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On the Green's function for $\Delta-\lambda^2$ with the boundary condition of the third kind in the exterior domain of a bounded obstacle

By

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0. Introduction

The boundary-value problems for the Helmholtz equation $\Delta u+k^2u=f$ play an important role in various problems of mathematical physics. In [2], [3], S. Mizohata constructed the Green's function for $\Delta-\lambda^2$ with Neumann or Dirichlet boundary condition in the exterior domain of a bounded obstacle, establishing the theory of integral equations with a holomorphic parameter. He showed the Green's function is meromorphic with respect to λ in the whole complex plane, and it is holomorphic in $\mathrm{Re}\,\lambda\geqslant 0$. In this note, we discuss the Green's function for $\Delta-\lambda^2$ with the boundary condition of the third kind $\frac{\partial u}{\partial n}+\sigma(s)u=0$. We try to represent the solution as the potential of a simple layer. In 1. we show the boundary-value problem can be reduced to an integral equation for the density on the boundary, and the methods in [2], [3] are applicable to our case. In 2. we see first the resolvent kernel for the integral operator is holomorphic in $\mathrm{Re}\,\lambda>0$ except for the positive real axis. If σ is sufficiently small, we can say the resolvent kernel is holomorphic on the non-negative real axis. The same thing holds, if $\sigma<\frac{(op,\, n)}{|op|^2}$, when the obstacle is star-shaped. In 3. we construct the Green's function, and show it is holomorphic on the positive and negative imaginary axis.

If we assume the above inequality, we can say the Green's function is holomorphic in $\mathrm{Re}\,\lambda\geqslant 0$. As is well known, such analyticity leads to the local decay of the solution of the exterior problem for the wave equation. When the boundary is a sphere with radius ρ, the inequality reads $\sigma<\frac{1}{\rho}$ (if we choose the origin as O). In 4. we construct the solution for the wave equation with the boundary condition $\frac{\partial u}{\partial r}+\frac{1}{\rho}u=0$ that never decays. When $\sigma<\frac{1}{\rho}$, T. Tokita showed, in [5], the solution decays exponentially. Finally, I wish to thank Prof. Mizohata for his valuable advice and incessant encouragement.

1. Preliminaries

Let S be a compact $C^{2+\alpha}$-surface in R^3, and $\Omega^+(\Omega^-)$ the exterior (the interior) of S. A $C^{2+\alpha}$-surface means there exists a $C^{2+\alpha}$ function φ in R^3 for every point on S, such that grad $\varphi \neq 0$ and S is represented as the null-set of φ in a neighbourhood of the point.

We consider the following problem.

$$(1.1)\qquad \begin{cases} \Delta u - \lambda^2 u = f & p \in \Omega^+ \quad \lambda \in \mathbf{C} \\ \dfrac{\partial u}{\partial n} + \sigma(p)u = 0 & p \in S. \end{cases}$$

Here, f is a bounded Hölder continuous function in Ω^+, and σ is a real-valued Hölder continuous function on S. From now on. P, Q,... denote the points in $\Omega^{\pm}$, p, q,... the points on S. dP denotes the volume element in R^3, dq the surface element of S. We say a function $\varphi \in C^{m+\alpha}(M)$ (M is S, $\Omega^{\pm}$ or R^3), when φ is m-times continuously differentiable, and its m-th derivatives are Hölder continuous with exponent α.

Difinition 1.1. We say $G(P, Q|\lambda)$ is the Green's function for (1.1) when it has the following properties.

i) If we fix a point Q in Ω^+, $G(P, Q|\lambda)$ is of class C^2 with respect to P in $\Omega^+\backslash\{Q\}$, and whose first derivatives are continuous up to the boundary of Ω^+.

ii) $(\Delta_p - \lambda^2)G(P, Q, |\lambda) = \delta(P-Q) \qquad P \in \Omega^+$, where δ is the Delta function.

iii) $\left[\dfrac{\partial}{\partial n} + \sigma(p)\right]G(p, Q|\lambda) = 0 \qquad p \in S.$

Since $-\dfrac{1}{4\pi}\dfrac{e^{-\lambda|P|}}{|P|} = -E(P|\lambda)$ is a fundamental solution for $\Delta - \lambda^2$, we set $G(P, Q|\lambda) = -E(P-Q|\lambda) + K(P, Q|\lambda)$. So we have only to consider (1.2)

$$(1.2)\qquad \begin{cases} \Delta u - \lambda^2 u = 0 & \text{in} \quad \Omega^+ \\ \dfrac{\partial u}{\partial n} + \sigma(p)u = g(p) & \text{on} \quad S \qquad g \in C^{\alpha}(S). \end{cases}$$

Definition 1.3. Let φ, ψ be continuous function on S, then $V(P) = 2\int E(P-Q|\lambda)\varphi(q)dq$ is called the simple layer potential with density φ, and $W(P) = 2\int \dfrac{\partial}{\partial n_q} E(P-q|\lambda)\psi(q)dq$ the double layer potential with density ψ.

