj(n)}B_k{}^n + B^i_{jn}B^n_{hk} - Q^i_{hjn}B_k{}^n\},$$

(3. 7H) $\bar{Q}^i_{hjk} = Q^i_{hjk} + 2S^i_{hmk}B_j{}^m - H^i_{hj(k)} - C^i_{mk}B^m_{hj} + B^i_{jn}C^n_{hk}$,

(3. 8H) $\bar{S}^i_{hjk} = S^i_{hjk}$,

where $H^i_{hj} = B^i_{hj} + C^i_{hm}B_j{}^m$.

(iv) *Case of* $B\Gamma$

(3. 3B) $\bar{R}^i_{jk} = R^i_{jk} - \mathfrak{S}_{jk}\{B^i_{j:k} + B^i_{jn}B_k{}^n\}$,

(3. 6B) $\bar{K}^i_{hjk} = K^i_{hjk} - \mathfrak{S}_{jk}\{B^i_{hj;k} + B^i_{hjn}B_k{}^n + B^i_{jn}B^n_{hk} - G^i_{hjn}B_k{}^n\}$,

(3. 7B) $\bar{G}^i_{hjk} = G^i_{hjk} - B^i_{hjk}$.

Proof. In the case that $\boldsymbol{A}^i_{jk}$ has the form $\boldsymbol{A}^i_{jk} = \boldsymbol{W}^i_{jk} + \boldsymbol{C}^i_{jm}\boldsymbol{B}_k{}^m$ for some tensor $\boldsymbol{W}^i_{jk}$ and $\boldsymbol{S}^i_{hjk} = \boldsymbol{C}^m_{hk}\boldsymbol{C}^i_{mj} - \boldsymbol{C}^m_{hj}\boldsymbol{C}^i_{mk}$, (3. 4*), (3. 6*) and (3. 7*) become as follows:

(3. 4**) $\bar{\boldsymbol{P}}^i_{jk} = \boldsymbol{P}^i_{jk} + \boldsymbol{W}^i_{kj} - \boldsymbol{B}^i_{j(k)}$,

(3. 6**) $\bar{\boldsymbol{R}}^i_{hjk} = \boldsymbol{R}^i_{hjk} + 2\boldsymbol{S}^i_{hmn}\boldsymbol{B}_j{}^m\boldsymbol{B}_k{}^n$

$$- \mathfrak{S}_{jk}\{\boldsymbol{A}^i_{hj|k} + \boldsymbol{A}^i_{hj(n)}\boldsymbol{B}_k{}^n + \boldsymbol{W}^i_{nj}\boldsymbol{W}^n_{hk} - \boldsymbol{P}^i_{hjn}\boldsymbol{B}_k{}^n\},$$

(3. 7**) $\bar{\boldsymbol{P}}^i_{hjk} = \boldsymbol{P}^i_{hjk} + 2\boldsymbol{S}^i_{hmk}\boldsymbol{B}_j{}^m - \boldsymbol{A}^i_{hj(k)} - \boldsymbol{C}^i_{mk}\boldsymbol{W}^m_{hj} + \boldsymbol{W}^i_{nj}\boldsymbol{C}^n_{hk}$.

Hence, the theorem follows immediately from Theorem 2. 1.

3. 5. It is important that these very complicated formulas have been directly derived from the structure equations. If we calculate these formulas from the definitions stated in Proposition 2.1, we might be at once lost in a maze. For example, the transformation formula of $\partial\Gamma^{*i}_{hj}/\partial x^k$ is very troublesome.

If we use the Ricci identities, we can obtain various expressions of the transformation formulas. For example, if we replace $B^{ir}_{hj;k}$ $(=B^{ir}_{j(h);k})$ in (3. 6B) by $B^{ir}_{j;k(h)}$ we have

$$\bar{K}^i_{hjk}=K^i_{hjk} \tag{3. 12}$$
$$-\mathfrak{S}_{jk}\{B^{ir}_{j;k(h)}\alpha_r+B^{ir}_{hj}\alpha_{r;k}+(B^{ir}_{jn}B^{ns}_{k})_{(h)}\alpha_r\alpha_s-G^r_{hmk}B^{im}_j\alpha_r\}.$$

This formula missing the last term has been derived by M. S. Knebelman.

