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Integral cohomology ring of the symmetricspace *EII*

By

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§1. Introduction

The simply connected Riemannian symmetric spaces have been classified. For classical cases, their cohomology rings are well known. For exceptional cases, those of ***FII, EIII, EIV*** and ***EVII*** are known [1], [14], [15], and in these cases they are torsion free. The remaining spaces ***G, FI, EI, EII, EV, EVI, EVIII*** and ***EIX*** have 2-torsions, and the cohomology rings of the first two are known [5], [12].

The purpose of this paper is to determine the integral cohomology ring of the compact Riemannian symmetric space ***EII***. As a homogeneous space, ***EII*** is expressed by $E_6/S^3 \cdot SU(6)$, where E_6 is the compact 1-connected exceptional Lie group of rank 6 and $S^3 \cap SU(6) = \boldsymbol{Z}_2$ [12].

In order to determine $H^*(\boldsymbol{EII})$, we first consider a homogeneous space E_6/C, where $C = T^1 \cdot SU(6)$ is the centralizer of a one-dimensional torus.

Our first result is

Theorem 3.2. $H^*(E_6/C) = \boldsymbol{Z}[t, u, v, w]/(r_{12}, r_{16}, r_{18}, r_{24})$,

where $\deg t = 2$, $\deg u = 6$, $\deg v = 8$, $\deg w = 12$, *and*

$$r_{12} = u^2 + 2w - 3vt^2 - ut^3 + 2t^6, \quad r_{16} = t^8 + 3wt^2 - 3v^2, \tag{2.6}$$

$$r_{18} = 2wu - wt^3 \quad \textit{and} \quad r_{24} = w^2 + 26v^3 - 15v^2t^4 - 21wvt^2 + 9wut^3.$$

Using the Gysin exact sequence for the S^1-bundle: $E_6/SU(6) \to E_6/C$ we have

Corollary 3.5. *$H^i(E_6/SU(6)) \cong \boldsymbol{Z}$ for $i = 0, 6, 8, 12, 14, 20, 23, 29, 31, 35, 37, 43$; $\cong \boldsymbol{Z}_2$ for $i = 18, 26$; $\cong \boldsymbol{Z}_3$ for $i = 16, 28$ and $= 0$ for the other i.*

Next applying the Gysin exact sequence for the fibering $S^2 \to E_6/C \to \boldsymbol{EII}$, we have the following theorems.

Theorem 5.2. $H^*(\boldsymbol{EII}; \boldsymbol{Z}_2) = \boldsymbol{Z}_2 < y_2^{i+3}, y'_{12} y_2^i; 0 \leq i \leq 11 >$

$$+ \boldsymbol{Z}_2 < 1, y_2, y_3, y_2^2, y_3 y_2, y_3^2, y_3^2 y_2 > \otimes \Delta(y_{12}, y_{20}),$$

where $\deg y_i = i$, $\deg y'_{12} = 12$.

The relations are given in Theorem 5.3.

Theorem 6.1. $H^*(\boldsymbol{EII}; \boldsymbol{Z}[1/2]) = \boldsymbol{Z}[1/2][a, b, c, d]/(q_{12}, q_{16}, q_{18}, q_{24})$,

where $\deg a = 4$, $\deg b = 6$, $\deg c = 8$, $\deg d = 12$, *and*

(2.8) $$q_{12} = b^2 + 8d - 6ca + a^3, \quad q_{16} = a^4 + 12da - 6ca^2 - 3c^2$$

$$q_{18} = db \quad \textit{and} \quad q_{24} = d^2 + c^3 - \frac{3}{2} dca.$$

Here we use the following notations. $\boldsymbol{Z}[1/2]$ indicates the subalgebra of $\boldsymbol{Q}$ generated by $1/2$ over $\boldsymbol{Z}$. $A < x_1, \dots, x_n >$ denotes the A-module spanned by linearly independent elements x_i's and $\Delta(x_1, \dots, x_n) = A < x_1^{a_1} \cdots x_n^{a_n} (a_i = 0, 1) >$, where $A = \boldsymbol{Z}_2$, $\boldsymbol{Z}[1/2]$ or $\boldsymbol{Z}$.

Remark that the elements a, b, c, d in Theorem 6.1 are integral cohomology classes, and they are uniquely determined by

$$p^*(a) = t^2, \ p^*(b) = 2u - t^3, \ p^*(c) = 2v - t^4,$$

$$p^*(d) = w \quad \text{and} \quad \rho_2(b) = y_2^3 + y_3^2,$$

where p^* is induced by the projection $p: E_6/C \to \boldsymbol{EII}$, and ρ_2 is the mod 2 reduction. There exist more integral cohomology classes χ, d', e and f such that

$$2\chi = 0, \ \rho_2(\chi) = y_3, \ d' = \frac{1}{2}(ca + a^3), \ e = \frac{1}{2}(cb + ba^2) \text{ and } f = \frac{1}{2} dc.$$

Using Theorems 5.2 and 6.1 we obtain the structure of $H^*(\boldsymbol{EII})$.

Theorem 6.3. Tors. $H^*(\boldsymbol{EII}) = \boldsymbol{Z}_2 < \chi, \chi^2 > \otimes \Delta(d, f)$ *and the Poincaré polynomial is* $P(\boldsymbol{EII}, t) = (1 + t^4 + t^8 + t^{12})(1 + t^6 + t^{12})(1 + t^8 + t^{16})$.

The ring structure will be given in Theorem 6.4 with the generators

$$\chi,\ a,\ b,\ c,\ d,\ d',\ e \ \text{ and } \ f$$

and various relations.

The paper is organized as follows. In §2 we calculate the invariant subalgebras of the Weyl groups in order to determine the rational cohomology of E_6/C and $\boldsymbol{EII}$, and in §3 $H^*(E_6/C)$ is determined. In §4 we discuss $H^*(\boldsymbol{EII})$ and $H^*(\boldsymbol{EII}; \boldsymbol{Z}_2)$ in low dimensions, and $H^*(\boldsymbol{EII}; \boldsymbol{Z}_2)$ is determined in §5. With these data the final section §6 completes the determination of the ring structures of $H^*(\boldsymbol{EII}; \boldsymbol{Z}[1/2])$ and $H^*(\boldsymbol{EII})$.

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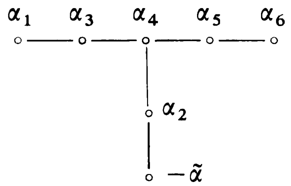
§2. Rational cohomology of $E_6/T^1\cdot SU(6)$ and EII

Let T be a maximal torus of E_6. The Dynkin diagram of E_6 is



where α_i $(1 \leq i \leq 6)$ are the simple roots and $\tilde{\alpha}=\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6$ is the highest root $(\tilde{\alpha}=w_2)$.

Let C and U be the identity components of the centralizers of $T^1=\{x \in T | \alpha_i(x)=0\ (i \neq 2)\}$ and of the element $x_2 \in T^1$ such that $\alpha_2(x_2)=\frac{1}{2}$, respectively. Then the Weyl groups $\Phi(\ \ )$ of E_6, C and U are generated by the following elements:

$$\Phi(E_6) = <R_i;\ i=1, 2, 3, 4, 5, 6>,$$

(2.1) $$\Phi(C) = <R_i;\ i \neq 2>,$$

$$\Phi(U) = <R_i, \tilde{R};\ i \neq 2>$$

where R_i (resp. $\tilde{R}$) denotes the refiection in the plane $\alpha_i=0$ (resp. $\tilde{\alpha}=0$) in the universal covering of T.

Recall from [12; Theorem 2.1]

(2.2) $$U=S^3\cdot SU(6),\quad C=T^1\cdot SU(6) \quad \text{and} \quad S^3 \cap SU(6) = T^1 \cap SU(6) \cong Z_2.$$

According to [5] we may consider that each weight is an element of $H^2(BT)$

$=H^1(T)$, then the fundamental weights w_i; $i=1,2,\dots,6$ form a basis of $H^2(BT)$, and $H^*(BT)=Z[w_1, w_2,\dots, w_6]$.

The reflections R_i's and $\tilde{R}$ act on $H^*(BT)$ as follows:

$$R_i(w_i)=w_i-\sum_j(2<\alpha_i, \alpha_j>/<\alpha_j, \alpha_j>)w_j, \qquad R_i(w_k)=w_k \quad \text{for} \quad k\neq i,$$

and $\quad \tilde{R}(w_i)=w_i-m_iw_2$ for $\tilde{\alpha}=\sum_i m_i\alpha_i$.

As in [14] we have the following isomorphism (2.3) and the table (2.4) of the action by taking following generators:

$$t_6=w_6, \quad t_i=R_{i+1}(t_{i+1})\ (i=5, 4, 3, 2), \quad t_1=R_1(t_2) \quad \text{and} \quad t=w_2.$$

(2.3) $\quad H^*(BT)=\boldsymbol{Z}[t, t_1, t_2,\dots, t_6]/(3t-c_1) \qquad \textit{for} \quad c_1=t_1+t_2+\cdots+t_6,$

(2.4)

	R_1	R_2	R_3	R_4	R_5	R_6	$\tilde{R}$
t_1	t_2	$t-t_2-t_3$					t_1-t
t_2	t_1	$t-t_1-t_3$	t_3				t_2-t
t_3		$t-t_1-t_2$	t_2	t_4			t_3-t
t_4				t_3	t_5		t_4-t
t_5					t_4	t_6	t_5-t
t_6						t_5	t_6-t
t		$t_4+t_5+t_6-t$					$-t$

where the blanks indicate the trivial action. Denote by

$$c_i=\sigma_i(t_1, t_2,\dots, t_6)$$

the i-th elementary symmetric function on the variables t_i's ($c_0=1$), then we have the following

Lemma 2.1. (i) $H^*(BT)^{\Phi(C)}=[t, c_1, c_2,\dots, c_6]/(3t-c_1)$.

