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On a characterization of finite groups of p-rank 1

By

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Dedicated to Professor A. Komatu on his 70th birthday

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§1. Introduction and statement of the results

Let G be a finite group. Let p be a prime number. Define the p-rank $r_p(G)$ of G by the maximal integer k such that G contains the elementary abelian p-group $(Z_p)^k$ of rank k.

It is obvious that G is of p-rank 0 if and only if the p-Sylow subgroup $G_{(p)}=e$. According to Cartan-Eilenberg [5], we see that G is of p-rank 1 if and only if $G_{(p)}$ is either a cyclic group Z_{p^r} or a generalized quoternionic group if $p=2$. It is also shown [5] that a finite group G with p-rank 0 or 1 for any p is characterized by having the periodic cohomology. Such a group is called an Artin-Tate group.

Now the purpose of the present note is to give a characterization of finite groups of p-rank 1 in terms of stable homotopy groups.

Let $|G|$ be the order of G and let Σ_n denote the symmetric group on n letters. We denote by $\rho=\rho_G\colon G\to\Sigma_{|G|}$ the regular permutation representation, and $B\rho\colon BG\to B\Sigma_{|G|}$ denotes the induced map on classifying spaces. Let

$$\omega\colon \coprod_n B\Sigma_n \longrightarrow \Omega B(\coprod_n B\Sigma_n)\simeq Q(S^0)$$

be the Barratt-Priddy-Quillen map [3], where $Q(S^0)=\lim_k \Omega^k S^k$. Then as the adjoint of the composition

$$BG_+ \xrightarrow{B\rho_+} B\Sigma_{|G|+} \subset \coprod_n B\Sigma_n \xrightarrow{\omega} Q(S^0)$$

we obtain a stable map of spectra

$$f\colon \mathbf{S}(BG_+) \longrightarrow \mathbf{S}$$

where $BG_+=BG\cup$ disjoint base point. Then we obtain a homomorphism

$$\phi=\phi_G\colon \pi_n^S(BG_+) \longrightarrow \pi_n^S(S^0)$$

of stable homotopy groups. Note that $\pi_n^S(BG_+)\cong\pi_n^S(BG)\oplus\pi_n^S(S^0)$, direct sum. The restriction $\phi|_{\pi_n^S(BG)}$ is also denoted by ϕ.

Now let $J\colon \pi_n(O)\to\pi_n^S(S^0)$ denote the J-homomorphism, where $O=\lim O(n)$.

Restricting J: $\pi_n(O)\to\pi_n^S(S^0)$ on $\pi_n(U)$ or $\pi_n(S_p)$, we obtain the complex J-homomorphism J_C or the quoternionic J-homomorphism J_H.

For a finite abelian group A, we denote by $A_{(p)}$ the p-component of A. Then we can state our theorems.

Theorem 1.1. *Let G be a finite group of p-rank* 1. *If p is odd, then*

$$\mathrm{Im}\,[\phi\colon \pi_*^S(BG) \longrightarrow \pi_*^S(S^0)] \supset (\mathrm{Im}\,J)_{(p)} = (\mathrm{Im}\,J_C)_{(p)}.$$

If $p=2$, then

$$\mathrm{Im}\,[\phi\colon \pi_*^S(BG) \longrightarrow \pi_*^S(S^0)] \supset (\mathrm{Im}\,J_H)_{(2)}.$$

Theorem 1.2. *Let G be a finite group. Then the p-rank of G is equal to* 1 *if and only if* ϕ: $\pi_{2p-3}^S(BG)_{(p)}\to\pi_{2p-3}^S(S^0)_{(p)}$ (ϕ: $\pi_3^S(BG)_{(2)}\to\pi_3^S(S^0)_{(2)}$ *if* $p=2$) *is an epimorphism.*

Concerning with the 2-component, it may be worth showing the following

Proposition 1.3. ϕ: $\pi_1^S(BG)\to\pi_1^S(S^0)$ *is an epimorphism if and only if the* 2-*Sylow subgroup* $G_{(2)}$ *is a non trivial cyclic group.*

From this proposition it follows immediately that if $G_{(2)}$ is non trivial cyclic, then G is not perfect, hence not simple unless $G=Z_2$ (Burnside's theorem).

If one uses the Feit-Thompson theorem [6], one can show the following

Corollary 1.4. *Let G be an Artin-Tate group. Suppose that* $H_i(G\colon \mathbf{Z})=0$, $1\le i\le 3$, then G is trivial.

Proof. By the assumption, $\pi_3^S(BG)=0$. Hence by Theorem 1.2, we see that $G_{(2)}=e$, i.e., G is of odd order. Then by the Feit-Thompson theorem, G is solvable. Then $H_1(G\colon \mathbf{Z})=0$ implies $G=e$. q.e.d.

Now for a finite group G of p-rank 1, Theorem 1.1 shows the non-triviality of $\pi_{2p-3}^S(BG)_{(p)}$ ($\pi_3^S(BG)_{(2)}$ if $p=2$). We remark that such a non-triviality of $\pi_i^S(BG)_{(p)}$ for $i<2p-3$ does not hold as the following examples show. If p is odd, then Σ_p is of p-rank 1. It is known [10] that $H_i(B\Sigma_p\colon Z_p)=0$ for $i<2p-3$. Then by Serre's class theory, $\pi_i^S(B\Sigma_p)_{(p)}=0$ if $i<2p-3$. For $p=2$, consider the binary icosahedral group I^*. This is a subgroup of order 120 of $Sp(1)=S^3$. Hence I^* is an Artin-Tate group and $I^*_{(2)}$ is the quoternionic group. It is well-known [16] that $H_1(BI^*)=H_2(BI^*)=0$. Hence $\pi_i^S(BI^*)=0$ for $i\le 2$.