The following properties for V, W are well known as "The jump relations".

Proposition 1.4.

i) $\dfrac{\partial}{\partial n} V(p)|_{\pm} = \mp\varphi + 2\displaystyle\int \dfrac{\partial}{\partial n_p} E(p-q|\lambda)\varphi(q)dq$

ii) $W(p)|_{\pm} = \pm\psi + 2\displaystyle\int \dfrac{\partial}{\partial n_q} E(p-q|\lambda)\psi(q)dq.$

Here $f(p)|_{\pm}$ denote $\lim_{\substack{P\to p \\ P\in\Omega^+}} f(P)$.

We try to represent the solution of (1.2) in the form of $u(P)=2\int E(P-q|\lambda)\varphi(q)dq$. Then the density should satisfy the following integral equation.

$$(1.5)\qquad -\varphi(p)+2\int\frac{\partial}{\partial n_p}E(p-q|\lambda)\varphi(q)dq+2\sigma(p)\int E(p-q|\lambda)\varphi(q)dq=g(p)$$

Conversely if we obtain a continuous function φ as a solution for (1.5), $u(P)=2\int E(P-q|\lambda)\varphi(q)dq$ is obviously the solution of (1.2). Hereafter, $K(p, q|\lambda)$ denotes $2\frac{\partial}{\partial n_p}E(p-q|\lambda)+2\sigma(p)E(p-q|\lambda)$. We notice a property of solution of the integral equation with the transposed kernel $K(q, p|\lambda)$, which can be proved easily.

Proposition 1.6. *Let g be a continuous function on S, ψ a solution of the integral equation with the transposed kernel $K(q, p|\lambda)$.*

$$(1.7)\qquad -\psi(p)+\int K(q, p|\lambda)\psi(q)dq=g(p)$$

Then $v(P)=2\int\frac{\partial}{\partial n_q}E(P-q|\lambda)\psi(q)dq$ is the solution of the following interior Dirichlet problem.

$$(1.8)\qquad \begin{cases} \Delta v-\lambda^2 v=0 & P\in\Omega^- \quad v\in C^2(\Omega^-)\cap C(\bar\Omega) \\ v|_-=g(p) & p\in S \end{cases}$$

We are going to solve the integral equations (1.5), (1.7) following Mizohata's argument. We briefly review the theory of integral equations with a holomorphic parameter.

i) The integral equations (1.5), (1.8) are solved by Neumann series in a domain $\mathscr{D}$ in **C**, and the resolvent kernel $R(p, q|\lambda)$ is meromorphically continued with respect to λ to the whole complex plane.

ii) If $R(p, q)$ is holomorphic in a neighbourhood of $\lambda_0\in C$, (1.5), (1.7) are solved uniquely for any f, g. And the solutions are expressed as follows.

$$\begin{cases} \varphi(p)=-f(p)-\int R(p, q|\lambda_0)f(q)dq \\ \psi(p)=-g(p)-\int R(q, p|\lambda_0)g(q)dq. \end{cases}$$

iii) If R has a pole at λ_0, there exist non-trivial eigenfunctions φ, ψ satisfying

$$\varphi(p)=\int K(p, q|\lambda_0)\varphi(q)dq$$

$$\psi(p)=\int K(q, p|\lambda_0)\psi(q)dq.$$

iv) Let $R(p, q|\lambda)=A_{-m}(p, q)(\lambda-\lambda_0)^{-m}+\cdots+A_0(p, q|\lambda)$ be the Laurent expansion of R at λ_0, then A_{-m} has a specific form, namely

$$A_{-m}(p, q)=\varphi_1(p)\psi_1(q)+\cdots+\varphi_k(p)\psi_k(q)$$

where $\{\varphi_j\}$ ($\{\psi_j\}$) are a system of linearly independent eigenfunctions for $K({}^tK)$.