(ii) $H^*(BT; \boldsymbol{Q})^{\Phi(U)}=\boldsymbol{Q}[t^2, c_2, c_3-2c_2t+5t^3, 2c_4-3c_3t, c_5-c_4t+c_2t^3-3t^5,$

$$4c_6-2c_5t+c_3t^3].$$

Proof. (i) follows easily from (2.1), (2.3) and (2.4). This and (2.1) imply that $H^*(BT; \boldsymbol{Q})^{\Phi(U)}$ consists of all $\tilde{R}$-invariant polynomials in $H^*(BT; \boldsymbol{Q})^{\Phi(C)}$ $=\boldsymbol{Q}[t, c_2,\dots, c_6]$. Applying $\tilde{R}$ to the equality $\sum c_i=\prod(1+t_j)$, we have $\sum\tilde{R}(c_i)$ $=\prod(1-t+t_j)=\sum(1-t)^{6-i}c_i$, and $\tilde{R}(c_2)=c_2,\ \tilde{R}(c_3)=c_3-4c_2t+10t^3,\ \tilde{R}(c_4)=c_4-3c_3t+6c_2t^2-15t^4,$

$$\tilde{R}(c_5)=c_5-2c_4t+3c_3t^2-4c_2t^3+9t^5,\quad \tilde{R}(c_6)=c_6-c_5t+c_4t^2-c_3t^3+c_2t^4-2t^6.$$

Since $\tilde{R}(t)=-t$ and $\tilde{R}^2=\text{identity}$, t^2 and $c_i+\tilde{R}(c_i)$ $(i=2, 3, 4, 5, 6)$ are $\tilde{R}$-invariant. It is easy to see that every polynomial f of $\boldsymbol{Q}[t, c_2,\dots, c_6]$ is written uniquely in the form $g+th$ for polynomials g and h in t^2 and $c_i+\tilde{R}(c_i)$. If f is $\tilde{R}$-invariant, then $g+th=f=\tilde{R}(f)=\tilde{R}(g)+\tilde{R}(t)\tilde{R}(h)=g-th$. It follows that f is $\tilde{R}$-invariant if and only if it is a polynomial in t^2 and $c_i+\tilde{R}(c_i)$. This proves (ii). q. e. d.

Putting

$$x_i=2t_i-t \qquad (i=1, 2,\dots, 6)$$

we have the following $\Phi(E_6)$-invariant set

$$S=\{x_i+x_j,\ t-x_k,\ -t-x_k;\ i, j, k=1,\dots, 6;\ i<j\}.$$

Thus we have invariant forms

$$I_n=\sum_{x\in S}x^n\in H^{2n}(BT;\boldsymbol{Q})^{\Phi(E_6)}.$$

Consider the following elements $(r_i\in H^i,\ u\in H^6,\ v\in H^8,\ w\in H^{12})$:

$$r_4=c_2-4t^2,$$

$$u=\frac{1}{2}c_3-t^3,\qquad v=\frac{1}{3}(c_4+2t^4)-ut,\qquad w=c_6,$$

$$r_{10}=c_5-3vt-ut^2+2t^5,$$

$$r_{12}=u^2+2w-3vt^2-ut^3+2t^6,$$

$$r_{16}=t^8+3wt^2-3v^2,$$

$$r_{18}=2wu-wt^3$$

and $$r_{24}=w^2+26v^3-15v^2t^4-21wvt^2+9wut^3.$$

Then we have the following

Lemma 2.2. *$H^*(BT;\boldsymbol{Q})^{\Phi(E_6)}=\boldsymbol{Q}[I_2, I_5, I_6, I_8, I_9, I_{12}]$ and as ideals $(I_2, I_5,$*

$I_6, I_8, I_9, I_{12})=(r_4, r_{10}, r_{12}, r_{16}, r_{18}, r_{24})$.

Proof. The first half is proved in Lemma 5.2, (i) of [14]. For the second half we shall show

(2.5) $$I_n \equiv k_n r_{2n} \bmod \mathfrak{a}_n \ (k_n \neq 0) \quad \textit{for} \quad n=2, 5, 6, 8, 9, 12,$$

where $\mathfrak{a}_n$ is the ideal generated by I_j's of $j<n$.

In §5 of [14], I_n is computed by the formula

$$I_n=\frac{1}{2}\sum_{i+j=n}\binom{n}{i}s_i s_j-2^{n-1}s_n+2\sum_{i+2j=n}(-1)^i\binom{n}{i}s_i t^{2j}$$

where $s_i=x_1^i+\cdots+x_6^i$, and it is described with t and $d_i=\sigma_i(x_1,\dots, x_6)$ by use of Newton formula. Then the first four of the following relations are already given in (5.10) of [14]:

$$I_2=-12I'_2 \qquad \text{for} \quad I'_2=d_2-t^2,$$

$$I_5\equiv -60I'_5 \bmod \mathfrak{a}_5 \qquad \text{for} \quad I'_5=d_5+d_3t^2,$$

$$I_6\equiv 144I'_6 \bmod \mathfrak{a}_6 \qquad \text{for} \quad I'_6=d_6-d_4t^2+\frac{1}{8}d_3^2,$$

$$I_8\equiv 80I'_8 \bmod \mathfrak{a}_8 \qquad \text{for} \quad I'_8=d_4^2-36d_6t^2+22d_4t^4+t^8,$$

$$I_9\equiv 756I'_9 \bmod \mathfrak{a}_9 \qquad \text{for} \quad I'_9=(d_6+d_4t^2+2t^6)d_3$$

and $$I_{12}\equiv 720I'_{12} \bmod \mathfrak{a}_{12} \qquad \text{for} \quad I'_{12}=39d_6d_4t^2-741d_6t^6+403d_4t^8+23t^{12}.$$

The last two relations are computed by continuing routine computations. The details are left to the readers.

Next, as on p. 275 of [14], we rewrite I_n in terms of t and c_i's by use of the formula

$$d_n=\sum_{i=0}^{n}(-1)^{n-i}2^i\binom{6-i}{n-i}c_i t^{n-i}, \qquad c_1=3t.$$

From $d_2=4c_2-15t^2$ it follows that

$$I'_2=d_2-t^2=4r_4.$$

Modulo $\mathfrak{a}_5=(I_2)=(r_4)$ we have

$$d_3\equiv 8c_3-24t^3=8(2u-t^3),$$

$$d_4 \equiv 16c_4 - 24c_3t + 51t^4 = 48v - 29t^4,$$

$$d_5 \equiv 32c_5 - 32c_4t + 24c_3t^2 - 40t^5 = 8(4c_5 - 12vt - 6ut^2 + 9t^5)$$

and $$d_6 - d_4t^2 \equiv 64c_6 - 32c_4t^2 - 2t^6 = 8(8w - 12vt^2 + 7t^4),$$

and then by direct computations

$$I'_5 \equiv 32r_{10}, \quad I'_6 \equiv 32r_{12}, \quad I'_8 \equiv -768r_{16}, \quad I'_9 \equiv 512r_{18}$$

and $$I'_{12} \equiv 768I''_{12} \qquad \text{for} \quad I''_{12} = 156wvt^2 - 273wt^6 + 208vt^8 - 120t^{12}.$$

Moreover we have

$$24r_{24} = I''_{12} - 3r_{18}(2u + 11t^3) + 8r_{16}(v - 6t^4) + 12r_{12}w.$$

Consequently (2.5) has been proved for $k_2 = -48$, $k_5 = -2^7 \cdot 3 \cdot 5$, $k_6 = 2^9 \cdot 3^2$, $k_8 = -2^{12} \cdot 3 \cdot 5$, $k_9 = 2^{11} \cdot 3^3 \cdot 7$ and $k_{12} = 2^{15} \cdot 3^4 \cdot 5$. q.e.d.