The non-triviality of $\pi_{2p-3}^S(BG)_{(p)}(\pi_3^S(BG)_{(2)})$ clearly fails for general finite groups as the following Quillen's example shows. Let $\mathbf{F}_q$ be the finite field with $q=p^d$ elements. Then Quillen has shown [11] that $H^i(BGL(n, \mathbf{F}_q)\colon Z_p)=0$ for $0<i<d(p-1)$. Thus $\pi_i^S(BGL(n, \mathbf{F}_q))_{(p)}=0$ for $i<d(p-1)$.

For a cyclic group Z_p of prime order, Theorem 1.1 is a direct consequence of the Kahn-Priddy theorem [7], that is ϕ: $\pi_*^S(BZ_p)\to\pi_*^S(S^0)_{(p)}$ is an epimorphism ($*>0$). We shall show that the Kahn-Priddy theorem fails for cyclic group of order 2^r, $r\ge 2$.

Theorem 1.5. *Let r be an integer ≥ 2. Let f: $\mathbf{S}BZ_{2^r} \to \mathbf{S}$ be an arbitrary stable map. Then f_*: $\pi_7^S(BZ_{2^r}) \to \pi_7^S(S^0)_{(2)}$ is not epimorphism.*

For an odd prime, the problem seems to be more difficult. For example, a direct computation shows that the element $\beta_1 \in \pi_{2p(p-1)-2}^S(S^0)_{(p)}$ is in the image of ϕ: $\pi_*^S(BZ_{p^r}) \to \pi_*^S(S^0)$ for any r.

The proof of Theorem 1.1, 1.2 and Proposition 1.3 will be given in §3, and that of Theorem 1.5 in §4.

§2. Factorization of the J-homomorphism

In this section, we review some results of Becker and Schultz [4]. For odd primary component, similar results can be obtained based on the algebraic K-theory which will be explained in the Appendix.

Let G be a finite group. Suppose that G has a free orthogonal representation W (G acts freely on the unit sphere $S(W)$). Let $kW = W \oplus \cdots \oplus W$, the direct sum of k copies of W. Set

$$F_G = \lim_k M(S(kW), S(kW)),$$

where $M(\ ,\ )$ denotes the set of G-maps with compact open topology.

For a topological space X, let $Q(X) = \lim_k \Omega^k S^k X$. Then one has

Theorem 2.1. ([4]). *There is a homotopy equivalence*

$$\lambda = \lambda_G\colon F_G \longrightarrow Q(BG_+).$$

For the naturality of λ_G with respect to G, one can see the following. Let H be a subgroup of G. Then there is a map

$$\psi\colon F_G \longrightarrow F_H$$

by regarding G-maps as H-maps. Next consider the finite covering map $BH \to BG$. According to Kahn and Priddy [7], one can associate to the finite covering a map

$$\tau\colon Q(BG_+) \longrightarrow Q(BH_+)$$

called the transfer map for $BH \to BG$. Then we have ([4], 6.10).

Proposition 2.2. *The following diagram is homotopy commutative*

$$\begin{array}{ccc} F_G & \xrightarrow{\lambda=\lambda_G} & Q(BG_+) \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\tau} \\ F_H & \xrightarrow[\lambda=\lambda_H]{} & Q(BH_+)\,. \end{array}$$

For the homotopy functor $[\ , F_G]$ and $[\ , Q(BG_+)]$, induced maps of maps in the above diagram are denoted by the same letter, e.g.,

$$\lambda\colon [\ , F_G] \longrightarrow [\ , Q(BG_+)]\,.$$

Now let us recall the definition of the transfer of Kahn and Priddy [7]. Let $k=[G, H]$. The left coset G/H is a G-set of order k with the standard left G-action. Let $g_1,\dots, g_k$ be a representatives of the coset G/H. Then the G-action on G/H determines a homomorphism

$$\gamma\colon G \longrightarrow \Sigma_k$$

by the formula $gg_i=g_{\gamma(g)(i)}h_i$, $h_i\in H$. Define a homomorphism

$$\mu\colon G \longrightarrow \Sigma_k \int H.$$

by $\mu(g)=(\gamma(g); h_1,\dots, h_k)$, where $\Sigma_k\int H$ denotes the wreath product. Let

$$B\mu\colon BG \longrightarrow B\left(\Sigma_k \int H\right)$$

be the induced map on classifying spaces. Note that $B\left(\Sigma_k\int H\right)$ is identified with $E\Sigma_k \otimes_{\Sigma_k}(BH)^k$, where $E\Sigma_k$ is a universal Σ_k-space and Σ_k acts on $(BH)^k$ by permutations of factors. It is known (see, e.g., [9]) that there is a canonical map

$$\omega\colon \coprod_n E\Sigma_n \times_{\Sigma_n}(BH)^n \longrightarrow Q(BH_+).$$

Then the composition

$$BG_+ \xrightarrow{B\mu} (E\Sigma_k \times_{\Sigma_k}(BH)^k)_+ \subset \coprod_n E\Sigma_n \times_{\Sigma_n}(BH)^n \xrightarrow{\omega} Q(BH_+)$$

extends to the transfer map.

$$\tau\colon Q(BG_+) \longrightarrow Q(BH_+).$$

using the natural transformation $QQ\to Q$ (see [7]).

The following lemma about the functoriality follows from the definition and a straightforward argument.