Those results are celebrated theorems of Fredholm when the kernel is continuous and has the form $K(p, q|\lambda)=\lambda K(p, q)$. The kernels introduced here may have certain singularity, but counting the inequality $|K(p, q|\lambda)|\leqslant C(1+|\lambda|\,|p-q|)|p-q|^{-1}e^{-\mathrm{Re}\lambda|p-q|}$, we can confine ourselves to the case where $K(p, q|\lambda)$ is continuous, considering the twicely iterated kernel $K^{(2)}(p, q|\lambda)$. Then we introduce another complex parameter μ artificially, and consider the following integral equation.

$$(1.9)\qquad -\varphi(p)+\mu\int K(p, q|\lambda)\varphi(q)dq=f(p) \qquad |\mu|\leqslant 1$$

As $|K^{(2)}(p, q|\lambda)|\leqslant C|\lambda|e^{-\mathrm{Re}\lambda|p-q|}$, we see easily the Neumann series converges uniformly with respect to μ and λ when $|\mu|\leqslant 1$ and λ in a sector in the complex plane. Then we have only to repeat the course of Fredholm's theory, considering μ as the eigenvalue λ as a parameter. Since the Fredholm determinant converges uniformly with respect to the parameter λ, Fredholm determinant is entire analytic with respect not only to μ but also to λ. Then we let μ be 1. The exposition of the theory is found in Mizohata [3].

We close this section noticing the regularity of the simple and double layer potentials.

i) The eigenfunctions for $K(p, q|\lambda)$ are of class $C^{\alpha}(S)$.

ii) The eigenfunctions for $K(q, p|\lambda)$ are of class $C^{1+\alpha}(S)$.

iii) The simple layer potential with a Hölder continuous density is of class $C^{\infty}(\Omega^{\pm})$ and $C^{1+\alpha}(\bar{\Omega}^{\pm})$.

iv) The double layer potential with a density of class $C^{1+\alpha}(S)$ is $C^{\infty}(\Omega^{\pm})$ and $C^{1+\alpha'}(\bar{\Omega}^{\pm})$.

Those facts are proved only for the Newtonian potentials in Günter [1]. But we need no essential modifications in proving them for our oscillating potentials.

2. Analytic continuation of the resolvent kernel

Now we discuss the properties of the resolvent kernel with respect to λ.

Lemma 2.1. *Let φ be a Hölder continuous function on S. We set $u(P)=2\int E(P-q|\lambda)\varphi(q)dq$. If $u(P)\equiv 0$ in Ω^+ and λ^2 is not an eigenvalue for the interior Dirichlet problem for Δ, then φ must be $\equiv 0$ on S.*

Proof. As u is continuous in R^3, $u\equiv 0$ especially on S. Therefore u is a solution of the interior Dirichlet problem

$$\begin{cases} \Delta u-\lambda^2 u=0 & \text{in } \Omega^- \\ u|_s=0 & \text{on } S \qquad u\in C^2(\Omega^-)\cap C(\bar{\Omega}^-) \end{cases}$$

Since λ^2 is not an eigenvalue, u must be $\equiv 0$ in Ω^-. So we have $-2\varphi(p)=\left.\frac{\partial u}{\partial n}\right|_+ - \left.\frac{\partial u}{\partial n}\right|_- =0$ by "The jump relation".

To begin with, we study the resolvent kernel in the case where the real part of λ is positive.

Lemma 2.2.
i) *The resolvent kernel R is holomorphic in $\{\lambda\in\mathbf{C}\,|\,\mathrm{Re}\,\lambda>0,\ \mathrm{Im}\,\lambda\neq 0\}$.*
ii) *There may be finitely many poles on the positive real axis.*

Proof. We assume $\lambda_0\in C$, $\mathrm{Re}\,\lambda_0>0$ is a pole of R, and set $u(P)=2\int E(P-q|\lambda_0)\varphi(q)dq$ for a non-trivial eigenfunction φ corresponding to λ_0. Then $u(P)$ satisfies $(\Delta-\lambda_0^2)u=0$ $P\in\Omega^+$, $\left(\frac{\partial}{\partial n}+\sigma(p)\right)u=0$ $p\in S$, and $\left(\frac{\partial}{\partial P}\right)^\alpha u(P)=O(e^{-\mathrm{Re}\lambda_0|P|})$. Applying the Green's identity to u and $\bar{u}$, we have $\int_{\Omega^+}\Delta u\bar{u}dP=-\int_{\Omega^+}|\nabla u|^2dP+\int_s\sigma|u|^2dq$. And we obtain

$$\int_{\Omega^+}|\nabla u|^2dP+\lambda_0^2\int_{\Omega^+}|u|^2dP=\int_s\sigma|u|^2dq \tag{2.3}$$

i) We set $\lambda_0=\mu_0+ik_0$ ($\mu_0>0$, $k_0\neq 0$, μ_0 and k_0 are real). Comparing the imaginary part of the both sides of (2.3), we have $2\mu_0k_0\int|u|^2dP=0$. Then $u\equiv 0$ in Ω^+ which is, by Lemma 2.2, contrary to the assumption that φ is non-trivial.

ii) When $\lambda_0=\mu_0$, to (2.3) we apply an interpolation inequality $\int_s|u|^2dq\leqslant\varepsilon\int_{\Omega^+}|\nabla u|^2dP+C_\varepsilon\int_{\Omega^+}|u|^2dP$ (for any $\varepsilon>0$). Then we obtain

$$(1-\varepsilon\max|\sigma|)\int|\nabla u|^2dP+(\lambda_0^2-C_\varepsilon\max|\sigma|)\int|u|^2dP\leqslant 0,$$

which shows $u\equiv 0$ in Ω^+ when $\lambda_0>(C_\varepsilon\max|\sigma|)^{1/2}$. This is again contrary to our assumption.