According to [2] we have $H^*(E_6/C; \boldsymbol{Q}) \cong H^*(BT; \boldsymbol{Q})^{\Phi(C)}/(H^+(BT; \boldsymbol{Q})^{\Phi(E_6)})$, $H^*(\boldsymbol{EII}; \boldsymbol{Q}) \cong H^*(BT; \boldsymbol{Q})^{\Phi(U)}/(H^+(BT; \boldsymbol{Q})^{\Phi(E_6)})$ and the homomorphism p^*: $H^*(\boldsymbol{EII}; \boldsymbol{Q}) \to H^*(E_6/C; \boldsymbol{Q})$ induced by the fibering $p: E_6/C \to \boldsymbol{EII} = E_6/U$ is equivalent to the natural map induced by the inclusion of $H^*(BT; \boldsymbol{Q})^{\Phi(U)}$ into $H^*(BT; \boldsymbol{Q})^{\Phi(C)}$. Then we have from Lemmas 2.1 and 2.2, by cancelling c_2, c_5 by r_4, r_{10} and by replacing c_3, c_4, c_6 with u, v, w, respectively,

Proposition 2.3. *$H^*(E_6/C; \boldsymbol{Q}) = \boldsymbol{Q}[t, u, v, w]/(r_{12}, r_{16}, r_{18}, r_{24})$ where the relations are given by*

(2.6) $$r_{12} = u^2 + 2w - 3vt^2 - ut^3 + 2t^6, \quad r_{16} = t^8 + 3wt^2 - 3v^2,$$

$$r_{18} = 2wu - wt^3 \quad \textit{and} \quad r_{24} = w^2 + 26v^3 - 15v^2t^4 - 21wvt^2 + 9wut^3.$$

Similarly, from Lemma 2.1, (ii) we have

$$p^*H^*(\boldsymbol{EII}; \boldsymbol{Q}) = \boldsymbol{Q}[t^2, 2u - t^3, v, w]/(r_{12}, r_{16}, r_{18}, r_{24}).$$

Define elements a, b, c, d of $H^*(\boldsymbol{EII}; \boldsymbol{Q})$ by

(2.7) $$p^*(a) = t^2, \; p^*(b) = 2u - t^3, \; p^*(c) = 2v - t^4 \quad \textit{and} \quad p^*(d) = w.$$

Then we have

Proposition 2.4. *$H^*(\boldsymbol{EII}; \boldsymbol{Q}) = \boldsymbol{Q}[a, b, c, d]/(q_{12}, q_{16}, q_{18}, q_{24})$ where the re-*

lations are given by

$$(2.8)\qquad q_{12}=b^2+8d-6ca+a^3,\quad q_{16}=a^4+12da-6ca^2-3c^2,$$

$$q_{18}=db \quad and \quad q_{24}=d^2+c^3-\frac{3}{2}dca.$$

We shall see that Propositions 2.3 and 2.4 are valid for the coefficients $\boldsymbol{Z}$ and $\boldsymbol{Z}[1/2]$ respectively.

§3. Integral cohomology of $E_6/T^1\cdot SU(6)$ and $E_6/SU(6)$

Lemma 3.1. *The subgroup* $C=T^1\cdot SU(6)$ *of* E_6 *has torsion free cohomology and the canonical projection* ρ: $BT\to BC$ *induces an isomorphism*

$$\rho^*\colon H^*(BC)\cong \boldsymbol{Z}[t, c_2, c_3, c_4, c_5, c_6]=H^*(BT)^{\Phi(C)}\subset H^*(BT).$$

Proof. As is seen is the proof of Proposition 3.5 of [12], we have a homeomorphism $C\cong SU(6)\times S^1$. Therefore $H^*(C)\cong H^*(SU(6))\otimes H^*(S^1)\cong \Lambda(s_1, s_3, s_5, s_7, s_9, s_{11})$, $\deg s_i=i$. Then the lemma follows from Lemma 2.1, (i) by the general method of Borel [2].

We identify $H^*(BC)$ with its image under ρ^*, then

$$H^*(BC)=\boldsymbol{Z}[t, c_2, c_3, c_4, c_5, c_6],\quad \rho^*t=t,\ \rho^*c_i=c_i.$$

We also use the same symbols $t, c_i\in H^*(E_6/C)$ for the images under the induced homomorphism

$$i^*\colon H^*(BC)\longrightarrow H^*(E_6/C),$$

where i: $E_6/C\to BC$ is a map classifying the bundle $E_6\to E_6/C$.
A main result in this section is the following

Theorem 3.2. *There exist elements* u *and* v *of* $H^*(E_6/C)$ *satisfying*

$$2u=c_3-2t^3 \quad and \quad 3v=c_4+2t^4-3ut.$$

We have $$H^*(E_6/C)=Z[t, u, v, w]/(r_{12}, r_{16}, r_{18}, r_{24})$$

for $w=c_6$ *and the relations* (2.6) *in Proposition* 2.3.

The proof of this theorem is analogous to that of Theorem 4.4 of [12].
The mod p cohomology of E_6 for each prime p is given as follows (see

e.g. [13]):

(3.1) $$H^*(E_6; \boldsymbol{Z}_2)=\Lambda(x_5, x_9, x_{15}, x_{17}, x_{23})\otimes\boldsymbol{Z}_2[x_3]/(x_3^4)$$

$$=\Delta(x_3, x_5, x_9, x_{15}, x_{17}, x_{23})\otimes\boldsymbol{Z}_2[x_6]/(x_6^2),$$

$$\text{where } x_5=\mathrm{Sq}^2x_3 \text{ and } x_6=x_3^2=\mathrm{Sq}^1x_5=\beta x_5;$$

$$H^*(E_6; \boldsymbol{Z}_3)=\Lambda(x_3, x_7, x_9, x_{11}, x_{15}, x_{17})\otimes\boldsymbol{Z}_3[x_8]/(x_8^3)$$

$$\text{where } x_7=\mathscr{P}^1x_3 \text{ and } x_8=\beta x_7;$$

and for $p\geqq 5$,

$$H^*(E_6; \boldsymbol{Z}_p)=\Lambda(x_3, x_9, x_{11}, x_{15}, x_{17}, x_{23}).$$

By direct computations we have

Lemma 3.3. *For* $r_4=c_2-4t^2\in H^4(BC)$,

$$\mathrm{Sq}^2r_2=c_3+c_2c_1=c_3+c_2t \quad \text{in} \quad H^*(BC; \boldsymbol{Z}_2)$$

and

$$\mathscr{P}^1r_2=c_4+c_2^2-2t^4 \quad \text{in} \quad H^*(BC; \boldsymbol{Z}_3).$$

We need also

Lemma 3.4. *Up to degree* 24, $\boldsymbol{Z}[t, u, v, w]/(r_{12}, r_{16}, r_{18}, r_{24})$ *is torsion free.*

Proof. Obviously, $\boldsymbol{Z}[t, u, v, w]/(r_{12}, r_{16})$ is free and has an additive base $\{w^iv^ju^kt^l\ (i, j\geq 0;\ k=0, 1;\ 8>l\geq 0)\}$. We add relations $-r_{18}t^i=wt^{i+3}-2wut^i$ $(i=0, 1, 2, 3)$, $r_{24}=w^2+26v^3-15v^2t^4-21wvt^2+9wut^3$ and $-r_{18}u=wut^3-2wu^2$. By cancelling wt^{i+3} and w^2 with $r_{18}t^i$ and r_{24}, we have that, up to degree 24, $\mathbf{Z}[t, u, v, w]/(r_{12}, r_{16}, r_{18}, r_{24})$ has a system of generators $\{w^iv^ju^kt^l\ (i=0, 1;\ j\geq 0;\ k=0, 1;\ 8>l+5i\geq 0;\ 6i+4j+3k+l\leq 12), wut^3\}$ with a single relation

$$104v^3-60v^2t^4-78wvt^2+29wut^3\ (=r_{18}u-4r_{18}t^3-2r_{12}w+4r_{24})=0$$

whose coefficients are relatively prime. So the lemma follows. q.e.d.

Proof of Theorem 3.2. We can apply Theorem 2.1 of [13] to the homogeneous space $E_6/C=E_6/T^1\cdot SU(6)$, and we have the following description of the integral cohomology of E_6/C:

$$H^*(E_6/C)=\boldsymbol{Z}[t,\ c_2,\ c_3,\dots,\ c_6,\ \gamma_6,\ \gamma_8]/(\rho_2,\ \rho_5,\ \rho_6,\ \rho_8,\ \rho_9,\ \rho_{12},\ \rho'_6,\ \rho'_8),$$

$$\rho'_6=2\gamma_6+\delta_6 \qquad \text{for} \quad \delta_6 \bmod 2=\tau(x_5)=\mathrm{Sq}^2(\tau(x_3)),$$

$$\rho'_8=3\gamma_8+\delta_8 \qquad \text{for} \quad \delta_8 \bmod 3=\tau(x_7)=\mathscr{P}^1(\tau(x_3)),$$

where τ indicates the transgression mod p ($=2$ or 3) with respect to the fibering

$$\text{(3.2)} \qquad E_6 \longrightarrow E_6/C \xrightarrow{\ i\ } BC$$

and the relation ρ_j is determined, up to sign, by the maximality of the integer n in

$$\text{(3.3)} \qquad n\cdot\rho_j\equiv I_j \bmod (\rho'_6,\ \rho'_8,\ \rho_i\ (i<j)).$$

At first consider the relation ρ_2. Since $I_2=-48(c_2-4t^2)$ and since $r_4=c_2-4t^2$ cannot be divisible by any integer >1, we may take $\rho_2=r_4=c_2-4t^2$. By Serre's exact sequence

$$0=H^3(E_6/C) \longrightarrow H^3(E_6) \xrightarrow{\ \tau\ } H^4(BC) \xrightarrow{\ i^*\ } H^4(E_6/C)$$

$H^3(E_6)\cong \boldsymbol{Z}$ and it is generated by an element x_3 such that $\tau(x_3)=r_4$. Obviously, the elements x_3's in (3.1) are the mod p reductions of this x_3 up to sign. Applying Lemma 3.3 we have

$$\delta_6 \ (\bmod 2)=\mathrm{Sq}^2(r_4)=c_3+c_2t=c_3 \qquad \text{in} \quad H^*(BC;\ \boldsymbol{Z}_2)/(r_4),$$

$$\delta_8 \ (\bmod 3)=\mathscr{P}^1(r_4)=c_4+c_2^2-2t^4=c_4-t^4 \qquad \text{in} \quad H^*(BC;\ \boldsymbol{Z}_3)/(r_4)$$

and relations $\rho'_6=2\gamma_6+c_3$ and $\rho'_8=3\gamma_8+c_4-t^4$.