Lemma 2.3. *Let $H\supset K$ be subgroups of G. Then the following diagram is homotopy commutative*

$$\begin{tikzcd} Q(BG_+) \arrow[rr, "\tau"] \arrow[dr, "\tau"'] & & Q(BH_+) \arrow[dl, "\tau"] \\ & Q(BK_+) & \end{tikzcd}$$

Suppose now that $H=e$. Then τ is clearly the extension of

$$BG_+ \xrightarrow{B\rho} B\Sigma_{|G|} \xrightarrow{\omega} Q(S^0).$$

hence $\tau_*\colon \pi_i(Q(BG_+))\cong\pi_i^s(BG_+)\to\pi_i(Q(S^0))\cong\pi_i^s(S^0)$ is just the homomorphism ϕ defined in §1.

Consider the homomorphism ϕ: $\pi_*^S(BG_+) \cong \pi_*^S(BG) \oplus \pi_*^S(S^0) \to \pi_*^S(S^0)$. Then we have

Lemma 2.4. $\phi|_{\pi_*^S(S^0)}$: $\pi_*^S(S^0) \to \pi_*^S(S^0)$ *is the multiplication with* $|G|$.

Proof. $\phi|_{\pi_*^S(S^0)}$ is induced from the adjoint of the composition map

$$g: S^0 \xrightarrow{i} BG_+ \longrightarrow B\Sigma_{|G|+} \longrightarrow Q(S^0)$$

where i: $S^0 \to BG_+$ maps the non base point of S^0 into BG. Hence g maps that point into the component of degree $|G|$ maps of $Q(S^0)$. Hence $Ad(g)$: $\mathbf{S} \to \mathbf{S}$ is a map of degree $|G|$. This shows the lemma. q.e.d.

Let $K = \mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$ be the field of real, complex or quoternionic numbers, respectively. Let $O_K(n)$ denote $O(n)$, $U(n)$ or $Sp(n)$ according to $K = \mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$, respectively. We let $O_K(1)$ $(= Z_2, S^1$ or $S^3)$ act on K^n as the scalar multiplication. Then any element $f \in O_K(n)$ gives an $O_K(1)$-equivariant map f: $S(K^n) \to S(K^n)$. Let G be a finite subgroup of $O_K(1)$. Then we obtain a map

$$j_G: O_K(\infty) \longrightarrow F_G.$$

By Theorem 2.1, there is an isomorphism λ: $[\ , F_G] \cong [\ , Q(BG_+)]$. Hence j_G induces a map

$$j_G: [\ , O_K(\infty)] \longrightarrow [\ , Q(BG_+)].$$

It is obvious that if $G = e$, then

$$j_e: [\ , O_K(\infty)] \longrightarrow [\ , Q(S^0)]$$

agrees with the (K-)J-homomorphism.

If $G \supset H$, then we see easily that

$$\psi j_G = j_H: O_K(\infty) \longrightarrow F_H$$

Then setting $H = e$, we have obtained

Proposition 2.5. *The following diagram is commutative.*

$$\begin{tikzcd}
\pi_n(O_K(\infty)) \arrow[rr, "j_G"] \arrow[dr, "J_K"'] & & \pi_n^S(BG_+) \arrow[dl, "\phi"] \\
& \pi_n^S(S^0) &
\end{tikzcd}$$

§3. Finite groups of p-rank 1

Proposition 3.1. *Let* p *be a prime number and* $a \geq 1$ *an integer. Then*

$$\operatorname{Im}[\phi: \pi_*^S(BZ_{p^a}) \longrightarrow \pi_*^S(S^0)] \supset (\operatorname{Im} J_{\mathbf{C}})_{(p)}.$$

Proof. Since $Z_{p^a}\subset S^1$, one can apply Proposition 2.5 for $K=\mathbf{C}$. Then one see that

$$\mathrm{Im}\,[\phi\colon \pi_*^S(BZ_{p^a+})\longrightarrow \pi_*^S(S^0)]\supset \mathrm{Im}\,J_{\mathbf{C}}.$$

By Lemma 2.4, $\phi|_{\pi_*^S(S^0)}(x)=p^a(x)$. Then clearly

$$\mathrm{Im}\,\phi\,|_{\pi_*^S(BZ_{p^a})}\supset(\mathrm{Im}\,J_{\mathbf{C}})_{(p)}.$$

q.e.d.

Let $Q(2^a)$ denote the generalized quoternionic group of order 2^{a+2}. Then $Q(2^a)\subset S^3$ and we have similarly

Proposition 3.2. $\mathrm{Im}\,[\phi\colon \pi_*^S(BQ(2^a))\to\pi_*^S(S^0)]\supset(\mathrm{Im}\,J_{\mathbf{H}})_{(2)}$.

Now we prove Theorem 1.1, 1.2 and Proposition 1.3.

Proof of Theorem 1.1. Let G be of p-rank 1. Let $i\colon G_{(p)}\to G$ be the inclusion of p-Sylow subgroup. Then the composition homomorphism

$$\pi_*^S(BG_{(p)+})\xrightarrow{Bi_*}\pi_*^S(BG_+)\xrightarrow{\phi_G}\pi_*^S(S^0)$$

is induced from adjoint map of the composite

$$BG_{(p)+}\xrightarrow{Bi}BG_+\xrightarrow{B\rho}B\Sigma_{|G|+}\xrightarrow{\omega}Q(S^0)\,.$$

Note that the restriction of the regular permutation of G on H is a direct sum of that of H. Thus we have a commutative diagram

$$\begin{array}{ccl} G_{(p)} & \xrightarrow{\rho\times\cdots\times\rho} & \Sigma_{|G_{(p)}|}\times\cdots\times\Sigma_{|G_{(p)}|}\ ([G\colon G_{(p)}]\text{ times}) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle\oplus} \\ G & \xrightarrow[\rho]{} & \Sigma_{|G|} \end{array}$$

where $\oplus$ is the homomorphism defined by the juxtaposition. Thus we have

$$\phi_G(Bi)_*=[G\colon G_{(p)}]\phi_{G_{(p)}}$$

Since G is of p-rank 1, $G_{(p)}$ is a cyclic group if p is odd, and a cyclic group or a generalized quoternionic group if $p=2$. Then since $[G\colon G_{(p)}]$ is prime to p, the theorem follows from Proposition 3.1 and 3.2.

