J. Math. Kyoto Univ. (JMKYAZ)
18-3 (1978) 535-541

On the cohomology mod 2 of the classifying space of AdE_7

By

Akira KONO and Mamoru MIMURA

Dedicated to Professor A. Komatu on his 70th birthday

(Received Sept. 12, 1977)

§1. Introduction

Let E_7 be the compact 1-connected exceptional Lie group and AdE_7 the quotient of E_7 by its center $\mathbf{Z}/2\mathbf{Z}$. The purpose of this paper is to determine the module structure of $H^*(BAdE_7; \mathbf{F}_2)$ where $\mathbf{F}_p$ is the prime field of characteristic p and $BAdE_7$ is the classifying space of AdE_7.

To determine this we make use of the Rothenberg-Steenrod spectral sequence ([8]):

$$(3.1) \qquad E_2 = \mathrm{Ext}_{H_*(AdE_7;\, \mathbf{F}_2)}(\mathbf{F}_2, \mathbf{F}_2) \Longrightarrow E_\infty = GrH^*(BAdE_7; \mathbf{F}_2).$$

In fact, we show

Theorem 3.5. *The spectral sequence* (3.1) *collapses.*

Corollary 3.6. *As a module over* $\mathbf{F}_2$

$$H^*(BAdE_7; \mathbf{F}_2) \cong \mathbf{F}_2[y_i;\ i \in \overline{M}]/R,$$

where R is the ideal generated by (3.9) *and* (3.10) *in* [6], $\deg y_i = i$ *and* $\overline{M} = \{2, 3, 6, 7, 10, 11, 18, 19, 34, 35, 64, 66, 67, 96, 112\}$.

Remark. $H^*(BAdE_7; \mathbf{F}_p) \cong H^*(BE_7; \mathbf{F}_p)$ *for any odd prime p.*

The paper is organized as follows. Certain indecomposable elements of $H^*(BAdE_7; \mathbf{F}_2)$ are constructed in §2, where the adjoint representation of E_7 plays an important role. In the next section, §3, the E_2-term of (3.1) is determined and the main theorem is proved. Throughout the paper we use the following

Notations. π: $E_7 \to AdE_7$ is the covering projection. For a homomorphism f: $G \to G'$ between compact Lie groups the induced map between the classifying spaces $BG \to BG'$ is denoted by the same symbol. $H^*(\ )$ is the mod 2 (not integral) singular cohomology functor. $\mathscr{A}_2$ is the mod 2 Steenrod algebra.

§ 2. Certain elements of $H^*(BAdE_7)$

Notation 2.1. $M_1=\{6, 10, 18, 34, 66\}$,

$M_2=\{7, 11, 19, 35, 67\}$,

$M = M_1 \cup M_2$,

$\tilde{M} = \{4\} \cup M \cup \{64, 96, 112\}$,

$\bar{M} = \{2, 3\} \cup M \cup \{64, 96, 112\}$.

Notation 2.2. $S_n = Sq^{2^n} Sq^{2^{n-1}} \cdots Sq^4 \in \mathscr{A}_2$ for $n \geq 2$,

$S'_n = Sq^1 S_n$ for $n \geq 2$.

Now recall the result of Kono-Mimura-Shimada [4]:

Theorem 2.3. *There exist indecomposable elements*

$$y_i \in H^i(BE_7) \quad \text{for} \quad i \in \tilde{M}$$

such that as an algebra

$$H^*(BE_7) \cong \mathbf{F}_2[y_i;\ i \in \tilde{M}]/\bar{R}$$

for some ideal $\bar{R}$ *of* $\mathbf{F}_2[y_i;\ i \in \tilde{M}]$. *Moreover*

$$y_6 = Sq^2 y_4, \qquad y_{2^n+2} = S_{n-1} y_6 \quad (3 \leq n \leq 6),$$

$$y_7 = Sq^1 y_6, \qquad y_{2^n+3} = Sq^1 y_{2^n+2} = S'_{n-1} y_6 \quad (3 \leq n \leq 6),$$

$$y_{96} = Sq^{32} y_{64}, \qquad y_{112} = Sq^{16} y_{96}.$$

Remark 2.4. $H^*(BE_7) \cong \mathbf{F}_2[y_4, y_6]$ *for* $* \leq 6$.

Lemma 2.5. *As an algebra over* $\mathbf{F}_2$

$$H^*(BAdE_7) \cong \mathbf{F}_2[e_2, e_3] \quad \text{for} \quad * \leq 5,$$

where $\deg e_i = i$ *and* $e_3 = Sq^1 e_2$.

