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On the Lax-Mizohata theorem in the analytic and Gevrey classes

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1. Introduction

In this paper, we consider the non-characteristic Cauchy problem for the differential operators with Gevrey or analytic coefficients.

L. Boutet de Monvel and P. Krée [2] have showed some fundamental properties of analytic and Gevrey symbols of pseudo-differential operators. In [1], L. Hörmander has localized the pseudo-differential operators with analytic symbols in a suitable way on the dual space to extend the regularity and uniqueness theorems and to study the propagation of the singularities.

Let $L(x, t; D_x, D_t)=P(x, t; D_x, D_t)Q(x, t; D_x, D_t)$ be a differential operator of order m with Gevrey or analytic coefficients, and $Lu=0$. If P is a elliptic differential operator of order ν, then the analytic-hypoellipticity means that Qu is a Gevrey or analytic function. Therefore, $D_t^j u(x, 0)$ $(\nu+\mu=m, \mu\leqslant j\leqslant m-1)$ are also Gevrey or analytic functions provided that $t=0$ is non characteristic for Q and $D_t^j u(x, 0)$ $(0\leqslant j\leqslant \mu-1)$ are in Gevrey or analytic class. This shows that, for the Cauchy problem of L, we cannot give the first $\mu+1$ initial data arbitrarily in C^∞ class.

Here, using the above localized differential operator, we shall generalize this simple example and give the same necessary relation between the admissible initial data and the number of real roots of the characteristic equation. And, as application of this relation, we extend the Lax-Mizohata theorem to the analytic and Gevrey classes.

2. Definitions and Results

Definition 2.1. *Let V be an open set in R^m, we shall denote by $\gamma^{(s)}(V)$ $(s\geqslant 1)$ the set of all $f\in C^\infty(V)$ such that for every compact set $K\subset V$, there are constants C, A with*

$$(2.1)\qquad |D^\alpha f(x)|\leqslant CA^{|\alpha|}|\alpha|!^s, \qquad x\in K,$$

for all multi-indexes α.

Definition 2.2. ([1]) Let $x_0 \in V \subset R^m$, $\xi_0 \in R^m \backslash 0$ and $u \in \mathscr{D}'(V)$. Then we shall say that (x_0, ξ_0) is in the complement of wave front set $WF_s(u)$, if and only if there are an open neighborhood U of x_0, an open conic neighborhood Γ of ξ_0 and a bounded sequence $u_N \in \mathscr{E}'(V)$ which is equal to u in U, such that

$$|\hat{u}_N(\xi)| \leqslant C(CN^s)^N |\xi|^{-N} \tag{2.2}$$

is valid for some constant C when $\xi \in \Gamma$.

Let $p(x, t; D_x, D_t) = D_t^m + \sum_{j=1}^{m} a_j(x, t; D_x) D_t^{m-j}$ be a differential operator with coefficients in $\gamma^{(s)}(W)$, where W is an open neighborhood of the origin in R^{n+1}, and the order of $a_j(x, t; D_x)$ is less than j. We shall denote

$$D_t = \frac{1}{i}\frac{\partial}{\partial t}, \quad D_x = \left(\frac{1}{i}\frac{\partial}{\partial x_1}, \dots, \frac{1}{i}\frac{\partial}{\partial x_n}\right), \quad x = (x_1, \dots, x_n), \quad \xi = (\xi_1, \dots, \xi_n)$$

and

$$p(x, t; \xi, \lambda) = \lambda^m + \sum_{j=1}^{m} a_j(x, t; \xi)\lambda^{m-j} = p_0(x, t; \xi, \lambda) + p_1(x, t; \xi, \lambda) + \cdots$$
$$\cdots + p_m(x, t; \xi, \lambda)$$

where, $p_j(x, t; \xi, \lambda)$ is a homogeneous polynomial of order $m-j$ in (ξ, λ).

Theorem 2.1. *Suppose that the characteristic equation* $p_0(0, 0; \hat{\xi}, \lambda) = 0$ $(|\hat{\xi}| \neq 0)$ *has* μ *real and* ν *non-real roots* (resp. μ *roots with* $\operatorname{Im}\lambda \geqslant 0$ *and* ν *roots with* $\operatorname{Im}\lambda < 0$) $(\mu + \nu = m,\ \nu \geqslant 1)$, *and* u *is a* C^∞ *solution of the equation* $p(x, t; D_x, D_t)u = 0$ *defined in a neighborhood of the origin in* R^{n+1} (resp. *in* $R^{n+1} \cap (t \geqslant 0)$) *such that* $D_t^j u(x, 0) = 0$ *for* $0 \leqslant j \leqslant \mu - 1$. *Then* $(0, \hat{\xi})$ *is in the complement of wave front set* $WF_s(D_t^j u(x, 0))$ $(\mu \leqslant j \leqslant m-1)$.