Next we impose a certain condition on the surface S.

Definition 2.3. An obstacle $\mathcal{O}$ is called star-shaped, when there exists an interior point $O\in\mathcal{O}$ such that $(n_p, \overrightarrow{op})\geqslant 0$ for every point p on $\partial\mathcal{O}$. Here n_p denotes the outward unit normal vector at $p\in S$. We may call "S is star-shaped" when the interior of S is star-shaped.

Proposition 2.4. *We assume S is star-shaped. If $\sigma(p)$ satisfies $\sigma(p)<|(n_p,\overrightarrow{op})|\,|\overrightarrow{op}|^{-2}$, where O is an interior point mentioned above, $R(p, q|\lambda)$ is holomorphic is a neighborhood of the non-negative real axis, containing the origin.*

Proof. Let $\lambda_0(\lambda_0\geqslant 0)$ be a pole of R, we set $u(P)=2\int E(P-q|\lambda_0)\varphi(q)dq$ for a non-trivial eigenfunction corresponding to λ_0. Then $u(P)$ satisfies the following.

$$
(2.4)\quad \begin{cases} (\Delta-\lambda_0^2)u=0 & \text{in} \quad \Omega^+ \\ \left(\dfrac{\partial}{\partial n}+\sigma\right)u=0 & \text{on} \quad S \qquad u\in C^2(\Omega^+)\cap C^1(\bar{\Omega}^+) \\ \left(\dfrac{\partial}{\partial P}\right)^{\alpha}u=O(e^{-\mathrm{Re}\lambda|P|}) & \text{if} \quad \lambda_0>0,\ |P| \ \text{sufficiently large} \\ u=O(|P|^{-1}),\ \left(\dfrac{\partial}{\partial P}\right)^{\alpha}u=O(|P|^{-1-|\alpha|}) & \text{if} \quad \lambda_0=0,\ |P| \ \text{sufficiently large} \end{cases}
$$

We introduce the polar coordinates with the centre at O, setting $\omega=op|op|^{-1}$, $\overrightarrow{op}=|\overrightarrow{op}|\omega=p(\omega)$. As $u(p(\omega))=-\int_{|p(\omega)|}^{\infty}\frac{\partial}{\partial r}u(r\omega)dr$, (The integral converges, according to (2.4). $\frac{\partial u}{\partial r}$ denotes the derivative along the radial direction.) we obtain the following by applying the Schwartz's inequality to the right hand side.

$$
(2.5)\qquad |u(P(\omega))|\leqslant\frac{1}{|p(\omega)|}\int_{|P(\omega)|}^{\infty}|\nabla u|^2r^2dr
$$

If $(n_p, \overrightarrow{op})\neq 0$, dp is represented as $\frac{|p(\omega)|^2}{|(n_\omega, \omega)|}d\omega$, where $d\omega$ is the surface element of the unit sphere, $n_\omega=n_{p(\omega)}$. At the point p where $(n_p, \overrightarrow{op})=0$, the condition reads $\sigma(p)<0$. So if we set $\omega_+=\{|\omega|=1;\ \sigma(p(\omega))\geqslant 0\}$ and $\omega_\delta=\{|\omega|=1;\ |(n_\omega, \omega)|\geqslant\delta\}$, there exists a positive δ such that $\omega_+\subset\omega_\delta$.

Multiplying $\sigma(p(\omega))$ to the both sides of (2.5) and integrating over ω_+ we have

$$
\int_{\omega_+}|u|^2dq\leqslant\int_{\omega}\frac{\sigma(p(\omega))\,|P(\omega)|^2}{|(n_p, op)|}\int_{|p(\omega)|}^{\infty}|\nabla u|^2r^2drdq
$$

$$
\leqslant\sup_{p\in\omega}\left|\frac{\sigma(p)\,|\overrightarrow{op}|^2}{(n_p, op)}\right|\int_{\Omega^+}|\nabla u|^2dP.
$$