Proof. According to Ishitoya-Kono-Toda [2]

$$H^*(AdE_7) \cong \mathbf{F}_2[x_1]/(x_1^4) \quad \text{for} \quad * \leq 4,$$

where $\deg x_1 = 1$. Clearly x_1 is universally transgressive and so is x_1^2. Put $e_2 = \tau(x_1)$ and $e_3 = \tau(x_1^2) = Sq^1 e_2$ where τ is the trangression operation. Then the result follows easily. Q.E.D.

Let

$$\varepsilon\colon BAdE_7 \longrightarrow K(\mathbf{Z}/2\mathbf{Z}, 2)$$

be the map corresponding to $e_2 \in H^2(BAdE_7)$ so that exists a fibering

$$(2.1) \qquad BE_7 \xrightarrow{\pi} BAdE_7 \xrightarrow{\varepsilon} K(\mathbf{Z}/2\mathbf{Z}, 2).$$

The cohomology Serre spectral sequence associated with the fibering (2.1) has the following E_2-term

$$E_2 \cong H^*(K(\mathbf{Z}/2\mathbf{Z}, 2)) \otimes H^*(BE_7)$$

and converges to $H^*(BAdE_7)$, where

$$H^*(K(\mathbf{Z}/2\mathbf{Z}, 2)) \cong \mathbf{F}_2[u_2, u_3, u_5] \qquad \text{for} \quad * \leq 8$$

with $\deg u_i = i$, $u_3 = Sq^1 u_2$ and $u_5 = Sq^2 Sq^1 u_2$. Clearly y_4 is transgressive. Since $H^4(BAdE_7)$ is decomposable, we have

$$\tau(y_4) = u_5 + \alpha u_2 u_3 \qquad \text{with} \quad \alpha \in \mathbf{F}_2.$$

Since $y_6 = Sq^2 y_4$, y_6 is also transgressive with

$$\tau(y_6) = Sq^2 \tau(y_4) = Sq^2 u_5 + \alpha Sq^2(u_2 u_3).$$

Then by the Adem relation we obtain

$$Sq^2 u_5 = Sq^2 Sq^2 Sq^1 u_2 = Sq^3 Sq^1 Sq^1 u_2 = 0$$

and

$$Sq^1 u_3 = Sq^1 Sq^1 u_2 = 0,$$

and hence

$$Sq^2 \tau(y_4) = \alpha u_2(u_5 + u_2 u_3) \in (\tau(y_4)), \text{ the ideal generated by } \tau(y_4).$$

So $(1 \otimes y_6)$ is a permanent cycle and $y_6 \in \operatorname{Im} \pi^*$.

Proposition 2.6. $y_i \in \operatorname{Im} \pi^*$ *for* $i \in M$.
This is an immediate consequence of the naturality of the Steenrod operations.

Let

$$\mu\colon E_7 \longrightarrow O(133)$$

be the adjoint representation of E_7. Then, as is well known, $\operatorname{Ker} \mu = \operatorname{Center} E_7$ ($=$the center of E_7). So μ induces a representation $\bar{\mu}$ of AdE_7:

$$\bar{\mu}\colon AdE_7 \longrightarrow O(133)$$

such that the diagram

$$\begin{tikzcd} E_7 \arrow[r, "\mu"] \arrow[d, "\pi"'] & O(133) \\ AdE_7 \arrow[ur, "\bar{\mu}"'] & \end{tikzcd}$$

is commutative.

Lemma 2.7. *E_7 contains $SU(7)$ as a closed subgroup such that* Center $E_7 \cap SU(7) = \{unit\}$.

For a proof see P. 279 of [4].

Let T^6 be a maximal torus of $SU(7)$ and T^7 a maximal torus of E_7 such that $T^6 \subset T^7$. The commutative diagram

$$\begin{tikzcd} & & T^7 \arrow[d, "i_4"] \\ T^6 \arrow[urr, "i_3"] \arrow[r, "i_1"'] & SU(7) \arrow[r, "i_2"'] & E_7 \end{tikzcd}$$

gives rise to a homotopy commutative one

$$\begin{tikzcd} & & BT^7 \arrow[d, "i_4"] \\ BT^6 \arrow[urr, "i_3"] \arrow[r, "i_1"'] & BSU(7) \arrow[r, "i_2"'] & BE_7 \end{tikzcd}$$

where i_j's are the inclusions. On the other hand, there exists a representation of T^7

$$\mu' \colon T^7 \longrightarrow O(126)$$

such that $\mu \circ i_4$ is equivalent to $\mu' \oplus \varepsilon_7$ as a representation of T^7, where ε_7 is the 7-dimensional trivial representation of T^7. Then we have the following homotopy commutative diagram

$$\begin{tikzcd} BT^7 \arrow[r, "\mu'"] \arrow[d, "i_4"'] & BO(126) \arrow[d, "i_5"] \\ BE_7 \arrow[r, "\mu"] & BO(133) \end{tikzcd}$$

where i_5 is the map corresponding to the natural inclusion: $O(126) \to O(133)$.