Consider the following problem

$$(P)_k = \begin{cases} p(x, t; D_x, D_t)u = 0 \\ D_t^j u(x, 0) = u_j(x) \qquad 0 \leqslant j \leqslant k-1 \quad (k \leqslant m) \end{cases}$$

then by the theorem 2.1, we have

Corollary 2.1. *If the problem* $(P)_k$ *has a* C^∞*-solution in a neighborhood of the origin for any given* $(u_0(x), \dots, u_{k-1}(x)) \in \prod^{k} \gamma^{(t)}(R^n)$ $(s < t)$, *then the characteristic equation* $p_0(0, 0; \xi, \lambda) = 0$ *must have more than* k *real roots for every* $\xi \neq 0$.

Corollary 2.2. *Assume that* $s = 1$ *and* $p_0(0, 0; \xi, \lambda) = 0$ *has at least* k *roots with negative imaginary parts for any* $\xi \neq 0$. *If* u *is a* C^∞*-solution of* $(P)_{m-k}$ *defined in a neighborhood of the origin in* $R^{n+1} \cap (t \geqslant 0)$ *with analytic* u_j $(0 \leqslant j \leqslant m-k-1)$, *then the Cauchy data of* u i.e. $D_t^j u(x, 0)$ $(0 \leqslant j \leqslant m-1)$ *is analytic at the origin and the radius of convergence only depends on* $p(x, t; D_x, D_t)$, u_j $(0 \leqslant j \leqslant m-k-1)$ *and the size of definition domain of* u.

Remark: When the case $s=1$, $k=m$ in the corollary 2.1, more detailed result is obtained in [4], by constructing the exact solutions.

Definition 2.3. *The Cauchy problem $(P)_m$ is said to be $\gamma^{(s)}$-well posed $(s\geqslant 1)$ in a neighborhood of the origin, if there exists a neighborhood D of the origin such that the problem*

$$(2.1)\qquad \begin{cases} p(x, t; D_x, D_t)u=0 & \text{in } D \\ D_t^j u(x, 0)=u_j(x) \quad 0\leqslant j\leqslant m-1 & \text{in } D\cap(t=0) \end{cases}$$

has a solution $u\in C^\infty(D)$ for any given initial data

$$(u_0(x),\dots, u_{m-1}(x))\in\prod^m \gamma^{(s)}(R^n).$$

Theorem 2.2. *For the Cauchy problem $(P)_m$ to be $\gamma^{(s)}$-well posed in a neighborhood of the origin, it is necessary that the characteristic equation $p_0(0, 0; \xi, \lambda)=0$ has only real roots for any $\xi\neq 0$.*

Corollary 2.3. (c.f. [5]) *Suppose that $s=1$, and the characteristic equation $p_0(0, 0; 1, 0,\dots, 0, \lambda)=0$ has at least one non-real root. Then for any open neighborhood W of the origin in R^{n+1}, there is an analytic initial data on R^n which is independent of $(x_2,\dots, x_n)$ such that the corresponding solution of the Cauchy problem $(P)_m$ cannot be continued analytically whole in W.*

3. Fundamental lemmas

Let W be an open set in R^{n+1}, and Γ be a conic open set in $R^{n+1}\backslash 0$. We write $y=(x, t)$, $\eta=(\xi, \lambda)$ and $|\eta|^2=|\xi|^2+|\lambda|^2$.

Definition 3.1. ([2]) We shall say that the formal sum $p=\sum_{k=0}^{\infty} p_k(y, \eta)$ is a symbol on $W\times\Gamma$ of class s with order (r_1, r_2), if each $p_k(y, \eta)$ is a smooth function on $W\times\Gamma$, homogeneous degree r_1+r_2-k with respect to η and then there exist constants C, A such that for any integer k, any multi-indexes α, β, any $(y, \eta)\in W\times\Gamma$, the following inequality holds

$$(3.1)\qquad |p_{k(\beta)}^{(\alpha)}(y, \eta)|\leqslant CA^{k+|\alpha+\beta|}|\eta|^{r_2}|\xi|^{r_1-k-|\alpha|}(k+|\beta|)!^s\alpha!,$$

where we have set

$$p_{k(\beta)}^{(\alpha)}(y, \eta)=\left(\frac{1}{i}\frac{\partial}{\partial y}\right)^\beta\left(\frac{\partial}{\partial\eta}\right)^\alpha p_k(y, \eta).$$