If we denote $\varepsilon=\sup_{p\in\omega}\left|\frac{\sigma(p)|\overrightarrow{op}|^2}{(n_p, op)}\right|$, we obtain

$$
(2.6)\qquad \int_S\sigma|u|^2dq\leqslant\varepsilon\int_{\Omega^+}|\nabla u|^2dP.
$$

i) In the case $\lambda_0>0$, substituting (2.6) in (2.3), we find

$$
\lambda_0^2\int_{\Omega^+}|u|^2dP+(1-\varepsilon)\int_{\Omega^+}|\nabla u|^2dP\leqslant 0.
$$

Since $\varepsilon<1$, $u\equiv 0$ in Ω^+, which is contrary to the assumption.

ii) In the case $\lambda_0=0$, we choose such a large sphere with radius R as to contain S in its interior. Integrating $\Delta u\bar{u}$ over the ball, we find

$$
\int_{|P|<R}\Delta u\bar{u}\,dP=-\int_{|P|<R}|\nabla u|^2dP-\int_S\frac{\partial u}{\partial n}\bar{u}\,dq-\int_{|P|=R}\frac{\partial u}{\partial r}\bar{u}\,dq.
$$

As $\Delta u=0$, $\partial u/\partial r\ u=O(R^{-3})$, we obtain

$$\int_{\Omega^+}|\nabla u|^2dP=\int_S\sigma|u|^2dq\leqslant\varepsilon\int_{\Omega^+}|\nabla u|^2dP,$$

if R goes to ∞. Since $\varepsilon<1$ by the condition, ∇u must be zero in Ω^+. Remembering $u=O(|P|^{-1})$, $u\equiv 0$ in Ω, which is impossible.

We give two Lemmas to study the resolvent kernel on the imaginary axis.

Lemma 2.5. i) *Let φ be a Hölder continuous function on S. We set* $u(P)=2\int E(P-q|ik)\varphi(q)dq$ $(k\in R,\ k\neq 0)$. *Then if* $\left(\frac{\partial}{\partial n}+\sigma\right)u=0$ *on S,* $u\equiv 0$ *in* Ω^+.

ii) *Let ψ be a function of class $C^{1+\alpha}(S)$. We set* $v(P)=2\int\frac{\partial}{\partial n_q}E(P-q|ik)\psi(q)dq+2\int E(P-q|ik)\psi(q)\sigma(q)dq$. *Then the same thing as* i) *holds.*

Proof. The proof is reduced to the uniqueness theorem of Rellich (see Mizohata [3]). In our case it is important that σ is real-valued.

Lemma 2.6. *Let ik_0 be a pole on the imaginary axis* $(k_0\neq 0)$, *then k_0^2 is an eigenvalue for $-\Delta$ in the interior Dirichlet problem.*

Proof. This Lemma is essentially the same one as Lemma 2.1.

Lemma 2.7. *Let ik_0 be a pole of R on the imaginary axis, $\{\varphi_j\}$ be the system of linearly independent eigenfuctions for $K(p, q|ik_0)$, $\{\psi_j\}$ be for $K(q, p|ik_0)$ corresponding to ik_0, and we set* $\Phi_j(P)=\int E(P-q\,|\,ik_0)\varphi_j(q)dq$, $\Psi_j(P)=\int\frac{\partial}{\partial n_q}E(P-q|ik_0)\psi_j(q)dq+\int E(P-q|ik_0)\sigma(q)\psi(q)dq=W_j+V_j$. *Then each of $\{\Phi_j\}$, $\{\Psi_j\}$ is a system of linearly independent eigenfunctions for $-\Delta$ in the interior Dirichlet problem.*

Proof. On Φ_j: Φ_j satisfies $\Delta\Phi_j+k_0^2\Phi_j=0$ in Ω^+, $\left(\frac{\partial}{\partial n}+\sigma\right)\Phi_j=0$ on S. By Lemma 2.5. $\Phi_j\equiv 0$ in Ω^+ and especially $\Phi_j\equiv 0$ on S. As Φ_j is continuous in R^3, Φ_j is an eigenfunction for $-\Delta$ in the interior Dirichlet problem. Now if we suppose $\Phi=\sum_j\alpha_j\Phi_j=0$ in Ω^- for some α_j, Φ is identically zero in $\Omega^+\cup\Omega^-$. Then $-2\sum_j\alpha_j\varphi_j(p)=\frac{\partial}{\partial n}\Phi|_+-\frac{\partial}{\partial n}\Phi|_-=0$, which is impossible.