§ 4. Spaces conformal to some special Finsler spaces.

4. 1. In this section we shall deal with the special Finsler spaces. We shall first take up the Landsberg space defined by $P_{hijk}=0$. Owing to (2. 19) and $P_{ijk}=y^hP_{hijk}$, $P_{hijk}=0$ is equivalent to $P_{ijk}=0$. So we shall pay attention to the transformation formula (3. 4C), that is,

$$\bar{P}^i_{jk}=P^i_{jk}+V^{ir}_{jk}\alpha_r, \tag{4. 1}$$

which yields immediately the following theorems.

Theorem 4. 1. *The condition that a space be conformal to a Landsberg space is that the following system of equations has a solution* α:

$$P^i_{jk}=-V^{ir}_{jk}\alpha_r. \tag{4. 2}$$

Theorem 4. 2. *A Landsberg space remains to be a Landsberg space by a conformal transformation* α *if and only if* α *satisfies the system of equations*

$$V^{ir}_{jk}\alpha_r=0. \tag{4. 3}$$

4. 2. We shall here consider (4. 3) from another viewpoint.

By (4.1), that a conformal transformation α statisfies (4.3) means $\bar{P}^i_{jk}=P^i_{jk}$. From (2.19) we have

Proposition 4.1. *$\bar{P}^i_{hjk}=P^i_{hjk}$ if and only if $\bar{P}^i_{jk}=P^i_{jk}$.*

On the other hand, $V^{ir}_{jk}=0$ if and only if (4.3) holds for any α. So we have

Proposition 4.2. *$V^{ir}_{jk}=0$ if and only if $\bar{P}^i_{hjk}=P^i_{hjk}$ for any conformal transformations.*

If we put

$$T_{hijk}:=L^{-1}g_{hm}g_{in}V^{mn}_{jk}, \tag{4.4}$$

we have from (1.40)

$$T_{hijk}=LC_{hij}|_k+C_{hij}l_k+C_{hik}l_j+C_{hjk}l_i+C_{ijk}l_h, \tag{4.5}$$

which is just the tensor treated by M. Matsumoto and H. Kawaguchi.

M. Matsumoto [21] showed that the condition that a two-dimensional Finsler space admits the strictly isometric V-rotations of maximal order 1 be $T_{hijk}=0$. All such spaces have been found by L. Berwald [3, 4] from the following proposition.

Proposition 4.3. *In two-dimensional Finsler spaces, $T_{hijk}=0$ if and only if the main scalar I is at most a point function.*

The proof follows from $T_{hijk}=I_{(k)}m_hm_im_j$. Further, M. Matsumoto [23] has recently considered three-dimensional Finsler spaces with $T_{hijk}=0$ and obtained many interesting results.

And, H. Kawaguchi [13] showed in general dimensions that P_{hijk} vanishes for any Finsler spaces which are conformal to the Minkowski space with $T_{hijk}=0$.

Since $V^{ir}_{jk}=0$ is equivalent to $T_{hijk}=0$, Proposition 4.2 is restated as follows.

Theorem 4.3. *The tensor T_{hijk} vanishes if and only if the tensor P^i_{hjk} be invariant under any conformal transformations.*

From this theorem we have immediately

Theorem 4.4. *A Landsberg space remains to be a Landsberg space by any conformal transformations if and only if* $T_{hijk}=0$.

Since $P_{hijk}=0$ for the Minkowski spaces, the above theorems are generalizations of Kawaguchi's result. The condition $T_{hijk}=0$ has such distinctive geometrical meanings, and should be thought to be very significant in the Finsler geometry.