Proof of Theorem 1.2. Note that $\mathrm{Im}\,J=\mathrm{Im}\,J_{\mathbf{C}}=\mathrm{Im}\,J_{\mathbf{H}}$ in $\pi_3^S(S^0)$, for canonical homomorphisms $\pi_3(Sp)\to\pi_3(U)\to\pi_3(O)$ are isomorphisms. Therefore the only if part of the theorem follows from Theorem 1.1.

We now prove the if part. Suppose that the p-rank of G is greater than 1. Then G contains a subgroup $H=Z_p\times Z_p$. Applying Lemma 2.3 for $G\supset H\supset e$, we obtain a commutative diagram

$$\begin{tikzcd} \pi^S_*(BG_+) \arrow[rr, "\tau_*"] \arrow[dr, "\phi"'] & & \pi^S_*(BH_+) \arrow[dl, "\phi"] \\ & \pi^S_*(S^0) & \end{tikzcd}$$

where τ_* is the transfer homomorphism induced from τ: $Q(BG_+)\to Q(BH_+)$.

Suppose first that p is odd. To prove the theorem, we have to show that ϕ: $\pi^S_{2p-3}(BG)\to\pi^S_{2p-3}(S^0)_{(p)}$ is not epimorphic. Assume contrary that ϕ is an epimorphism. Then by the above diagram,

$$\phi\colon \pi^S_{2p-3}(B(Z_p\times Z_p)) \longrightarrow \pi^S_{2p-3}(S^0)_{(p)}$$

is an epimorphism. Note that the regular permutation representation of $Z_p\otimes Z_p$ can be given by the composite

$$Z_p\times Z_p \xrightarrow{\rho\times\rho} \Sigma_p\times\Sigma_p \xrightarrow{\otimes} \Sigma_{p^2}$$

where ρ is the regular permutation representation of Z_p and $\otimes$ is defined by the standard action of $\Sigma_p\times\Sigma_p$ on $\{1,\dots, p\}\times\{1,\dots, p\}\cong\{1,\dots, p^2\}$. Let γ: $\mathbf{S}(B(\Sigma_p\otimes \Sigma_p)_+)\to\mathbf{S}$ be the stable map adjoint to

$$B(\Sigma_p\times\Sigma_p)_+ \xrightarrow{B\otimes} B\Sigma_{p^2+} \xrightarrow{\omega} Q(S^0)$$

Then one has easily the following commutative diagram

$$\begin{tikzcd} \pi^S_{2p-3}(B(Z_p\times Z_p)_+) \arrow[rr, "B(\rho\times\rho)_*"] \arrow[dr, "\phi"'] & & \pi^S_{2p-3}(B(\Sigma_p\times\Sigma_p)_+) \arrow[dl, "\gamma_*"] \\ & \pi^S_{2p-3}(S^0) & \end{tikzcd}$$

Note that $\gamma_*|_{\pi^S_{2p-3}(S^0)}(x)=px^2$ for $x\in\pi^S_{2p-3}(S^0)\subset\pi^S_{2p-3}(B(\Sigma_p\times\Sigma_p)_+)$. It is known [10] that $H_*(B\Sigma_p\colon \mathbf{Z}_p)=0$ for $*<2p-3$, and $H_{2p-3}(B\Sigma_p)_{(p)}=Z_p$. Hence one see easily that $\pi^S_{2p-3}(B\Sigma_p\times B\Sigma_p)_{(p)}\cong Z_p\oplus Z_p$ generated by $j_{1*}(u)$ and $j_{2*}(u)$, where j_i: $B\Sigma_p\to B\Sigma_p\times B\Sigma_p$ is the canonical inclusion, $i=1, 2$, and $u\in\pi^S_{2p-3}(B\Sigma_p)$ is a generator. Let d: $\Sigma_p\to\prod^p\Sigma_p$ be the diagonal map. Then for $i=1, 2$, the following diagram is commutative up to some inner automorphism of Σ_{p^2}.

$$\begin{tikzcd} \Sigma_p \arrow[r, "j_i"] \arrow[d, "d"'] & \Sigma_p\times\Sigma_p \arrow[d] \\ \prod^p\Sigma_p \arrow[r, "\oplus"'] & \Sigma_{p^2} \end{tikzcd}$$

Remark that an inner automorphism induces the identity on stable homotopy groups. Then we see that

$$\gamma_*j_{i*}=\mu_*d_*\colon \pi^S_{2p-3}(B\Sigma_{p+}) \longrightarrow \pi^S_{2p-3}(S^0)\,.$$

where μ: $\mathbf{S}(B(\prod^p \Sigma_p)_+)\to\mathbf{S}$ is the adjoint of $B(\prod^p \Sigma_p)_+)\xrightarrow{B\oplus} B\Sigma_{p^2+}\xrightarrow{\omega} Q(S^0)$. Now it is easy to see that $\mu_* d_*(x)=p\phi(x)$ for $x\in\pi_*^S(B\Sigma_{p+})$. Hence $\gamma_* j_{i*}=0$ in $\pi_{2p-3}^S(B\Sigma_{p+})$ and this contradicts to the assumption. Hence ϕ: $\pi_{2p-3}^S(BG)\to\pi_{2p-3}^S(S^0)$ is not epimorphism.