These relations and ρ_2 are cancelled with the generators c_3, c_4 and c_2 respectively, and ($w=c_6$)

$$H^*(E_6/C)=\boldsymbol{Z}[t,\ c_5,\ w,\ \gamma_6,\ \gamma_8]/(\rho_5,\ \rho_6,\ \rho_8,\ \rho_9,\ \rho_{12}).$$

Here we replace γ_6 and γ_8 by $u=-\gamma_6-t^3=\frac{1}{2}c_3-t^3$ and $v=-\gamma_8-ut+t^4=\frac{1}{3}(c_4+2t^4)-ut$, then we may take $\rho_5=r_{10}=c_5-3vt-ut^2+2t^5$ (Lemma 2.2) since the coefficient of c_5 is 1. Then c_5 is cancelled with ρ_5:

$$H^*(E_6/C)=\boldsymbol{Z}[t,\ u,\ v,\ w]/(\rho_6,\ \rho_8,\ \rho_9,\ \rho_{12}).$$

Since $H^*(E_6/C)$ is torsion free r_{12}, r_{16}, r_{18}, $r_{24}\in \boldsymbol{Z}[t,\ u,\ v,\ w]$ are relations in $H^*(E_6/C)$ by Lemma 2.2. Thus there is a natural ring homomorphism $\boldsymbol{Z}[t,\ u,\ v,\ w]/(r_{12},\ r_{16},\ r_{18},\ r_{24})\rightarrow H^*(E_6/C)$. So we have a natural homomorphism $\eta: \boldsymbol{Z}[t, u, v, w]/(r_{12}, r_{16}, r_{18}, r_{24}) \to H^*(E_6/C)$ which is surjective. By Lemma 2.2, $\operatorname{Ker}\eta$ is finite. Then it follows from Lemma 3.4 that η is isomorphic for degree ≤ 24. This shows that we can replace the relations ρ_j's by r_{2j}'s, and this completes the proof of the theorem.

Corollary 3.5. (i) *The projection* $p: E_6/SU(6) \to E_6/T^1 \cdot SU(6)$ *induces an isomorphism* $H^{\text{even}}(E_6/SU(6)) \cong \boldsymbol{Z}[u, v, w]/(2w+u^2, 3v^2, 2wu, w^2-v^3)$
(ii) $H^i(E_6/SU(6)) \cong \boldsymbol{Z}$ *for* $i=0, 6, 8, 12, 14, 20, 23, 29, 31, 35, 37, 43$; $\cong \boldsymbol{Z}_2$ *for* $i=18, 26$; $\cong \boldsymbol{Z}_3$ *for* $i=16, 28$ *and* $=0$ *for other* i.

Proof. Since the fibre $C/SU(6)$ of the fibering p is a circle, we have a Gysin exact sequence which splits into the short exact sequences

$$0 \longrightarrow H^{2i-1}(E_6/SU(6)) \longrightarrow H^{2i-2}(E_6/C)$$
$$\xrightarrow{h} H^{2i}(E_6/C) \xrightarrow{p^*} H^{2i}(E_6/SU(6)) \longrightarrow 0,$$

where $h(x)=x\cdot\Omega$, and $\Omega=\pm t$ since $E_6/SU(6)$ is 2-connected. From the exactness of the sequence follows that $H^{\text{even}}(E_6/SU(6)) \cong \operatorname{Coker} h$, and the first assertion holds as $\operatorname{Im} h=(t)$. So the second assertion holds for i even. Note that the odd dimensional part is torsion free by the above exactness. Then (ii) holds for i odd by Poincaré duality (and the universal coefficient theorem).

§4. Low dimensional cohomology of the symmetric space $\boldsymbol{EII}=E_6/S^3\cdot SU(6)$

According to [12, Theorem 2.1], we have $\boldsymbol{EII}=E_6/U$, $U=S^3\cdot SU(6)$.
Consider the fibering

$$U/C \longrightarrow E_6/C \longrightarrow E_6/U=\boldsymbol{EII}.$$

Since $U/C \cong S^3/T^1$ is a 2-sphere, we have a Gysin exact sequence which is reduced to exact sequences

$$(4.1)_i\colon\quad 0 \longrightarrow H^{2i-3}(\boldsymbol{EII}; A) \xrightarrow{h} H^{2i}(\boldsymbol{EII}; A) \xrightarrow{p^*} H^{2i}(E_6/C; A)$$
$$\xrightarrow{\theta} H^{2i-2}(\boldsymbol{EII}; A) \xrightarrow{h} H^{2i+1}(\boldsymbol{EII}; A) \longrightarrow 0,$$

where $A=\boldsymbol{Z}, \boldsymbol{Z}[1/2]$ or $\boldsymbol{Z}_2$, the homomorphisms θ and h satisfy

$$\theta(p^*(x)y)=x\theta(y) \quad \textit{and} \quad h(x)=x\cdot\chi \tag{4.2}$$

for some $\chi \in H^3(\boldsymbol{EII}; A)$ such that $2\chi=0$. The sequences commute with the mod 2 reduction ρ_2.

Since $H^{2i}(E_6/C)$ is free, it follows from (4.1) that

(4.3) $H^{\mathrm{odd}}(\boldsymbol{EII})=\chi\cdot H^{\mathrm{even}}(\boldsymbol{EII})\subset \mathrm{Im}\, h=\mathrm{Tors.}\, H^*(\boldsymbol{EII})\cong \boldsymbol{Z}_2+\cdots+\boldsymbol{Z}_2$ *(finite sum)*

and $\rho_2\colon H^{\mathrm{odd}}(\boldsymbol{EII})\to H^{\mathrm{odd}}(\boldsymbol{EII};\boldsymbol{Z}_2)$ *is injective.*

In particular $H^3(\boldsymbol{EII})\cong \boldsymbol{Z}_2$ or 0 according to $\chi\neq 0$ or $\chi=0$. On the other hand, since E_6 is 2-connected, $\pi_1(\boldsymbol{EII})\cong\pi_0(U)=0$ and $H_2(\boldsymbol{EII})\cong\pi_2(\boldsymbol{EII})\cong \pi_1(U)\cong \boldsymbol{Z}_2$. This and $(4.1)_1$ show that

(4.4) $$H^3(\boldsymbol{EII})=\boldsymbol{Z}_2<\chi>,\quad H^2(\boldsymbol{EII})=H^1(\boldsymbol{EII})=0 \quad \textit{and} \quad \theta(t)=2.$$

Here we change θ to $-\theta$ if it is necessary.

First we consider low dimensional cases.

Lemma 4.1. *There exist unique elements* $a, b, c\in H^*(\boldsymbol{EII})$ *and* $y_i\in H^i(\boldsymbol{EII}; \boldsymbol{Z}_2)$, $(i=2, 3)$, $\deg a=4$, $\deg b=6$, $\deg c=8$, *such that*

$$p^*(a)=t^2,\quad p^*(b)=2u-t^3,\quad p^*(c)=2v-t^4,$$

$$p^*(y_2)=\rho_2(t),\quad \rho_2(\chi)=y_3 \quad \textit{and} \quad \rho_2(b)=y_2^3+y_3^2.$$

Then, up to degree 9, *we have*

$$H^*(\boldsymbol{EII})=\boldsymbol{Z}[a, b, c]+\boldsymbol{Z}_2<\chi, \chi^2>,\quad a\chi=b\chi=\chi^3=0,$$

$$H^*(\boldsymbol{EII};\boldsymbol{Z}_2)=\boldsymbol{Z}_2[y_2, y_3]/(y_3y_2^2, y_3^3),\quad \rho_2(a)=y_2^2,\quad \rho_2(c)=y_2^4+y_3^2y_2,$$

$$\mathrm{Sq}^1y_2=y_3 \quad \textit{and} \quad \mathrm{Sq}^2y_3=y_3y_2.$$

Proof. From (4.4) and $(4.1)_2$ it follows that $H^5(\boldsymbol{EII})=0$ and $p^*\colon H^4(\boldsymbol{EII}) =\boldsymbol{Z}<a>\to H^4(E_6/C)=\boldsymbol{Z}<t^2>$ is an isomorphism for $a=p^{*-1}(t^2)$. Next consider $(4.1)_3$:

$$0\longrightarrow \boldsymbol{Z}_2<\chi>\xrightarrow{h} H^6(\boldsymbol{EII})\xrightarrow{p^*}\boldsymbol{Z}<u, t^3>\xrightarrow{\theta}\boldsymbol{Z}<a>\xrightarrow{h} H^7(\boldsymbol{EII})\longrightarrow 0.$$

By Proposition 2.4 the image of p^* contains $m(2u-t^3)$ for some non-zero integer m. Then $m\theta(2u-t^3)=0$. By (4.2), $\theta(t^3)=\theta(p^*(a)t)=2a$. Since $H^4(\boldsymbol{EII})=\boldsymbol{Z}<a>$ is torsion free we have

(4.5) $$\theta(2u-t^3)=0 \quad \textit{and} \quad \theta(u)=a.$$

From the exactness of the above sequence

$$H^7(\boldsymbol{EII})=0, \quad a\chi=0 \quad \text{and} \quad H^6(\boldsymbol{EII})=\boldsymbol{Z}_2<\chi^2>+\boldsymbol{Z}<b>$$

for some element b satisfying $p^*(b)=2u-t^3$.