4.3. We consider, in particular, the two-dimensional case. Specifying Theorem 4.4, we have

Theorem 4.5. *If a two-dimensional Landsberg space remains to be a Landsberg space by a non-homothetic conformal transformation, the main scalar I is at most a point function.*

Proof. From Theorem 4.2 we have $V^{ir}_{jk}\alpha_r=0$. On the other hand, $V^{ir}_{jk}y_r=0$ owing to (1.40). Hence, in the two-dimensional case $V^{ir}_{jk}\neq 0$ implies $\alpha_r=\lambda y_r$, which contradicts the following Lemma. Therefore, $V^{ir}_{jk}=0$, i.e., $T_{hijk}=0$. Thus, the conclusion follows from Proposition 4.3.

Lemma. *Let α be a conformal transformation in a Finsler space of dimension $n>1$. If there exists a scalar field $\lambda(x, y)$ satisfying $\alpha_r=\lambda y_r$, then α is homothetic.*

Proof. Suppose that such a field λ exists. Differentiating by y^k, we have $0=\lambda_{(k)}y_r+\lambda g_{kr}$. Since λ must be positively homogeneous of degree -1, we have $(n-1)\lambda=0$ by the contraction with g^{kr}. Thus $\lambda=0$ for $n>1$, so $\alpha_r=0$.

4.4. The condition $P_{hijk}=0$ is equivalent to $C^i_{jk|0}=0$. If a Finsler space satisfies a stronger condition $C^i_{jk|l}=0$ (resp. $C^i_{jk|l}=0$, $R^i_{hjk}=0$), it is called an *affinely connected space* (resp. a *Minkowski space*). In the two-dimensional case, $C^i_{jk|l}=0$ is equivalent to $I_{|i}=0$. Hence, if I is at most a point function, then I is constant and all such spaces are found by L. Berwald [3, 4]. Thus we have from Theorem 4. 5

Theorem 4. 6. *If a two-dimensional affinely connected (esp. Minkowski) space remains to be a Landsberg space by a non-homothetic conformal transformation, the main scalar is constant.*

Returning to the cases of general dimension, we shall find the condition that a space be conformal to an affinely connected (resp. Minkowski) space. Since it is known that $C^i_{jk|l}=0$ be equivalent to $G^i_{jkl}=0$, we have from (3.7B)

Theorem 4. 7. *The condition that a space be conformal to an affinely connected space is that the following system of equations has a solution* α:

$$G^i_{jkl}=B^{ir}_{jkl}\alpha_r. \tag{4. 6}$$

4. 5. As to the Minkowski space we may easily conclude:

Proposition 4. 4. *The following conditions are mutually equivalent*:

(i) $C^i_{jk|l}=0,\ R^i_{hjk}=0,$ (ii) $\Gamma^{*i}_{jkl}=0,\ R^{*i}_{hjk}=0,$

(iii) $G^i_{jkl}=0,\ K^i_{hjk}=0,$ (iv) $G^i_{jkl}=0,\ R^i_{jk}=0.$

Each of these conditions characterizes the Minkowski space. ((ii) (resp. (iii)) means that all curvature tensors vanish with respect to $R\Gamma$ (resp. $B\Gamma$).) Thus, we have various conditions that a space be conformal to a Minkowski space. If we use (iv) we have from (3. 3B), (3. 7B)

Theorem 4. 8. *The condition that a space be conformal to a Minkowski space is that the following system of equations has a solution* α:

$$G^i_{jkl}=B^{ir}_{jkl}\alpha_r, \tag{4. 6}$$

$$R^i_{jk}=\mathfrak{S}_{jk}\{(B^{ir}_j\alpha_r)_{;k}+B^{ir}_{jm}B^{ms}_k\alpha_r\alpha_s\}. \tag{4. 7}$$

§5. *C*-conformal transformations.

5. 1. We shall finally treat the special transformation named *C-conformal*, which is, by definition, non-homothetic conformal transformation satisfying

$$C_{jk}=0. \tag{5.1}$$

If a space admits a *C*-conformal transformation α, the vector field α_j is defined on the space. Since α_j depends on point only, it follows $\alpha_j|_k=-C_{jk}$ and $\alpha^i_{(k)}=-2C_k{}^i$. Thus we have

Proposition 5. 1. *The following conditions are mutually equivalent:*

(1) $C_{jk}=0$, (2) $C_k{}^i=0$, (3) $\alpha_j|_k=0$, (4) $\alpha^i_{(k)}=0$.