Next suppose that $p=2$. We have an isomorphism

$$\pi_3^S(BZ_2\times BZ_2)\cong\pi_3^S(BZ_2)\oplus\pi_3^S(BZ_2)\oplus\pi_3^S(BZ_2\wedge BZ_2)$$

By the homomorphism ϕ: $\pi_3^S(BZ_2\times BZ_2)\to\pi_3^S(S^0)$, the first and the second summands are mapped onto $2\pi_3^S(S^0)_{(2)}$ by the same reason as for p odd. Since $\pi_3^S(S^0)_{(2)}\cong Z_8$, it suffices to show that $\pi_3^S(BZ_2\wedge BZ_2)$ contains no element of order 8.

Let $M=S^1\cup{}_2e^2$ be the Moore space mod 2. Then $\pi_3^S(BZ_2\wedge BZ_2)\to\pi_3^S(M\wedge M)$, for the 3-skeleton of $\mathbf{S}BZ_2$ is $\mathbf{S}M\vee\mathbf{S}S^3$. But it is easy to see that $\pi_3^S(M\wedge M)\cong Z_4$. This completes the proof. q.e.d.

Proof of Proposition 1.3. Let ε: $\Sigma_n\to Z_2$ be the sign homomorphism. We easily see that $H_1(B\Sigma_n\colon \mathbf{Z}_2)\cong Z_2$, and ε^*: $H_1(B\Sigma_n\colon \mathbf{Z}_2)\to H_1(BZ_2\colon \mathbf{Z}_2)$ is an isomorphism for $n\geq 1$. Let G be a finite 2-group. Then one can easily see that $\varepsilon\rho$: $G\to Z_2$ is an epimorphism if and only if G is a cyclic group, where ρ: $G\to\Sigma_{|G|}$ is the regular permutation representation. For if G is not cyclic, for any $g\in G$, the restriction $\varepsilon\rho|_{<g>}$ on the subgroup generated by g is trivial. Therefore we see that

$$\rho^*\colon H^1(B\Sigma_{2^a}\colon \mathbf{Z}_2)\longrightarrow H^1(BZ_{2^a}\colon \mathbf{Z}_2)$$

is an isomorphism. Hence

$$\rho_*\colon H_1(BZ_{2^a})\longrightarrow H_1(B\Sigma_{2^a})\cong H_1(QS^0)$$

is an epimorphism. This implies

$$\phi\colon \pi_1^S(BZ_{2^a})\longrightarrow \pi_1^S(S^0)\cong Z_2$$

is an epimorphism. Then for a finite group G with $G_{(2)}\cong Z_{2^a}$ $(a>0)$, we see that

$$\phi\colon \pi_1^S(BG)\longrightarrow \pi_1^S(S^0)$$

is an epimorphism as in the proof of Theorem 1.1.

Next let G be a finite group such that $G_{(2)}$ is not cyclic. Then

$$\rho_*\colon H_1(BG_{(2)})\longrightarrow H_1(B\Sigma_{2^a})$$

is trivial, and hence

$$\phi\colon \pi_1^S(BG_{(2)})\longrightarrow \pi_1^S(S^0)$$

is trivial. Then so is ϕ: $\pi_1^S(BG)\to\pi_1^S(S^0)$. q.e.d.

§4. Proof of Theorem 1.5

Let f: $\mathbf{S}BZ_{2^a}\to\mathbf{S}$, $a\geq 2$ be a stable map. Let $\sigma\in\pi_7^S(S^0)$ be the element of the

Hopf invariant one. Suppose that there is an element $u\in\pi_7^S(BZ_{2^a})$ such that $f_*(u)=\sigma$. Then since σ is Hopf invariant one, we see easily that

$$u_*\colon H_7(S^7\colon \mathbf{Z}_2)\longrightarrow H_7(BZ_{2^a}\colon \mathbf{Z}_2)$$

is essential. Let $L^n(2^a)=S^{2n+1}/Z_{2^a}$ be the standard lens space mod 2^a. Then $L^n(2^a)$ is the $2n+1$ skeleton of BZ_{2^a}. The stable map $u\colon \mathbf{S}(S^7)\to \mathbf{S}BZ_{2^a}$ is then factored through a stable map

$$u'\colon \mathbf{S}(S^7)\longrightarrow \mathbf{S}L^3(2^a)$$

such that

$$u'_*\colon H_7(S^7\colon \mathbf{Z}_2)\longrightarrow H_7(L^3(2^a)\colon \mathbf{Z}_2)$$

is an isomorphism. Let τ be the stable tangent bundle of $L^3(2^a)$. Then by the results of Atiyah [2] and by the mod k Dold theorem [1], we see that $\tilde{J}(\tau)\in\tilde{J}(L^3(2^a))$ is of odd order (may be zero). Thus in order to prove the theorem, it suffices to show the following

Lemma 4.1. *Let $a\geq 2$ be an integer. Then $\tilde{J}(\tau)\in\tilde{J}(L^3(2^a))$ is a non zero element of even order.*

Proof. First we determine the tangent bundle of $L^n(2^a)$. Applying Theorem 1.1 of [14] to the principal bundle $Z_{2^a}\to S^{2n+1}\to L^n(2^a)$, we see that

$$\tau(L^n(2^a))\oplus\varepsilon\cong(n+1)\eta$$

where ε is the trivial line bundle and $\eta=S^{2n+1}\times_{Z_{2^a}}\mathbf{C}^1$ is the canonical complex line bundle (Z_{2^a} acts on $\mathbf{C}^1$ via the canonical inclusion $Z_{2^a}\subset S^1$). Let

$$i\colon L^n(2^a)\longrightarrow L^n(2^{a+1})$$

be the canonical map. Then it is obvious that $i^*(\tau(L^n(2^{a+1}))\oplus\varepsilon)\cong\tau(L^n(2^a))\oplus\varepsilon$. Hence we are enough to prove the lemma for $a=2$. Now it is known (Corollary 4.6, [8]) that the order of $\tilde{J}(r(\eta-1))\in\tilde{J}(L^3(4))$ is 8. Thus

$$\tilde{J}(\tau)=J(4r(\eta-1))$$

is an element of order 2. This completes the proof.