ii) On Ψ_j: Ψ_j satisfies, by Proposition 1.6. $\Delta\Psi_j+k_0^2\Psi_j=0$ in $\Omega^-\Psi_j|_-=0$ on S. We note $\Psi_j|_+=W_j|_++V_j|_+=2\psi_j+W_j|_-+V_j|_-=2\psi_j$, and $\frac{\partial}{\partial n}\Psi_j|_+=\frac{\partial}{\partial n}W_j|_++\frac{\partial}{\partial n}V_j|_+=\frac{\partial}{\partial n}W_j|_+-2\sigma\psi_j+\frac{\partial}{\partial n}V_j|_--\frac{\partial}{\partial n}\Psi_j|_--\sigma\Psi_j|_+$. By the way, we suppose $\Psi=\sum_j\alpha_j\Psi_j=0$ in Ω^- for some α_j. Since $\frac{\partial}{\partial n}\Psi_+=\frac{\partial}{\partial n}\Psi|_+-\sigma\Psi|_+=-\sigma\Psi|_+$, Ψ satisfies $\Delta\Psi+k_0^2\Psi=0$ in $\Omega^+\left(\frac{\partial}{\partial n}+\sigma\right)\Psi=0$ on S, again $\Psi\equiv 0$ in Ω^+ by Lemma 2.5. Therefore $\Psi|_+-\Psi_-|=2\sum_j\alpha_j\psi_j=0$; which is impossible.

3. Construction of the Green's function

In this section we construct the Green's function, and study its properties. To construct the compensating function K, we consider the integral equation

$$(3.1)\qquad -\varphi(p, Q|\lambda)+\int K(p, q|\lambda)\varphi(q, Q)dq=\frac{\partial}{\partial n_p}E(p-Q|\lambda)+\sigma(p)E(p-Q|\lambda)$$

According to previous considerations, (3.1) is solved uniquely as the following, when $\mathrm{Re}\,\lambda>0$, $\mathrm{Im}\,\lambda\neq 0$.

$\varphi(p, Q)=-\frac{\partial}{\partial n_p}E(p-Q|\lambda)+\sigma(p)E(p-Q|\lambda)-2\int R(p, r|\lambda)\left\{\frac{\partial}{\partial n_r}E(r-Q|\lambda)+\sigma(r)E(r-Q|\lambda)\right\}dr$. Thus the compensating function $K(P, Q|\lambda)$ is expressed as $K(P, Q|\lambda)=2\int E(P-q|\lambda)\varphi(q, Q)dq$. Then we have

Proposition 3.1. *When* $\mathrm{Re}\,\lambda>0$, $\mathrm{Im}\,\lambda\neq 0$, *the Green's function of* (1.1) *is expressed as*

$$(3.2)\quad G(P, Q|\lambda)=-E(P-Q|\lambda)-2\int E(P-q|\lambda)\left\{\frac{\partial}{\partial n_q}E(q-Q|\lambda)+\sigma(q)E(q-Q|\lambda)\right\}dq$$

$$+4\iint E(P-q|\lambda)R(q, r|\lambda)\left\{\frac{\partial}{\partial n_r}E(r-Q|\lambda)+\sigma(r)E(r-Q|\lambda)\right\}drdq.$$

And $G(P, Q|\lambda)$ *has the meromorphic continuation in* λ *to the whole complex plane.*

Proposition 3.2. *We denote* $\tilde{G}(P, Q|\lambda)=G(Q, P|\lambda)$ *for* $P, Q\in\Omega^-$. *Then* G *is the Green's function for* $\Delta-\lambda^2$ *with Dirichlet boundary condition in the interior domain.*

Proof. Changing the order of integration in (3.2), we find the proposition a consequence of Proposition 1.6.

From now on, we carry out the same discussions as in Mizohata [3].

Proposition 3.3. *Let* ik_0 *be a pole of* $R(p, q|\lambda)$ *on the imaginary axis. Then the pole is simple.*

Proposition 3.4. $G(P, Q|\lambda)$ *is holomorphic in a neighbourhood of the imaginary axis without the origin.*

To prove Proposition 3.3. we need the fact the Green's operator for the interior Dirichlet problem for $\Delta-\lambda^2$ has a simple pole at every eigenvalue for $-\Delta$. We substitute to (3.2) the Laurent expansion of the resolvent kernel at a pole on the imaginary axis. Then we obtain Proposition 3.4 easily.

We summarize what we have obtained.

Theorem 3.5. *Let* S *be a compact* $C^{2+\alpha}$ *surface in* R^3. *Then the Green's function for* (1.1) *is meromorphic with respect to the parameter* λ *in the whole complex plane. Especially, if the obstacle is star-shaped, and the coefficient* σ *satisfies* $\sigma(p)<\frac{(n, \overrightarrow{op})}{|\overrightarrow{op}|^2}$, $G(P, Q|\lambda)$ *is holomorphic in a neighbourhood of* $\{\lambda|$

$\mathrm{Re}\,\lambda\geq 0\}$.