5. 2. S. Tachibana [27] has generalized the concept of the *concurrent vector field* on a Riemannian space ([26], [28]) and derived the conditions that a vector field α^i be concurrent as follows:

(i) α^i *depends on point only,* (ii) $\alpha^i_{|j}+\delta_j{}^i=0$,

(iii) $\alpha^j C^i_{jk}=0$.

From (4) and (2) in Proposition 5. 1, our field α^i is a generalization of a concurrent one, in which the condition (ii) is weakened to (ii′) $\alpha^i\neq 0$ *and* α_i *is gradient.*

5. 3. In their recent paper [24], M. Matsumoto and K. Eguchi have given an elegant definition for the concurrent vector field on a Finsler space and obtained many interesting results. Some of them hold also in the space admitting a *C*-conformal transformation.

Theorem 5. 1. *If each of the following Finsler spaces admits a C-conformal transformation, then the space is Riemannian*:

(1) *two-dimensional spaces,*

(2) *three-dimensional spaces satisfying the condition of Brickel's theorem* ([7]),

(3) *spaces with* $T_{hijk}=0$,

(4) *C-reducible spaces* ([20]),

(5) *spaces with* (α, β)*-metric* ([20]).

The proofs are given in the same ways as in [24]. For example, in the two-dimensional case, $C^r\alpha_r=0$ and $C^r y_r=0$ yield $C^r=0$ or $\alpha_r=\lambda y_r$. The latter contradicts Lemma, since our transformation is assumed to be not homothetic. So, $C^r=0$, which means that the space is Riemannian.

5.4. Now, we shall treat the transformation formulas. We have at once

Proposition 5.2. *In the C-conformal case, the tensors* $B_j{}^i$, B^i_{jk}, B^i_{jkl}, U^i_{jk}, V^i_{jk}, A^i_{jk} *and* H^i_{jk} *are reduced to the following forms*:

$$B_j{}^i=y_j\alpha^i-\delta_j{}^i\alpha_0-y^i\alpha_j, \tag{5.2}$$

$$B^i_{jk}=g_{jk}\alpha^i-\delta_j{}^i\alpha_k-\delta_k{}^i\alpha_j, \tag{5.3}$$

$$B^i_{jkl}=2C_{jkl}\alpha^i, \tag{5.4}$$

$$U^i_{jk}=B^i_{jk}+\alpha_0 C^i_{jk}, \tag{5.5}$$

$$V^i_{jk}=\alpha_0 C^i_{jk}, \tag{5.6}$$

$$A^i_{jk}=B^i_{jk}, \tag{5.7}$$

$$H^i_{jk}=B^i_{jk}-\alpha_0 C^i_{jk}. \tag{5.8}$$

5.5. From (5.6) the transformation formula (4.1) becomes

$$\bar{P}^i_{jk}=P^i_{jk}+\alpha_0 C^i_{jk}, \tag{5.9}$$

from which we have by putting $\lambda=-\alpha_0$

Theorem 5.2. *If a space is C-conformal to a Landsberg space, the tensor* P^i_{jk} *is proportional to the tensor* C^i_{jk}, *i.e.*,

$$P^i_{jk}=\lambda C^i_{jk}. \tag{5.10}$$

Theorem 5.2 gives an example of the spaces treated by the author [11]. On the other hand, $\bar{P}^i_{jk}=P^i_{jk}$ implies $C^i_{jk}=0$ owing to (5.9), because we have assumed the transformation α to be non-homothetic. Thus we have

Theorem 5.3. *If the tensor P^i_{hjk} is unchanged by a C-conformal transformation, then the space is Riemannian.*

Theorem 5.3 with Theorem 4.3 gives another proof for the case (3) in Theorem 5.1. From Theorem 5.3 we have immediately

Theorem 5.4. *If a Landsberg space remains to be a Landsberg space by a C-conformal transformation, then the space is Riemannian. Especially, if a Minkowski space is C-conformal to a Landsberg space, then the space is Euclidean.*

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