Appendix

The theory of infinite loop spaces says that a small category $\mathscr{C}$ with a coherent associative and commutative bifunctor $\boxtimes\colon \mathscr{C}\times\mathscr{C}\to\mathscr{C}$ (a symmetric monoidal category) defines a generalized cohomology theory ([9] and [13]). More precisely, if $\mathscr{C}$ is a symmetric monoidal category, one can associate a spectrum $\mathbf{B}\mathscr{C}=\{B^nB\mathscr{C}\}_{n=0,1,2,\dots}$ such that

i) *$B^0B\mathscr{C}=B\mathscr{C}$ is the classifying space of $\mathscr{C}$*
ii) *$\{B^nB\mathscr{C}\}_{n\geq 1}$ is an Ω-spectrum*

iii) *if $B\mathscr{C}$ is of the homotopy type of countable CW complex, then the structure map g_0: $B\mathscr{C}\to B^1B\mathscr{C}$ is the "group completion", i.e.,*

$$g_{0*}\colon H_*(B\mathscr{C};\ k)[\pi_0(B\mathscr{C})^{-1}] \longrightarrow H_*(\Omega B^1B\mathscr{C};\ k)$$

is an isomorphism for any field k. (see [9]).

The Barratt-Priddy-Quillen theorem asserts that the cohomology theory defined by the category of finite sets is equivalent to the stable cohomotopy theory (see [13]). Here we consider an equivariant version of the Barratt-Priddy-Quillen theorem, essentially due to Segal [12].

Let $\mathscr{C}$ be a symmetric monoidal category such that any morphism is invertible. Then $B\mathscr{C}$ is a homotopy commutative H-space, and the abelian (additive) monoid $\pi_0(B\mathscr{C})$ is identified with the set of isomorphism classes of $\mathrm{Ob}\,\mathscr{C}$. Given an object X, one has a functor

$$L_X\colon \mathscr{C} \longrightarrow \mathscr{C}$$

by $L_X(Y)=X\times Y$. This induces a continuous map

$$l_X\colon B\mathscr{C} \longrightarrow B\mathscr{C}$$

If X and Y are objects in the same component of $B\mathscr{C}$, then clearly $l_X\sim l_Y$ (homotopic). Let $\alpha\in\pi_0(B\mathscr{C})$. Choose a representative X of α, and put $\phi_\alpha=l_X$. Then clearly $\phi_{\alpha+\beta}\sim\phi_\alpha\phi_\beta$.

Now regard $\pi_0(B\mathscr{C})$ as a directed set by setting $\alpha<\beta$ if $\beta=\gamma+\alpha$ for some γ. Suppose that $\pi_0(B\mathscr{C})$ is countable. Then one can choose $d_1, d_2,\dots, \in\pi_0(B\mathscr{C})$ such that the sequence $\{\alpha_i=d_1+\cdots+d_i\}_{i=1,2,\dots}$ is cofinal in $\pi_0(B\mathscr{C})$. Consider the direct system

$$B\mathscr{C} \xrightarrow[\phi_{\alpha_1}]{} B\mathscr{C} \xrightarrow[\phi_{\alpha_2}]{} B\mathscr{C} \longrightarrow \cdots$$

it is clear that connected components of the direct limit $\lim\{B\mathscr{C}, \phi_{\alpha_i}\}$ are homotopy equivalent to each other. So put

$$B\mathscr{C}_\infty = \text{a component of } \lim\{B\mathscr{C}, \phi_{\alpha_i}\}.$$

Theorem A.1. *There is a map*

$$\omega\colon B\mathscr{C}_\infty \longrightarrow (\Omega B^1B\mathscr{C})_0$$

such that ω_*: $H_*(B\mathscr{C}_\infty)\to H_*(\Omega B^1B\mathscr{C})_0)$ is an isomorphism, where the subscript 0 means the 0-component.

Proof. Letting S be the multiplicative subset of $\pi_0(B\mathscr{C})$ generated by $d_1, d_2,\dots,$

$$\begin{aligned} H_*(B\mathscr{C};k)[\pi_0(B\mathscr{C})^{-1}] &= H_*(B\mathscr{C};k)[S^{-1}] \\ &= \lim\{\to H_*(B\mathscr{C};k) \xrightarrow{(\phi_{\alpha_i})_*} H_*(B\mathscr{C};k)\to\cdots\} \\ &= H_*(\lim\{B\mathscr{C}, \phi_{\alpha_i}\};k). \end{aligned}$$

for any field k. For $\beta \in \pi_0(B\mathscr{C})$, let

$$\omega_\beta \colon B\mathscr{C}_\beta \xrightarrow[g_0]{} (\Omega B^1 B\mathscr{C})_\beta \xrightarrow[\phi_{-\beta}]{} (\Omega B^1 B\mathscr{C})_0 .$$

Then clearly $\omega_{\alpha+\beta}\phi_\alpha \sim \omega_\beta$, and hence by the telescope argument, we obtain a map

$$\omega = \lim \omega_\beta \colon B\mathscr{C}_\infty \longrightarrow (\Omega B^1 B\mathscr{C})_0 .$$

The fact that ω^* is an isomorphism follows from that

$$g_{0*} \colon H_*(B\mathscr{C} : k)[S^{-1}] \longrightarrow H_*(\Omega B^1 B\mathscr{C} : k)$$

for any field k. q.e.d.