Remark. i) The spectral function for $-\Delta$ with the boundary condition of the third kind in the exterior domain is real-analytic, if the conditions in Theorem 3.5. are satisfied.

ii) By virtue of i), the solution of the following problem for the wave equation decays when t goes to ∞.

$$\begin{cases} u_{tt}-\Delta u=0 & \text{in} \quad (0,\infty)\times\Omega^+ \\ u=f_1,\ u_t=f_2 & \text{on} \quad \{t=0\}\times\Omega^+ \qquad f_1, f_2\in C_0^\infty(\Omega^+) \\ \left(\frac{\partial}{\partial n}+\sigma\right)u=0 & \text{on} \quad (0,\infty)\times S \end{cases}$$

iii) When the surface is a sphere with radius ρ, the condition on σ becomes $\sigma(p)<\frac{1}{\rho}$ if we choose the origin as O. This coinsides the Tokita's condition for exponential decay of the solution.

4. An example

In this section we are going to construct the solution of an exterior problem for the wave equation that never decays. The construction is based on Tokita's paper [5].

We consider the following problem.

$$(4.1)\quad \begin{cases} u_{tt}(x,t)-\Delta u(x,t)=\Box u(x,t)=0 & \text{in} \quad \Omega\times(0,\infty) \quad \Omega=\{x\mid |x|>\rho\} \\ u(x,0)=f_0(x),\ u_t(x,0)=f_1(x) & \text{on} \quad \Omega\times\{t=0\} \\ \left(\frac{\partial}{\partial r}+\sigma\right)u(x,t)=0,\ \sigma \text{ is const.} & \text{on} \quad \{x\mid |x|=\rho\}\times(0,\infty) \end{cases}$$

As we assume the compatibility condition of infinite order, we have only to consider the following.

$$(4.2)\quad \begin{cases} \Box u(x,t)=0 & \text{in} \quad \{x\mid |x|>\rho\}\times(0,\infty) \\ u(x,0)=u_t(x,0)=0 & \text{on} \quad \{x\mid |x|>\rho\}\times\{t=0\} \\ \left(\frac{\partial}{\partial r}+\sigma\right)u(x,t)=g(x,t) & \text{on} \quad \{x\mid |x|=\rho\}\times(0,\infty) \\ \operatorname{supp} u\subset\{x\mid |x|>\rho\}\times\{t\geqslant 0\},\ \operatorname{supp} g\subset\{x\mid |x|=\rho\}\times\{0\leqslant t\leqslant T\} \end{cases}$$

We solve (4.2) by the method of separation of variables. We set formally $u(x,t)=\sum_n\sum_m u_{nm}(r,\rho,t)Y_{nm}(\omega)$ and $f(x,t)=\sum_n\sum_m f_{nm}(t,\rho)Y_{nm}(\omega)$, where Y_{nm} is the spherical harmonics of order n. Then $u_{nm}(r,\rho,t)=\sum_{\lambda_n^{(s)}}\int_0^{t-r+\rho}e^{\lambda_n^{(s)}(t-\tau)}f_{nm}(\tau,\rho)d\tau\times \operatorname{Res}_{\lambda_n^{(s)}}\Phi_n(r,\rho|\chi)$, $\Phi_n=\dfrac{\rho(\lambda r)^{-1/2}K_{n+1/2}(\lambda r)}{(\lambda\rho)^{1/2}K'_{n+1/2}(\lambda\rho)+(\rho\sigma-1/2)(\lambda\rho)K_{n+1/2}(\lambda\rho)}$. Here $K_{n+1/2}(z)$

is the modified Bessel function of order $n+1/2$ and $\lambda_n^{(s)}$ are the poles of Φ_n in λ. For the next Lemma, we note $\Phi_0=-\rho^2e^{-\lambda(r-\rho)}(\lambda-\sigma+1/\rho)^{-1}r^{-1}$. We can see easily the formal solution converges uniformly in any compact set, if $f(x,t)$ is sufficiently smooth. On the location of the poles of Φ_n in λ we can say

Lemma 4.1. i) *When $\sigma\leqslant 1/\rho$, the poles of Φ_n are located only in* $\mathrm{Re}\,\lambda<0$ *or at* 0.

ii) *When $\sigma>0$, all the poles of Φ_n are simple.*

iii) 0 *is not a pole of Φ_n $(n\geqslant 1)$.*

iv) *When $\sigma=1/\rho$, 0 is the pole of Φ_0.*

Proof. i), ii), and iii) are essentially proved in Tokita [5]. iv) is obvious, to see the form of Φ_0.

Next Lemma shows the distribution of poles of Φ_n for large n.