Lemma 2.8. *$H^*(BSU(7))$ is a finite $H^*(BO(133))$-module under $(\mu \circ i_2)^*$: $H^*(BO(133)) \to H^*(BE_7) \to H^*(BSU(7))$.*

Proof. Since $\mathrm{Ker}\,\mu \cap \mathrm{Im}\, i_2 = \{\text{unit}\}$, $\mu \circ i_2$ is an injection. So the result follows from the Quillen's finiteness theorem (cf. §2 of [7]). Q.E.D.

Recall that

$$H^*(BO(n))=\mathbf{F}_2[w_1,\dots,w_n],$$
$$H^*(BSU(7))=\mathbf{F}_2[c_2,c_3,c_4,c_5,c_6,c_7],$$

with $\deg w_i=i$ and $\deg c_i=2i$. Let I be the ideal of $H^*(BSU(7))$ generated by c_2, c_3, c_5 and J the subalgebra of $H^*(BO(133))$ generated by w_i, $i\leq 126$. Then the Wu-formula indicates

Lemma 2.9. *I is $\mathscr{A}_2$-invariant and J is generated by w_{2^i} for $0\leq i\leq 6$ over $\mathscr{A}_2$.*

Lemma 2.10. $(i_2^*\circ\mu^*)(w_i)=0$ *for* $i\geq 127$.

Proof. Since i_1^* is injective, it suffices to show that $(i_1^*\circ i_2^*\circ\mu^*)(w_i)=0$ for $i\geq 127$, which follows from

$$(i_1^*\circ i_2^*\circ\mu^*)(w_i)=(i_3^*\circ\mu'^*\circ i_5^*)(w_i)=0 \qquad \text{for} \quad i\geq 127.$$

Q.E.D.

Proposition 2.11. *The element μ^*w_{64} is not decomposable.*

Proof. Since $H^*(BE_7)$ for $*\leq 63$ is generated by y_4 over $\mathscr{A}_2$, $\mathrm{Im}(\mu\circ i_2)^*$ for degrees ≤ 63 is contained in I. So, if μ^*w_{64} were decomposable, the element $(i_2^*\circ\mu^*)w_{2^i}$ would belong to I for $0\leq i\leq 6$. Thus $(i_2^*\circ\mu^*)(J)\subset I$ and so $\mathrm{Im}(i_2^*\circ\mu^*)=\mathrm{Im}(\mu\circ i_2)^*\subset I$ by Lemma 2.10, which would contradict to Lemma 2.8. Q.E.D.

Now Lemma 2.5, Propositions 2.6 and 2.11 yield the following

Theorem 2.12. *There exists indecomposable elements*

$$e_i\in H^i(BAdE_7) \qquad \textit{for} \quad i\in\overline{M}$$

such that $\pi^*(e_i)=y_i$ for $i\in M$ and $\pi^*(e_i)\equiv y_i$ mod decomposables for $i=64$, 96, 112.

Proof. The elements e_2, e_3 are as in Lemma 2.5. By Proposition 2.6 there exists e_i such that $\pi^*(e_i)=y_i$ for $i\in M$. Moreover, put $e_{64}=\bar{\mu}^*(w_{64})$, $e_{96}=Sq^{32}e_{64}$ and $e_{112}=Sq^{16}Sq^{32}e_{64}$. Then $\pi^*(e_{64})=\mu^*(w_{64})\equiv y_{64}$ mod decomposables and so $\pi^*(e_i)\equiv y_i$ mod decomposables for $i=96$, 112. These are not decomposable by Proposition 2.11. Q.E.D.

Remark 2.13. The element e_i $(i\in M)$ is not uniquely determined. But we can rechoose e_i such that $e_7=Sq^1e_6$, $e_{2^n+2}=S_{n-1}e_6$ $(3\leq n\leq 6)$ and $e_{2^n+3}=S'_{n-1}e_6$ $(3\leq n\leq 6)$.

§3. Proof of the main theorem

First recall from [2] (see also [3] and [4]):