Let $p=\sum_{k=0}^{\infty} p_k(y, \eta)$ be a symbol on $W\times\Gamma$ of class s with order (r_1, r_2). Following [2] we set

$$(3.2)\qquad N(p, T)=\sum_{k,\alpha,\beta}\frac{2(2n)^{-k}k!}{(k+|\alpha|)!(k+|\beta|)!^s}\|p_{k(\beta)}^{(\alpha)}\|T^{2k+|\alpha+\beta|},$$

where $\|p_{k(\beta)}^{(\alpha)}\|=\sup_{W\times\Gamma}\{|\eta|^{-r_2}|\xi|^{-r_1+k+|\alpha|}|p_{k(\beta)}^{(\alpha)}(y, \eta)|\}$, then the definition 3.1 is equivalent to that the power series (3.2) converges for small $T>0$. If we define $r=p\circ q$ by

$$(3.3)\qquad r=\sum_{k=0}^{\infty} r_m(y, \eta), \quad r_m(y, \eta)=\sum_{k+l+|\gamma|=m} \frac{1}{\gamma!} p_k^{(\gamma)} q_{l(\gamma)}$$

then we have by lemma 1.2 in [2] (or rather by its proof)

Lemma 3.1. *Let p, q be symbols on $W\times\Gamma$ of class s with order (r, m), then $p+q$ is a symbol on $W\times\Gamma$ of class s with order (r, m) and we have*

$$(3.4)\qquad N(p+q, T)\ll N(p, T)+N(q, T).$$

Let p, q be symbols on $W\times\Gamma$ of class s with order (r_1, m_1), (r_2, m_2) respectively, then the inequality

$$(3.5)\qquad N(p\circ q, T)\ll N(p, T)N(q, T)$$

holds, i.e. *$p\circ q$ is a symbol on $W\times\Gamma$ of class s with order (r_1+r_2, m_1+m_2).*

Remark: If we set $s=pq$, $s=\sum_{m=0}^{\infty} s_m(y, \eta)$, $s_m=\sum_{k+l=m} p_k p_l$, the proof of the lemma 3.1 shows

$$(3.6)\qquad N(p\circ q-pq, T)\ll 2N'(p, T)N'(q, T),$$

where $$N'(p, T)=\sum_{k,|\alpha+\beta|\geq 1} \frac{2(2n)^{-k}k!}{(k+|\alpha|)!(k+|\beta|)!^s} \| p_{k(\beta)}^{(\alpha)} \| T^{2k+|\alpha+\beta|}.$$

Furthermore, obvious inequality $N'(p, T)\ll N(p, T)$ means

$$(3.7)\qquad N(pq, T)\ll 2N(p, T)N(q, T).$$

Lemma 3.2. *Suppose that $p_0(0, 0; \hat{\xi}, \lambda)=0$ $(|\hat{\xi}|\neq 0)$ has μ real roots and ν non-real roots $(\mu+\nu=m)$. Then there are a neighborhood W of 0 in R^{n+1}, a conic neighborhood Γ of $\hat{\xi}$ in $R^n\backslash 0$ and symbols a^j $(1\leqslant j\leqslant\mu)$, b^i $(1\leqslant i\leqslant\nu)$ on $W\times(\Gamma\times R)$ which are independent of λ, of class s with order $(j, 0)$, $(i, 0)$ respectively, and satisfy the equation*

$$(3.8)\qquad p(x, t; \xi, \lambda)=(\lambda^\mu+\sum_{j=1}^{\mu} a^j(x, t; \xi)\lambda^{\mu-j})\circ(\lambda^\nu+\sum_{i=1}^{\nu} b^i(x, t; \xi)\lambda^{\nu-i})$$

as symbols on $W\times(\Gamma\times R)$. Where, $\lambda^\mu+\sum_{j=1}^{\mu} a_0^j(0, 0; \hat{\xi})\lambda^{\mu-j}=0$, $\lambda^\nu+\sum_{i=1}^{\nu} b_0^i(0, 0; \hat{\xi})\lambda^{\nu-i}=0$ has only real and non-real roots respectively.

Lemma 3.3. *Under the same condition in the lemma 3.2 there exists a neighborhood W of 0 in R^{n+1}, a conic neighborhood Γ of $\hat{\xi}$ in $R^n\backslash 0$, and symbols q, r on $W\times(\Gamma\times R)$ which satisfy followings.* i.e. *$p\circ q=r$, where $r=\lambda^\mu+\sum_{j=1}^{\mu} a^j(y, \xi)\lambda^{\mu-j}$ is the same one in the lemma 3.2 and q is of class s with order $(0, -\nu)$. Moreover, for $k+|\alpha|\geqslant 1$, $(y, \eta)\in W\times(\Gamma\times R)$, the inequality*

$$(3.9)\qquad |q_{k(\beta)}^{(\alpha)}(y, \eta)|\leqslant CA^{k+|\alpha+\beta|}|\eta|^{-\nu-1}|\xi|^{1-k-|\alpha|}(k+|\beta|)!^s\alpha!$$

holds.