Next applying the universal coefficient theorem, we have

$$H^i(\boldsymbol{EII};\boldsymbol{Z}_2)=\boldsymbol{Z}_2<y_i> \qquad (i=2, 3, 4, 5)$$

and

$$H^6(\boldsymbol{EII};\boldsymbol{Z}_2)=\boldsymbol{Z}_2<\rho_2(\chi^2), \rho_2(b)>,$$

where $\mathrm{Sq}^1 y_2=y_3=\rho_2(\chi)$, $y_4=\rho_2(a)$ and $\mathrm{Sq}^1 y_5=\rho_2(\chi^2)=y_3^2$. By $(4.1)_1$, $p^*(y_2)\neq 0$, and $p^*(y_2)=\rho_2(t)$. Then $p^*(y_2^2)=\rho_2(t^2)\neq 0$, and $y_2^2=y_4$. Since $\mathrm{Sq}^1(y_3y_2)=y_3\mathrm{Sq}^1 y_2=y_3^2$, $y_5=y_3y_2$.

From $p^*(\rho_2(b))=\rho_2(2u-t^3)=\rho_2(t^3)=p^*(y_2^3)$ it follows $\rho_2(b)=y_2^3+n\cdot y_3^2$ for some $n\in\boldsymbol{Z}_2$. We replace b by $b+(n+1)\cdot\chi^2$. Then the relations $\rho_2(b)=y_2^3+y_3^2$ and $p^*(b)=2u-t^3$ hold, and such b is unique.

By $(4.1)_3$, $H^7(\boldsymbol{EII};\boldsymbol{Z}_2)$ is generated by $h(\rho_2(a))=h(y_2^2)=y_3y_2^2$. On the other hand, from (4.5) reduced mod 2, $h(\rho_2(a))=h(\theta(\rho_2(u)))=0$. Thus

$$H^7(\boldsymbol{EII};\boldsymbol{Z}_2)=0 \quad \text{and} \quad y_3y_2^2=0.$$

Since $\mathrm{Sq}^1(\mathrm{Sq}^2 y_3)=\mathrm{Sq}^3 y_3=y_3^2\neq 0$, $\mathrm{Sq}^2 y_3$ does not vanish and $\mathrm{Sq}^2 y_3=y_3y_2$. Moreover $0=\mathrm{Sq}^2(y_3y_2^2)=(\mathrm{Sq}^2 y_3)y_2^2+y_3(\mathrm{Sq}^1 y_2)^2=y_3y_2^3+y_3^3=y_3^3$.

Consider $(4.1)_4$ for $A=\boldsymbol{Z}_2$:

$$0\longrightarrow Z_2<y_3y_2>\xrightarrow{h} H^8(\boldsymbol{EII};\boldsymbol{Z}_2)\xrightarrow{p^*}\boldsymbol{Z}_2<v, ut, t^4>$$

$$\xrightarrow{\theta}\boldsymbol{Z}_2<y_3^2, y_2^3>\xrightarrow{h} H^9(\boldsymbol{EII};\boldsymbol{Z}_2)\longrightarrow 0,$$

in which $p^*(y_2^4)=\rho_2(t^4)$, $\theta(\rho_2(ut))=\theta(p^*(y_2)\rho_2(u))=y_2^3$ and $h(y_3^2)=y_3^3=0$. By the exactness of the sequence we have $H^9(\boldsymbol{EII};\boldsymbol{Z}_2)=0$, $\dim. H^8(\boldsymbol{EII};\boldsymbol{Z}_2)=2$ and hence $H^8(\boldsymbol{EII};\boldsymbol{Z}_2)=\boldsymbol{Z}_2<y_3^2y_2, y_2^4>$. We have determined the ring $H^*(\boldsymbol{EII};\boldsymbol{Z}_2)$ up to degree 9.

(4.3) and $H^9(\boldsymbol{EII};\boldsymbol{Z}_2)=0$ imply $H^9(\boldsymbol{EII})=0$ and $b\chi=\chi^3=0$. Then $(4.1)_4$ is reduced to

$$0\longrightarrow H^8(\boldsymbol{EII})\xrightarrow{p^*}\boldsymbol{Z}<v, ut, t^4>\xrightarrow{\theta}\boldsymbol{Z}<b>+\boldsymbol{Z}_2<\chi^2>\longrightarrow 0.$$

From $2\theta(ut)=\theta(2ut-t^4)=\theta(p^*(b)t)=2b$, we have $\theta(ut)=b+m\cdot\chi^2$ for some $m\in\boldsymbol{Z}_2$. Applying ρ_2 we have $\rho_2\theta(ut)=y_2^3+(m+1)\cdot y_3^2$ and this equals to $\theta(\rho_2(ut))=y_2^3$ as above. Thus $m=1$ and

(4.6) $$\theta(ut)=b+\chi^2.$$

By Proposition 2.4, v is a p^*-image in rational coefficient, and we have $\theta(v)$

$=n\cdot\chi^2$ $(n\in \boldsymbol{Z}_2)$. Then (4.6) and $\theta(t^4)=\theta(p^*(a^2))=0$ show that $n=1$. Thus

$$\theta(v)=\chi^2. \tag{4.7}$$

By the exactness of the above sequence we have $H^8(\boldsymbol{EII})=\boldsymbol{Z}<c, a^2>$ for an element c which is uniquely determined by $p^*(c)=2v-t^4$.

Finally $p^*\rho_2(c)=\rho_2(t^4)$ implies $\rho_2(c)=y_2^4+m\cdot y_3^2y_2$ $(m\in \boldsymbol{Z}_2)$. But $\rho_2(a^2)=y_2^4$, and ρ_2 induces an isomorphism: $H^8(\boldsymbol{EII})\otimes \boldsymbol{Z}_2\rightarrow H^8(\boldsymbol{EII}; \boldsymbol{Z}_2)$. So we have $\rho_2(c)=y_2^4+y_3^2y_2$. q.e.d.

Since $H^8(\boldsymbol{EII})$ is free and $2\theta(vt)=\theta(p^*(c+a^2)t)=2(c+a^2)$ we have

$$\theta(vt)=c+a^2. \tag{4.8}$$

From (4.2), $p^*(a)=t^2$, (4.5), (4.6), (4.7) and (4.8) we have $(i\geq 0)$

$$\theta(t^{2i})=0,\quad \theta(t^{2i+1})=2a^i,\quad \theta(ut^{2i})=a^{i+1},\quad \theta(ut^{2i+1})=(b+\chi^2)a^i, \tag{4.9}$$

$$\theta(vt^{2i})=a^i\chi^2\ (=0\ \textit{if}\ i>0)\quad \textit{and}\quad \theta(vt^{2i+1})=(c+a^2)a^i.$$

We continue the computation up to degree 13.

Lemma 4.2. (i) *We have* $H^{10}(\boldsymbol{EII})=\boldsymbol{Z}<ba>$, $H^{11}(\boldsymbol{EII})=H^{13}(\boldsymbol{EII})=0$ *and* $H^{12}(\boldsymbol{EII})=\boldsymbol{Z}<d, d', a^3>$ *where* d *and* d' *are uniquely determined by the relations*

$$p^*(d)=w\quad \textit{and}\quad p^*(d')=vt^2.$$

The following relations hold:

$$c\chi=0,\quad 2d'=ca+a^3\quad \textit{and}\quad 8d=6ca-b^2-a^3.$$

(ii) *Putting* $y_{12}=\rho_2(d)$ *and* $y'_{12}=\rho_2(d')$ *we have* $H^{10}(\boldsymbol{EII}; \boldsymbol{Z}_2)=Z_2<y_2^5>$, $H^{11}(\boldsymbol{EII}; \boldsymbol{Z}_2)=H^{13}(\boldsymbol{EII}; \boldsymbol{Z}_2)=0$ *and* $H^{12}(\boldsymbol{EII}; \boldsymbol{Z}_2)=\boldsymbol{Z}_2<y_{12}, y'_{12}, y_2^6>$.

(iii) $\theta(vu)=d'$ *and* $d'\chi=0$.

Proof. (i) From $(4.1)_5$ and $H^7(\boldsymbol{EII})=0$ we have an exact sequence

$$0\longrightarrow H^{10}(\boldsymbol{EII})\xrightarrow{p^*}\boldsymbol{Z}<vt, ut^2, t^5>\xrightarrow{\theta}\boldsymbol{Z}<c, a^2>\xrightarrow{h}H^{11}(\boldsymbol{EII})\longrightarrow 0.$$

By (4.9), θ is onto and $\operatorname{Ker}\theta$ is generated by $2ut^2-t^5=p^*(ba)$. So, we have $H^{10}(\boldsymbol{EII})=\boldsymbol{Z}<ba>$, $H^{11}(\boldsymbol{EII})=0$ and $c\chi=0$. Similarly from $H^9(\boldsymbol{EII})=0$ and $h(ba)=ba\chi=0$ we have $H^{13}(\boldsymbol{EII})=0$ and an exact sequence

$$0 \longrightarrow H^{12}(\boldsymbol{EII}) \xrightarrow{p^*} \boldsymbol{Z}<w, vt^2, ut^3, t^6> \xrightarrow{\theta} \boldsymbol{Z}<ba> \longrightarrow 0.$$

Obviously $p^*(a^3)=t^6$. By Proposition 2.4, $m \cdot w \in \operatorname{Im} p^*$ for an integer $m \neq 0$. Then $m\theta(w)=0$ in $\boldsymbol{Z}<ba>$, and $\theta(w)=0$. Thus there exists $d=p^{*-1}(w)$. Similarly $d'=p^{*-1}(vt^2)$ exists. By (4.9), $\theta(ut^3)=ba$. By the exactness of the above sequence $H^{12}(\boldsymbol{EII})=\boldsymbol{Z}<d, d', a^3>$. By use of the relation $r_{12}=0$ in Theorem 3.2, $p^*(ca)=2vt^2-t^6=p^*(2d'-a^3)$ and $p^*(b^2)=(2u-t^3)^2=-8w+12vt^2-7t^6=p^*(-8d+6ca-a^3)$. Since p^* is injective, the last two relations in (i) follow.