Lemma 4.2. *Let $\lambda_n^{(s)}(1\leqslant s\leqslant n+1)$ be the poles of Φ_n, then there exist a positive integer n_0 and positive numbers A, B, such that*

i) $\mathrm{Re}\,\lambda_n^{(s)}<-An^{1/3}$ *for* $n\geqslant n_0$ $1\leqslant s\leqslant n+1$

ii) $|\lambda_n^{(s)}|\leqslant Bn$ *for* $n\geqslant n_0$ $1\leqslant s\leqslant n+1$

The proof of this lemma is found in Tokita's paper, where the profound results on the distribution of zeros of Bessel functions obtained by Olver [4] are needed. From now on, we confine ourselves to the case $\sigma=1/\rho$. We separate the first term of the formal solution from the others. Then we have a proposition.

Proposition 4.3.

$$u(x,t)=-(4\pi r)^{-1}\rho\int_0^{t-r+\rho}f_0(\rho,\tau)\,d\tau+O(e^{-\mu t})$$

The estimate of the remainder terms is valid uniformly with respect to x in any compact set.

Proof. Since $u(x,t)=-\rho(4\pi r)^{-1}\int_0^{t-r+\rho}f_0(\rho,\tau)\,d\tau+\sum_n\sum_m u_{nm}Y_{nm}$, we apply Tokita's argument showing the decay of the solution only to the second terms of the expansion.

Now we can construct the solution of an exterior problem that never decays.

Proposition 4.4. *Consider the problem*

$$\begin{cases}\Box u(x,t)=0 & \text{in}\quad \{x \mid |x|>\rho\}\times(0,\infty)\\ u(x,0)=0,\ u_t(x,0)=f(x) & \text{on}\quad t=0,\quad \text{supp } f \text{ is compact}\\ \left(\dfrac{\partial}{\partial r}+\dfrac{1}{\rho}\right)u(x,t)=0 & \text{on}\quad \{x \mid |x|=\rho\}\times(0,\infty)\end{cases}$$

Then either the solution, or local energy of the solution never decays if f is *suitably chosen.*

Proof. First we remark $g(\rho\omega, t)=-(4\pi)^{-1}\left(\frac{\partial}{\partial r}+\frac{1}{\rho}\right)\int_{|\theta|=1} f(r\omega+t\theta)d\theta$. We choose a spherically symmetric function f such that $\operatorname{supp} f \subset \{x|\rho+a<|x|<\rho+b\}$. Then $W(x, t)=(4\pi)^{-1}\int_{|\theta|=1} f(x+t\theta)d\theta$ and $g(\rho\omega, t)$ are also spherically symmetric. We set $U(r)=\int_0^T W(r, t)dt$ for large T fixed. Then

$$\Delta U=\frac{\partial W}{\partial t}(r, T)-f(r) \tag{4.3}$$

By virtue of Huyghens' principle, we see $U\equiv 0$ for $|x|\geqslant R$, for some $R>0$. Integrating the both sides of (4.3), we see $\int_{|x|\geqslant\rho} \Delta U dx=-\int_{|x|=\rho}\frac{\partial U}{\partial r}ds=-4\pi\rho^2\frac{\partial U}{\partial r}(\rho)=\int_{|x|\geqslant\rho}\left(\frac{\partial W}{\partial t}-f\right)dx$. Since $W(x, T)\equiv 0$ $|x|\leqslant 2\rho$ for large T, we have proved $-4\pi\rho^2\frac{\partial U}{\partial r}(\rho)=\int_{R^3}\frac{\partial W}{\partial t}(x, T)dx-\int_{R^3} f(r)dx$. If we caluculate $\frac{\partial W}{\partial t}$, we see easily $\int_{R^3}\frac{\partial W}{\partial t}dx=\int_{R^3} f(r)dx$. And therefore $\frac{\partial U}{\partial r}(\rho)=0$. As we can take beforehand large T such that $\int_0^{t-r+\rho} W(r, \tau)d\tau|_{r=\rho}=\int_0^T W(r, \tau)d\tau|_{r=\rho}$ for $t-r+\rho\geqslant T$,

$$\begin{aligned}\int_0^{t-r+\rho} g(\rho, \tau)d\tau &= -\left(\frac{\partial}{\partial r}+\frac{1}{\rho}\right)\int_0^T W(r, \tau)d\tau\,|_{r=\rho} \\ &= -\left(\frac{\partial}{\partial r}+\frac{1}{\rho}\right)U(r)\,|_{r=\rho} \\ &= -\rho^{-1}U(\rho).\end{aligned}$$

So, if $f>0$, $U(\rho)=(4\pi)^{-1}\int_0^T\tau\int_{|\theta|=1} f(\rho\omega+t\theta)d\theta d\tau$ is positive. Then $\lim_{t\to\infty} u(x, t)=-\rho r^{-1}\int_0^T g(\rho, \tau)d\tau$, which indicates the solution never decays. Looking closely at the expression, we find the local energy neither decays.

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