Now let G be a finite group. By $\mathscr{S}_G$ we denote the category of finite G-sets and G-isomorphisms. The direct sum and the direct product of finite G-sets give rise to binary functors $\oplus$ and $\otimes$, respectively. It is easy to see that $(\mathscr{S}_G, \oplus)$ and $(\mathscr{S}_G, \otimes)$ are both symmetric monoidal categories.

Let $\Omega_G = \{(H_1), \dots, (H_k)\}$ be the set of conjugacy classes of subgroups of G. Let $\mathscr{S}_i$ be the full subcategory of $\mathscr{S}_G$ consisting of objects of the form $nG/H_i = G/H_i \cdots G/H_i$.

Lemma A.2. *The category $\mathscr{S}_G$ is equivalent to the product category $\mathscr{S}_1 \times \cdots \times \mathscr{S}_k$, and*

$$B\mathscr{S}_i = \coprod_n E\Sigma_n \times_{\Sigma_n} (BW_{H_i})^n$$

where $E\Sigma_n$ is a universal Σ_n-space and $W_{H_i} = N(H_i)/H_i$.

Proof. The first statement is clear. By definition $B\mathscr{S}_i = \coprod_n B\,\mathrm{Aut}_G(nG/H_i)$. Clearly we see that $\mathrm{Aut}_G(G/H_i) = W_{H_i}$. Hence $B\,\mathrm{Aut}_G(nG/H_i) = B\left(\Sigma_n \int W_{H_i}\right) = E\Sigma_n \times_{\Sigma_n} (BW_{H_i})^n$. q.e.d.

Note that $B^1 B\mathscr{S}_i = Q((BW_{H_i})_+)$ by the Barratt-Priddy-Quillen theorem and by Lemma A.2. Thus we have obtained

Theorem A.3. *There is a map*

$$\omega \colon (B\mathscr{S}_G)_\infty \longrightarrow \prod_{i=1}^{k} Q((BW_H)_+)_0$$

which induces an isomorphism of homology.

Next we shall consider subcategories of $\mathscr{S}_G$ which represent "free part" of $\mathscr{S}_G$. Let $\mathscr{F}_G$ and $\mathscr{F}_G^{\otimes}$ be the fall subcategories of $\mathscr{S}_G$ consisting of free G-sets, and consisting of G-sets of the form free G set $\cup + *$, respectively. Clearly $\mathscr{F}_G$ is a symmetric monoidal subcategory of $\mathscr{S}_G$, whereas $\mathscr{F}_G^{\otimes}$ is symmetric monoidal with respect to $\otimes$.

Theorem A.4. *There is a map*

$$\lambda \colon (B\mathscr{F}_G^{\otimes})_\infty \longrightarrow Q_0(BG_+)_{P(G)}$$

inducing an isomorphism on homology, where $X_{P(G)}$ denotes the localization of X at the set of primes $P(G)=\{p;\ p \mid |G|\}$.

Proof. Note that $\pi_0(B\mathscr{F}_G^{\otimes})$ is the multiplicative submonoid of $\pi_0(B\mathscr{S}_G)$ consisting of $1+nG$, where G is regarded as a free G set, hence $\pi_0(B\mathscr{F}_G^{\otimes})$ is a free abelian monoid with countable generators.

For S, $T \in \mathrm{Ob}\,\mathscr{S}_G$, let

$$F(S, T):\ \mathscr{S}_G \longrightarrow \mathscr{S}_G$$

be the functor defined by $F(S, T)(X)=S\oplus(T\otimes X)$. Let $f(S, T)$: $B\mathscr{S}_G \to B\mathscr{S}_G$ be the induced map. Put $\gamma_m=f(mG,\ 1+m|G|)$. Then restricting to $B\mathscr{F}_G$, we have a map

$$\gamma_m:\ B\mathscr{F}_G \longrightarrow B\mathscr{F}_G$$

Also put $\beta_m=f(\phi,\ 1+mG)$: $B\mathscr{F}_G^{\otimes} \to B\mathscr{F}_G^{\otimes}$. Note that we have

$$\begin{aligned}(1+mG)(1+nG) &= 1+(n+m+nm|G|)G \\ &= 1+mG+(1+m|G|)nG\end{aligned}$$

in $\pi_0(B\mathscr{F}_G^{\otimes})$. Hence we have a homotopy commutative diagram

$$\begin{array}{ccc} B\mathscr{F}_G & \xrightarrow{\gamma_m} & B\mathscr{F}_G \\ {\scriptstyle f(1,\phi)}\downarrow & & \downarrow{\scriptstyle f(1,\phi)} \\ B\mathscr{F}_G^{\otimes} & \xrightarrow[\beta_m]{} & B\mathscr{F}_G^{\otimes} \end{array}$$

Similarly we have a homotopy commutative diagram

$$\begin{array}{ccc} (B\mathscr{F}_G)_\alpha & \xrightarrow{\gamma_m} & (B\mathscr{F}_G)_{\gamma_m(\alpha)} \\ {\scriptstyle \omega_\alpha}\downarrow & & \downarrow{\scriptstyle \omega_{m(\alpha)}} \\ (\Omega B^1 B\mathscr{F}_G)_0 & \xrightarrow[\times(1+m|G|)]{} & (\Omega B^1 B\mathscr{F}_G)_0 \end{array}$$

Note that if m tends to infinity, then so does $\gamma_m(\alpha)$. Hence $(B\mathscr{F}_G)_{\gamma_m(\alpha)}$ approximates $(\Omega B^1 B\mathscr{F}_G)_0$ homologically. Also note that the multiplicative set $M=\{1+m|G|;\ m=1, 2,\dots\}$ is cofinal in the multiplicative monoid of positive integers prime to $|G|$ because of the Dirichlet theorem on arithmetic progression.