(ii) Recall that $\rho_2(a)=y_2^2$, $\rho_2(b)=y_2^3+y_3^2$ and use the universal coefficient theorem. Then we have the assertion of (ii) provided that $\operatorname{Tors.} H^{14}(\boldsymbol{EII})=h(H^{11}(\boldsymbol{EII}))=0$, which follows from (4.3).

(iii) $2\theta(vu)=\theta(p^*(c+a^2)u)=(c+a^2)\theta(u)=(c+a^2)a=2d'$ by Lemma 4.1, (4.2) and (4.5). Since $H^{12}(\boldsymbol{EII})$ is torsion free, $\theta(vu)=d'$ and $d'\chi=h(d')=h\theta(vu)=0$ by exactness.

q.e.d.

§5. Mod 2 cohomology of the symmetric space *EII*

We shall discuss the mod 2 cohomology of ***EII***. First about mod 2 cohomology of E_6/C, we have

Lemma 5.1. (i) *$H^*(E_6/C; \boldsymbol{Z}_2)=\Delta(u, v)\otimes \boldsymbol{Z}_2<1, t, t^2,\dots, t^{14}, w, wt, wt^2>$ and the following relations hold*:

(5.1) $$u^2=vt^2+ut^3,\ v^2=wt^2+t^8,\ wt^3=0 \quad \textit{and} \quad w^2=wvt^2+t^{12}.$$

(ii) $\mathrm{Sq}^2(v+ut)=0$, $\mathrm{Sq}^4(v+ut)=w$, $\mathrm{Sq}^6(v+ut)=wt$;

$\mathrm{Sq}^2 w=wt$, $\mathrm{Sq}^4 w=\mathrm{Sq}^6 w=0$, $\mathrm{Sq}^8 w=w(v+ut)$, $\mathrm{Sq}^{10} w=w(v+ut)t$.

Proof. (i) follows from Theorem 3.2. Recall that, in $H^*(E_6/C; \boldsymbol{Z}_2)$, $c_1=t$, $c_2=c_3=0$, $c_4=v+ut$, $c_5=(v+ut)t$ and $c_6=w$. Then (ii) follows from Wu formulas: $\mathrm{Sq}^2 c_4=c_4c_1+c_5$, $\mathrm{Sq}^4 c_4=c_4c_2+c_6$, $\mathrm{Sq}^6 c_4=c_4c_3+c_5c_2+c_6c_1$ and $\mathrm{Sq}^{2i}c_6=c_6c_i$ $(i=1, 2,\dots, 6)$. q.e.d.

The following relations follow from (5.1).

(5.2) (i) $t^{15}=0$, $w^2v=vt^{12}$, $w^2ut=ut^{13}$;

(ii) $wvu=w(v+ut)u$, $wvut+t^{14}=w(v+ut)v$, $vt^{12}+ut^{13}=w^2(v+ut)$,

$vut^{12}=w^2(v+ut)u$, $vut^{13}=w^2(v+ut)v$.

Define an element y_{20} of $H^{20}(\boldsymbol{EII}; \boldsymbol{Z}_2)$ by

$$y_{20} = \mathrm{Sq}^8 y_{12}.$$

Then from Lemmas 4.1, 4.2 and 5.1, (ii) and from (4.5), (4.7)

(5.3) *we have elements* $y_i \in H^i(\boldsymbol{EII}; \boldsymbol{Z}_2)$ $(i=2, 3, 12, 20)$ *and* $y'_{12} \in H^{12}(\boldsymbol{EII}; \boldsymbol{Z}_2)$ *such that*

(i) $p^*(y_2)=t,\ p^*(y_3)=0,\ p^*(y_{12})=w,\ p^*(y'_{12})=vt^2,\ p^*(y_{20})=w(v+ut)$;

(ii) $\theta(1)=0,\ \theta(u)=y_2^2,\ \theta(v)=y_3^2,\ \theta(vu)=y'_{12}$;

(iii) $h(\alpha)=y_3\alpha \qquad (\alpha \in H^*(\boldsymbol{EII}; \boldsymbol{Z}_2))$;

(iv) $y_3y_2^2=0,\ y_3^3=0,\ y'_{12}y_3=0$;

(v) $\mathrm{Sq}^1 y_2=y_3,\ \mathrm{Sq}^1 y_3=\mathrm{Sq}^1 y_{12}=\mathrm{Sq}^1 y'_{12}=0$,

where the homomorphisms p^*, θ *and* h *are those in* $(4.1)_*$ *for* $A=\boldsymbol{Z}_2$.

The main purpose of this section is to prove the following theorems.

Theorem 5.2. *The additive base of* $H^*(\boldsymbol{EII}; \boldsymbol{Z}_2)$ *is given by*

$$H^*(\boldsymbol{EII}; \boldsymbol{Z}_2) = \boldsymbol{Z}_2 < y_2^{i+3},\ y'_{12}y_2^i;\ 0 \leq i \leq 11 >$$
$$+\boldsymbol{Z}_2 < 1,\ y_2,\ y_3,\ y_2^2,\ y_3y_2,\ y_3^2,\ y_3^2y_2 > \otimes \Delta(y_{12}, y_{20}).$$

Theorem 5.3. $H^*(\boldsymbol{EII}; \boldsymbol{Z}_2)=\boldsymbol{Z}_2[y_2, y_3, y_{12}, y'_{12}, y_{20}]/J$ *for the ideal* J *generated by the following elements*:

$$y_3y_2^2,\ \ y_3^3,\ \ y'_{12}y_3,\ \ y_{12}(y_2^3+y_3^2),\ \ y_{12}^2+y_{20}y_2^2+y_2^{12},\ \ y_{12}y'_{12}+y_{20}y_2^2,$$
$$(y'_{12})^2+y_2^{12},\ \ y_{20}(y_2^3+y_3^2),\ \ y_2^{15},\ \ y_{20}y'_{12},\ \ y_{20}^2+y_{20}y_{12}y_3^2y_2.$$

We consider the following graded $\boldsymbol{Z}_2$-modules:

$$B_0^* = \boldsymbol{Z}_2 < y_2^{i+3},\ y'_{12}y_2^i;\ 0 \leq i \leq 11 > + \boldsymbol{Z}_2 < y_2^2 > \otimes \Delta(y_{12}, y_{20}),$$

$$B_1^* = \boldsymbol{Z}_2 < 1,\ y_2 > \otimes \Delta(y_{12}, y_{20}),\ \ B_2^* = \boldsymbol{Z}_2 < y_3^2,\ y_3^2y_2 > \otimes \Delta(y_{12}, y_{20}),$$

$$B^* = B_0^* + B_1^* + B_2^* \quad \text{and} \quad C^* = \boldsymbol{Z}_2 < y_3,\ y_3y_2 > \otimes \Delta(y_{12}, y_{20}).$$

Lemma 5.4. *The following sequence is exact:*

$$0 \longrightarrow C^{2n-3} \xrightarrow{h} B^{2n} \xrightarrow{p^*} H^{2n}(E_6/C; \boldsymbol{Z}_2) \xrightarrow{\theta} B^{2n-2} \xrightarrow{h} C^{2n+1} \longrightarrow 0,$$

where, for each basic monomial of Theorem 5.2, h *is defined by* (iii), (iv) *of* (5.3), p^* *is defined by* (i) *of* (5.3) *and the multiplicativity* $p^*(\alpha\beta)=p^*(\alpha)p^*(\beta)$, *and* θ *is defined by* $(0\leq i\leq 14, 0\leq j\leq 2, 0\leq k\leq 1)$

(5.4), (i) $\theta(t^i)=\theta(wt^j)=0,$

$\theta(ut^i)=y_2^{i+2}$ *for* $i\leq 12,$ $\theta(vut^i)=y'_{12}y_2^i$ *for* $i\leq 11,$

$\theta(vt^i)=y_3^2y_2^i$ $(=0$ *if* $i\geq 2),$ $\theta(wvt^j)=y_{12}y_3^2y_2^j$ $(=0$ *if* $j=2),$

$\theta(wu)=y_{12}y_2^2,$ $\theta(wut^{k+1})=y_{12}y_3^2y_2^k;$

(ii) $\theta(wvu)=y_{20}y_2^2,$ $\theta(wvut^{k+1})=y_{20}y_3^2y_2^k,$ $\theta(ut^{k+13})=0,$

$\theta(vut^{12})=y_{20}y_{12}y_2^2$ *and* $\theta(vut^{k+13})=y_{20}y_{12}y_3^2y_2^k.$

Proof. For $h\colon B^*\to C^*$, h is surjective and $\operatorname{Ker} h=B_0^*+B_2^*$ by (5.3), (iv). By (5.4), $\operatorname{Im}\theta=\operatorname{Ker} h$ and $\operatorname{Ker}\theta$ has a base

$$t^l\ (0\leq l\leq 14),\ wt^j\ (0\leq j\leq 2),\ vt^{i+2}\ (0\leq i\leq 11),$$

$$ut^{13},\ ut^{14},\ vt^{14},\ w(v+ut),\ w(v+ut)t \text{ and } wvt^2.$$

Obviously $h\colon C^*\to B^*$ is injective and $h(C^*)=B_2^*$. Under p^*, the base of $B_0^*+B_1^*$ is mapped as follows:

$$p^*(y_2^l)=t^l(0\leq l\leq 14),\ p^*(y'_{12}y_2^i)=vt^{i+2}(0\leq i\leq 11),$$

$$p^*(y_{12}y_2^j)=wt^j,\ p^*(y_{20}y_2^j)=w(v+ut)t^j \text{ and } p^*(y_{20}y_{12}y_2^j)=w^2(v+ut)t^j.$$

Using (5.1) and (5.2), (i) we see that p^* is an isomorphism of $B_0^*+B_1^*$ onto $\operatorname{Ker}\theta$. Thus the exactness of the sequence is proved. q.e.d.