Theorem 3.1. *As an algebra*

$$H^*(E_7)\cong \mathbf{F}_2[x_3, x_5, x_9]/(x_3^4, x_5^4, x_9^4)\otimes\Lambda(x_{15}, x_{17}, x_{23}, x_{27}),$$

$$H^*(AdE_7)\cong \mathbf{F}_2[x_1, x_5, x_9]/(x_1^4, x_5^4, x_9^4)\otimes\Lambda(x_6, x_{15}, x_{17}, x_{23}, x_{27}),$$

where $\deg x_i = i$ *and the coalgebra structure is given by*

$$\bar{\phi}(x_i)=0 \qquad i=1, 3, 5, 6, 9, 17,$$

$$\bar{\phi}(x_{15})=\begin{cases} x_3^2\otimes x_9+x_5^2\otimes x_5 & \textit{for}\ \ E_7 \\ x_6\otimes x_9+x_5^2\otimes x_5 & \textit{for}\ \ AdE_7, \end{cases}$$

$$\bar{\phi}(x_{23})=\begin{cases} x_9^2\otimes x_5+x_3^2\otimes x_{17} & \textit{for}\ \ E_7 \\ x_9^2\otimes x_5+x_6\otimes x_{17} & \textit{for}\ \ AdE_7, \end{cases}$$

$$\bar{\phi}(x_{27})=x_9^2\otimes x_9+x_5^2\otimes x_{17}.$$

Put $A=H^*(E_7)//(x_3)$ (for the notation see [5]). Since x_3 is primitive, A is also a Hopf algebra. Then it follows from Theorem 3.1

Lemma 3.2 *As an algebra*

$$H_*(E_7)\cong\Lambda(s_3)\otimes A^*,$$

$$H_*(AdE_7)\cong\Lambda(s_1, s_2)\otimes A^*,$$

where $\deg s_i = i$ *and* A^* *is the dual of* A.

The following is Proposition 3.2 of [6].

Lemma 3.3 *As an algebra*

$$\mathrm{Ext}_{H_*(E_7)}(\mathbf{F}_2, \mathbf{F}_2)\cong\mathbf{F}_2[y_4]\otimes(\mathbf{F}_2[y_i, i\neq 4, i\in\tilde{M}]/R)$$

where R *is the ideal generated by the elements of* (3.9) *and* (3.10) *in* [6].

Thus we have

Proposition 3.4. *As an algebra*

$$\mathrm{Ext}_{H_*(AdE_7)}(\mathbf{F}_2, \mathbf{F}_2)\cong\mathbf{F}_2[y_i;\ i\in\bar{M}]/R.$$

Now consider the Rothenberg-Steenrod spectral sequence [8]:

(3.1) $$E_2\cong\mathrm{Ext}_{H_*(AdE_7)}(\mathbf{F}_2, \mathbf{F}_2)\Longrightarrow E_\infty\cong GrH^*(BAdE_7).$$

Using Theorem 2.12, we can easily get

Theorem 3.5. *The spectral sequence* (3.1) *collapses.*

Corollary 3.6. *As a module over* $\mathbf{F}_2$

$$H^*(BAdE_7)\cong\mathrm{Ext}_{H_*(AdE_7)}(\mathbf{F}_2, \mathbf{F}_2).$$

Corollary 3.7. *As an algebra* $H^*(BAdE_7)$ *is generated by the elements* e_i*'s*

$i \in \overline{M}$. *Moreover*,

$$e_3 = Sq^1 e_2,\ e_7 = Sq^1 e_6,$$

$$e_{2^n+2} = S_{n-1} e_6 \qquad (3 \leq n \leq 6),$$

$$e_{2^n+3} = S'_{n-1} e_6 \qquad (3 \leq n \leq 6),$$

$$e_{64} = \mu^*(w_{64}),\ e_{96} = Sq^{32} e_{64} \quad \textit{and} \quad e_{112} = Sq^{16} Sq^{32} e_{64}.$$

DEPARTMENT OF MATHEMATICS
KYOTO UNIVERSITY

MATHEMATICAL INSTITUTE
YOSHIDA COLLEGE
KYOTO UNIVERSITY

References

[1] J. F. Adams: Lectures on Lie groups, Benjamin, 1969.
[2] K. Ishitoya-A. Kono-H. Toda: Hopf algebra structure of mod 2 cohomology of simple Lie groups, Publ. RIMS Kyoto Univ., 12 (1975), 141–167.
[3] A. Kono: Hopf algebra structure and cohomology operations of the mod 2 cohomology of the exceptional Lie groups, Japanese J. Math., 3 (1977), 49–55.
[4] A. Kono-M. Mimura-N. Shimada: On the cohomology mod 2 of the classifying space of the 1-connected exceptional Lie group E_7, J. Pure and Applied Alg., 8 (1976), 267–283.
[5] J. Milnor-J. C. Moore: On the structure of Hopf algebras, Ann. Math., 81 (1965), 211–264.
[6] M. Mimura-M. Mori: The squaring operations in the Eilenberg-Moore spectral sequence and the classifying space of an associative H-space, I, Publ. RIMS Kyoto Univ., 13 (1977), 755–776.
[7] D. Quillen: The spectrum of an equivariant cohomology ring I, Ann. Math., 94 (1971), 549–572.
[8] M. Rothenberg-N. E. Steenrod: The cohomology of the classifying spaces, Bull. A.M.S., 71 (1965), 872–875.