Now since $(\Omega B^1 B\mathscr{F}_G)_0$ is an H-space, the limit of

$$\longrightarrow (\Omega B^1 B\mathscr{F}_G)_0 \xrightarrow[\times(1+m|G|)]{} (\Omega B^1 B\mathscr{F}_G)_0 \cdots$$

is $((\Omega B^1 B\mathscr{F}_G)_0)_{P(G)}$ the localization at the set of primes $P(G)$. By Theorem A.3, $((\Omega B^1 B\mathscr{F}_G)_0)_{P(G)} \cong Q_0(BG_+)_{P(G)}$. Note that $(B\mathscr{F}_G^{\otimes})_\infty$ is a component of $\lim\limits_{m}\{B\mathscr{F}_G^{\otimes},\ \beta_m\}$, and

$$f(1, \phi):\ B\mathscr{F}_G \longrightarrow B\mathscr{F}_G^{\otimes}$$

is a homotopy equivalence. Hence taking the limit of the above diagrams, we have a map

$$\lambda\colon (B\mathscr{F}_G^{\otimes})_\infty \longrightarrow Q_0(BG_+)_{P(G)}$$

which induces an isomorphism of homology. q.e.d.

Finally we sketch briefly a reproof of Theorem 1.1 for odd primes using algebraic K-theory.

Let $\mathbf{F}_q$ be a finite field with $q=p^r$ elements. Let $\mathscr{V}$ be the category of (finite dimensional) vector spaces over $\mathbf{F}_q$ and linear isomorphisms. Let $\mathscr{V}_{\otimes}^{<p>}$ be the full subcategory of $\mathscr{V}$ consisting of vector spaces of a power of p dimension. Let $\mathscr{S}_{\otimes}^{<p>}$ be the full subcategory of $\mathscr{S}$ (the category of finite sets) consisting of finite sets of cardinal p^n. Here the symmetric monoidal structures of $\mathscr{V}_{\otimes}^{<p>}$ and $\mathscr{S}_{\otimes}^{<p>}$ come from the tensor product and the direct product, respectively.

There are two functors

$$\mathscr{V} \xrightarrow{U} \mathscr{S}_{\otimes}^{<p>} \xrightarrow{L} \mathscr{V}_{\otimes}^{<p>}$$

where U forgets the vector space structure and L sends a finite set to a vector space generated by the set.

Let A be a subgroup of $\mathbf{F}_q^*$. By the scalar multiplication, A acts on a F_q-vector space V and the set $U(V)$. Since $\mathrm{Aut}_{\mathbf{F}_q}(V)$ comutes with A, $U(\mathrm{Aut}_{F_q}(V))\subset \mathrm{Aut}_A(U(V))$. Note that $U(V)\in \mathrm{Ob}\,\mathscr{F}_A^{\otimes}$. Thus we have a commutative diagram of functors

$$\begin{tikzcd} \mathscr{V} \arrow[rr, "U'"] \arrow[dr, "U"'] & & \mathscr{F}_A^{\otimes} \arrow[dl, "F"] \\ & \mathscr{S}_{\otimes}^{<p>} & \end{tikzcd}$$

where F is the functor forgetting the A-action.

Now we apply the functor $\Omega B^1 B$ on the above categories. According to Quillen [11], $(\Omega B^1 B\mathscr{V})_0 \cong BGL(\mathbf{F}_q)^+$ and the algebraic K-group $\tilde{K}_{\mathbf{F}_q}(X)$ is defined by $[X, BGL(\mathbf{F}_q)^+]$. It is easy to see that $(\Omega B^1 B\mathscr{V}_{\otimes}^{<p>})_0 \cong (BGL(\mathbf{F}_q)^+)_{<p>}$ and $(\Omega B^1 B\mathscr{S}_{\otimes}^{<p>})_0 \cong Q_0(S^0)_{<p>}$, where $X_{<p>}$ denotes the localization away from p. The functor U; $\mathscr{V}\to\mathscr{S}_{\otimes}^{<p>}$ induces a natural transformation

$$j\colon \tilde{K}_{\mathbf{F}_q}(X) \longrightarrow [X, Q_0(S^0)_{<p>}] \cong \pi^0(X)\left[\frac{1}{p}\right]$$

which refer to the algebraic J-homomorphism. Then by Theorem A.4, we have obtained

Proposition A.5. *There is a commutative diagram*

$$\begin{tikzcd} \tilde{K}_{F_q}(X) \arrow[rr, "j_A"] \arrow[dr, "j"'] & & {[X, Q_0(BA_+)_{P(A)}]} \arrow[dl, "\phi"] \\ & {[X, Q_0(S^0)_{<p>}]} & \end{tikzcd}$$

where ϕ is the forgetting homomorphism induced from F.

Now Theorem 1.1 for an odd prime follows from

Proposition A.6. *Let $A=Z_{l^a}$, l odd. Then there is an integer $q=p^r$ such that $\mathbf{F}_q^* \supset Z_{l^a}=A$ and*

$$j: \tilde{K}_{\mathbf{F}_q}(S^{2n-1})_{(l)} \longrightarrow \pi_{2n-1}^S(S^0)_{(l)}$$

is an epimorphism onto $(\operatorname{Im} J)_{(l)}$.

Proof. The existence of such q is well known. For $\operatorname{Im} j$, see Tornehave [15].

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