Proof of Theorem 5.2. We prove that the natural maps $B^{2n}\to H^{2n}(\boldsymbol{EII}; \boldsymbol{Z}_2)$ and $C^{2n+1}\to H^{2n+1}(\boldsymbol{EII}; \boldsymbol{Z}_2)$ are isomorphisms by induction on n. To do so, by virtue of Lemma 5.4 and the exactness of $(4.1)_n$, it is sufficient to prove that the formulas (5.4) hold for $\theta\colon H^{2n}(E_6/C; \boldsymbol{Z}_2)\to H^{2n-2}(\boldsymbol{EII}; \boldsymbol{Z}_2)$ provided the inductive assumption on $H^{2n-2}(\boldsymbol{EII}; \boldsymbol{Z}_2)$. (5.4), (i) is proved by (i), (ii), (iv) of (5.3) and the property (4.2) $\theta(p^*(x)y)=x\theta(y)$. Moreover the relations of

(5.4), (ii) are proved by applying the relations of (5.2), (ii), respectively, to $\theta(w(v+ut)y)=y_{20}\theta(y)$. q.e.d.

As a corollary of Theorem 5.2,

(5.5) *the kernel of* $p^*: H^*(\boldsymbol{EII}; \boldsymbol{Z}_2)\to H^*(E_6/C; \boldsymbol{Z}_2)$ *coincides with* $C^*+B_2^*=\boldsymbol{Z}_2<y_3, y_3y_2, y_3^2, y_3^2y_2>\otimes\Delta(y_{12}, y_{20})$, *in particular* p^* *is injective at degrees* 14, 24, 30, 32, 34 *and* 36.

Proof of Theorem 5.3. The first three relations are already given in (5.3), (iv). By use of (5.4)

$$y_{12}(y_2^3+y_3^2)=\theta(wv)+y_{12}\theta(ut)=\theta(w(v+ut))=\theta p^*(y_{20})=0$$

and
$$y_{20}(y_2^3+y_3^2)=\theta(wvu)y_2+\theta(wvut)=\theta(wvut)+\theta(wvut)=0.$$

By (5.1),

$$p^*(y_{12}^2+y_{20}y_2^2+y_2^{12})=w^2+w(v+ut)t^2+t^{12}=0.$$

Then it follows from (5.5) that $y_{12}^2+y_{20}y_2^2+y_2^{12}=0$. Similarly the elements $y_{12}y'_{12}+y_{20}y_2^2$, $(y'_{12})^2+y_2^{12}$, y_2^{15} and $y_{20}y'_{12}$ vanish.

In order to prove the triviality of the last element we prepare

(5.6) $$\mathrm{Sq}^{12}y_{20}=\mathrm{Sq}^{14}y_{20}=0 \quad \textit{and} \quad \mathrm{Sq}^{16}y_{20}=y_{20}y_{12}y_2^2.$$

By (5.5), (5.6) follows from $\mathrm{Sq}^{12}(w(v+ut))=\mathrm{Sq}^{14}(w(v+ut))=0$ and $\mathrm{Sq}^{16}(w(v+ut))=w^2(v+ut)t^2$ which are computed directly by Lemma 5.1 and by Cartan formula. Now, by use of Cartan formula and (5.6),

$$y_{20}^2=(\mathrm{Sq}^8y_{12})^2=\mathrm{Sq}^{16}(y_{12}^2)=\mathrm{Sq}^{16}(y_{20}y_2^2+y_2^{12})=\mathrm{Sq}^{16}(y_{20})y_2^2$$
$$=y_{20}y_{12}y_2^4=y_{20}y_{12}y_3^2y_2,$$

These relations show that J vanishes in $H^*(\boldsymbol{EII}; \boldsymbol{Z}_2)$. By use of these relations in J, we see that every monomial in $y_2, y_3, \dots, y_{20}$ is a linear combination of the base in Theorem 5.2. Thus Theorem 5.3 is established.

q.e.d.

Since $H^{21}(\boldsymbol{EII}; \boldsymbol{Z}_2)=0$, we have

(5.7) $$\mathrm{Sq}^1y_{20}=0.$$

By the derivativity of Sq^1, the following (5.8) is computed from Theorem

5.2, (5.7) and (iv), (v) of (5.3).

(5.8) $$\mathrm{Im}\,Sq^1 = Z_2\langle y_3, y_3^2\rangle \otimes \Delta(y_{12}, y_{20})$$

and $$\mathrm{Ker}\,Sq^1 = \mathrm{Im}\,Sq^1 + Z_2\langle 1, y_3^2 y_2\rangle \otimes \Delta(y_{12}, y_{20}) + B_0^*.$$

Since Sq^1 is the mod 2 Bockstein homomorphism, (5.8) and (4.3) yield

Proposition 5.4. *The* mod 2 *reduction* ρ_2: $H^*(\boldsymbol{EII}) \to H^*(\boldsymbol{EII}; Z_2)$ *induces isomorphisms*

$$\mathrm{Tors.}\, H^*(\boldsymbol{EII}) \cong Z_2\langle y_3, y_3^2\rangle \otimes \Delta(y_{12}, y_{20})$$

and $$(H^*(\boldsymbol{EII})/\mathrm{Tors.}\, H^*(\boldsymbol{EII})) \otimes Z_2 \cong Z_2\langle y_2^{i+3}, y'_{12} y_2^i;\ 0 \le i \le 11\rangle$$
$$+ Z_2\langle 1, y_2^2, y_3^2 y_2\rangle \otimes \Delta(y_{12}, y_{20}).$$

This and (4.3) determine the additive structure of $H^*(\boldsymbol{EII})$.

§6. Integral cohomology of the symmetric space *EII*

Consider the exact sequence (4.1) for $A = Z[1/2]$. Since $\chi = \frac{1}{2}(2\cdot\chi) = 0$ in $H^3(\boldsymbol{EII}; Z[1/2])$, (4.1) is reduced to the short exact sequence

(6.1) $$0 \longrightarrow H^*(\boldsymbol{EII}; Z[1/2]) \xrightarrow{p^*} H^*(E_6/C; Z[1/2])$$
$$\xrightarrow{\theta} H^*(\boldsymbol{EII}; Z[1/2]) \longrightarrow 0.$$

Theorem 6.1. *For the integral classes a, b, c, d of* $H^*(\boldsymbol{EII})$,

$$H^*(\boldsymbol{EII}; Z[1/2]) = Z[1/2][a, b, c, d]/(q_{12}, q_{16}, q_{18}, q_{24}),$$

where the relations q_j's are given in (2.8).

Proof. By Theorem 3.2, $H^*(E_6/C; Z[1/2]) = Z[1/2][t, u, v, w]/(r_{12}, r_{16}, r_{18}, r_{24})$. By Lemmas 4.1 and 4.2,

(6.2) $$p^*(a) = t^2,\quad p^*(b) = 2u - t^3,\quad p^*(c) = 2v - t^4 \quad \textit{and} \quad p^*(d) = w.$$

Hence an arbitrary element x of $H^*(E_6/C; Z[1/2])$ is written in the from $x = p^*(f) + p^*(g)t$ for some polynomials f and g in a, b, c, d. By (4.2), $\theta(x) = \theta(p^*(f)) + g\theta(t) = 2g$. Since θ is surjective, this shows that $H^*(\boldsymbol{EII}; Z[1/2])$ is multiplicatively generated by a, b, c, d. The coefficient homomorphism $H^*(\boldsymbol{EII}; Z[1/2]) \to H^*(\boldsymbol{EII}; Q)$ is injective since $H^*(\boldsymbol{EII})$ is odd torsion free by (4.3). Then the theorem follows easily from Proposition 2.4.

q.e.d.

Recall from Lemmas 4.1 and 4.2

(6.3) $\rho_2(a)=y_2^2,\ \rho_2(b)=y_2^3+y_3^2,\ \rho_2(c)=y_3^2y_2+y_2^4,\ \rho_2(d)=y_{12}$ *and*

$\rho_2(d')=y'_{12},\ p^*(d')=vt^2,\ 2d'=ca+a^3$ *for* $d'\in H^{12}(\boldsymbol{EII})$.

Lemma 6.2. *There exist elements* $e\in H^{14}(\boldsymbol{EII})$ *and* $f\in H^{20}(\boldsymbol{EII})$ *satisfying*

(6.4) $\rho_2(e)=y'_{12}y_2,\quad p^*(e)=v(2u-t^3),\quad 2e=cb+ba^2,$

$\rho_2(f)=y_{20}+\varepsilon y_{12}y_2^4(\varepsilon\in\boldsymbol{Z}_2),\quad p^*(f)=w(v-ut)$ *and* $2f=dc.$

Proof. By (6.3) and Theorem 5.3, $\rho_2(cb+ba^2)=y_3^2y_2^4=0$ and $\rho_2(dc)=y_{12}(y_2^3+y_3^2)y_2=0$. Thus there exist $e, f\in H^*(\boldsymbol{EII})$ such that $2e=cb+ba^2$ and $2f=dc$. Then, $p^*(e)=\frac{1}{2}p^*(c+a^2)p^*(b)=v(2u-t^3)$ and $p^*(f)=\frac{1}{2}p^*(d)p^*(c)=\frac{1}{2}w(2v-t^4)=w(v-ut)$ as $H^*(E_6/C)$ is torsion free. Next, by (i) of (5.3), $p^*(\rho_2(e))=\rho_2(p^*(e))=vt^3=p^*(y'_{12}y_2)$ and $p^*(\rho_2(f))=\rho_2\ (p^*(f))=w(v+ut)=p^*(y_{20})$. Then it follows from (5.5) that $\rho_2(e)=y'_{12}y_2$ and $\rho_2(f)=y_{20}$ or $\rho_2(f)=y_{20}+y_{12}y_3^2y_2=y_{20}+y_{12}y_2^4$. q.e.d.

The structure of $H^*(\boldsymbol{EII})$ is determined by the following theorems.

Theorem 6.3. Tors. $H^*(\boldsymbol{EII})=\boldsymbol{Z}_2<\chi,\ \chi^2>\otimes\Delta(d, f)$ *and the Poincaré polynomial is* $P(\boldsymbol{EII}, t)=(1+t^4+t^8+t^{12})(1+t^6+t^{12})(1+t^8+t^{16})$.

This follows directly from Proposition 5.4.

Theorem 6.4. $H^*(\boldsymbol{EII})$ *is multiplicatively generated by the elements*

(6.5) $\chi,\ a,\ b,\ c,\ d,\ d',\ e$ *and* $f,$

and $H^*(\boldsymbol{EII})=\boldsymbol{Z}[\chi,\ a,\ b,\ c,\ d,\ d',\ e,\ f]/I$ *for the ideal* I *generated by the following elements*:

(6.6) $2\chi,\quad a\chi,\quad \chi^3,\quad b\chi,\quad c\chi,\quad q_{12}=b^2+8d-6ca+a^3,\quad 2d'-ca-a^3,$

$2e-cb-ba^2,\quad d'\chi,\quad q_{16}=a^4+12da-6ca^2-3c^2,\quad e\chi,\quad q_{18}=db,$

$ea-d'b,\quad dc-2f,\quad 3d'c+3d'a^2-6da^2-2a^5,$

$eb+7d'a^2+8f-8da^2-4a^5,\quad 3ec+3d'ba-2ba^4,$

$q_{24}=d^2+c^3-3fa,\quad dd'+5fa-4c^3,\quad 3d'^2-24c^3+36fa-a^6,$

$3ed'-ba^5,\quad fb,\quad ed+f\chi^2,\quad 3fc-2d^2a,\quad 3e^2+8da^4-12d'a^4+7a^7,$

$3fd'+12fd-7d^2a^2,\quad fe,\quad 9f^2-fda^2.$

We denote the ρ_2-image of the elements of (6.5) by the same letters.

Lemma 6.5. $\operatorname{Im}(\rho_2: H^*(\boldsymbol{EII})\to H^*(\boldsymbol{EII};\boldsymbol{Z}_2))=\boldsymbol{Z}_2[\chi, a, b, c, d, d', e, f]/I_2$ *where I_2 is the ideal generated by the following elements:*

(6.7) $a\chi,\ \chi^3,\ b\chi,\ c\chi,\ b^2+a^3,\ ca+a^3,\ cb+ba^2,\ d'\chi,\ c^2+a^4,\ e\chi,\ db,$

$ea+d'b,\ dc,\ d'c+d'a^2,\ eb+d'a^2,\ ec+d'ba,\ d^2+fa+a^6,\ dd'+fa,$

$d'^2+a^6,\ ed'+ba^5,\ fb,\ ed+f\chi^2,\ fc,\ e^2+a^7,\ fd',\ fe,\ f^2+fda^2.$

Proof. $\operatorname{Im}\rho_2=\operatorname{Ker}Sq^1=B_0^*+\boldsymbol{Z}_2\langle 1, y_3, y_3^2, y_3^2y_2\rangle\otimes\Delta(y_{12}, y_{20})$ by (5.8). Rewrite this by the present notation, then

(6.8)
$$\operatorname{Im}\rho_2=\Delta(d')\otimes\boldsymbol{Z}_2\langle 1, a, b, a^2, ba,\dots, a^5, ba^4\rangle$$
$$+\boldsymbol{Z}_2\langle\chi, \chi^2, c, e, a^6, ba^5, a^7\rangle$$
$$+\boldsymbol{Z}_2\langle d, f, fd\rangle\otimes\boldsymbol{Z}_2\langle 1, \chi, a, \chi^2, a^2\rangle,$$

where $\chi=y_3$, $a=y_2^2$, $b=y_2^3+y_3^2$, $c=y_3^2y_2+y_2^4$, $d=y_{12}$, $d'=y'_{12}$, $e=y'_{12}y_2$ and $f=y_{20}+\varepsilon y_{12}y_2^4$. Then it is directly verified by Theorem 5.3 that the elements in (6.7) vanish in $H^*(\boldsymbol{EII};\boldsymbol{Z}_2)$. Moreover we see that the following elements are in I_2:

(6.9) $da^3,\ dba,\ fa^3,\ fba,\ ba^6,\ a^8,\ d'a^6+fda,\ d'ba^5+fd\chi^2.$

For example, $a^8=ca^6=d^2c+fca=fa^3$ and $fa^3=fb^2=0$.

By use of the triviality of the elements in (6.7) and (6.9), we see that every element of $\boldsymbol{Z}_2[\chi, a, b,\dots, f]$ is congruent modulo I_2 to an element of $\operatorname{Im}\rho_2$. This proves Lemma 6.5.

Proof of Theorem 6.4. Put $P=\boldsymbol{Z}[\chi, a, b, c, d, d', e, f]$. Since $\chi=0$, $d'=\frac{1}{2}(ca+a^3)$, $e=\frac{1}{2}(cb+ba^2)$ and $f=\frac{1}{2}dc$ in $H^*(\boldsymbol{EII};\boldsymbol{Z}[1/2])$, direct computations show that each element of (6.6) vanishes in $H^*(\boldsymbol{EII};\boldsymbol{Z}[1/2])$. Moreover, the basic relations q_{2j}'s are covered by some of (6.6). Thus we have a natural isomorphism

(6.10) $$(P/I)\otimes \boldsymbol{Z}[1/2] \xrightarrow{\cong} H^*(\boldsymbol{EII}; \boldsymbol{Z}[1/2]) = H^*(\boldsymbol{EII})\otimes \boldsymbol{Z}[1/2].$$

We see also mod 2 reductions of the elements of (6.6), except the first one, coincide with those of (6.7) modulo (6.9). Thus

(6.11) $$(P/I)\otimes \boldsymbol{Z}_2 \cong \mathrm{Im}(\rho_2: H^*(\boldsymbol{EII}) \longrightarrow H^*(\boldsymbol{EII}; \boldsymbol{Z}_2)) \cong H^*(\boldsymbol{EII})\otimes \boldsymbol{Z}_2.$$

Consider the natural ring homomorphism

$$g: P/I \longrightarrow H^*(\boldsymbol{EII}).$$

By tensoring the identity of $\boldsymbol{Z}[1/2]$ and $\boldsymbol{Z}_2$ with g we obtain the isomorphisms (6.10) and (6.11). So, by a simple algebraic consideration, together with that P/I is of finite type, we have that g is surjective and

(6.12) Ker g *is contained in* Tors.(P/I), *which is a finite* 2*-group and* $g\otimes 1$ *maps* Tors.$(P/I)\otimes \boldsymbol{Z}_2$ *isomorphically onto* Tors.$H^*(\boldsymbol{EII})\otimes \boldsymbol{Z}_2$.

The subgroup T of Tors.(P/I) generated by $\{\chi^{i+1}d^jf^k;\ i, j, k=0, 1\}$ is mapped, under g, isomorphically onto Tors.$H^*(\boldsymbol{EII}) = \boldsymbol{Z}_2 < \chi, \chi^2 > \otimes \Delta(d, f)$. Thus T is a direct summand of Tors.(P/I) and $(g\otimes 1:)T\otimes \boldsymbol{Z}_2 \cong$ Tors.$H^*(\boldsymbol{EII})\otimes \boldsymbol{Z}_2$. This and (6.12) show that (Tors.$(P/I)/T)\otimes \boldsymbol{Z}_2 = 0$, $T=$ Tors.(P/I) and Ker $g=0$. Consequently we have proved that g is an isomorphism. q.e.d.

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