4. Proof of the theorem 2.1

Admitting the lemma 3.3, we shall prove the theorem 2.1 when u is defined in a neighborhood of the origin in R^{n+1}. For the symbol $p=\sum_{k=0}^{\infty} p_k(y, \eta)$, following [1], we define the differential operator $P_l(y, \eta; D)$ by

$$P_l(y, \eta; D)=\sum_{k+|\alpha|=l} \frac{1}{\alpha!} p_k^{(\alpha)}(y, \eta) D^\alpha. \tag{4.1}$$

If q is another symbol and $p\circ q=r$, then the identity

$$R_l(y, \eta; D)=\sum_{j=0}^{l} P_{l-j}(y, \eta; D) Q_j(y, \eta; D) \tag{4.2}$$

is easily verified.

Remark: In the case when p is a polynomial of order m,

$$e^{-i<y,\eta>} p(y, D)(e^{i<y,\eta>} v)=\sum_{j=0}^{m} P_j(y, \eta; D) v$$

is valid.

By the lemma 2.2 in [1], for any compact sets K, $\tilde{K}$ ($K \Subset \tilde{K}$) in R^{n+1}, we can find a sequence $v_N(y) \in C_0^\infty(R^{n+1})$ such that $\operatorname{supp}[v_N] \subset \tilde{K}$, $v_N=1$ on K and

$$|D^\alpha v_N(y)| \leqslant C A^{|\alpha|} N^{|\alpha|} \tag{4.3}$$

when $|\alpha| \leqslant N$, where C, A is independent of N ($N=1, 2, \ldots$).

Applying the lemma 3.3 to the transposed operator ${}^t p$ of p, we get two symbols q, r on $W \times (\Gamma \times R)$. If u is a solution in the theorem 2.1 we take v_N such that $\operatorname{supp}[v_N] \subset W$, $p(y, D)u=0$ on $\operatorname{supp}[v_N]$, $D_t^j u(x, 0)=0$ on $\operatorname{supp}[v_N] \cap (t=0)$ ($0 \leqslant j \leqslant \mu-1$) and we set $w_N=\sum_{j=1}^{N-m} Q_j v_N$. Then (4.2) shows

$$e^{-i<y,\eta>} p(y, D)(e^{i<y,\eta>} w_N)=\sum_{j=0}^{N-m} R_j v_N+\sum_{\substack{N>k+l>N-m \\ N-m>k,\ m>l}} P_l Q_k v_N$$

After the integration by parts, we have

$$\int e^{i<y,\eta>}\Big(\sum_{j=0}^{N-m} R_j v_N\Big) u \, dy=-\int e^{i<y,\eta>}\Big(\sum_{\substack{N>k+l>N-m \\ N-m>k,\ m>l}} P_l Q_k v_N\Big) u \, dy, \tag{4.4}$$

Now estimate the right hand side in (4.4). We set $h=Q_k v_N$ and consider the integral

$$\int e^{i<y,\eta>}(P_l h) u \, dy.$$

For the term in $P_l h$ which includes λ^k ($k \geqslant 1$), we replace λ^k by $D_t^k(e^{i\lambda t})$ and integrate by parts, then, noting p is a polynomial of order m with coefficients in $\gamma^{(s)}$, we obtain the following estimate

$$\left|\int e^{i<y,\eta>}(P_l h)u\mathrm{d}y\right| \leqslant C|\xi|^m \sup_{\tilde{K}\, j\leqslant m} |D_t^j u| \sup_{\tilde{K}\, j+|\alpha|\leqslant m} |D_x^\alpha D_t^j h| \tag{4.5}$$

for $|\xi|\geqslant 1$, where constant C is only depends on $p(y, D)$ and $\tilde{K}$. In virtue of (3.9) in lemma 3.3 and (4.3), $D^\gamma(Q_k v_N)$ can be estimated as follows

$$|D^\gamma(Q_k v_N)| \leqslant CA^{2N}|\eta|^{-\nu-1}|\xi|^{1-k}N^{sN} \tag{4.6}$$

for any integer N, any k $(1\leqslant k\leqslant N-m)$, any $\gamma(|\gamma|\leqslant m)$, and any $\xi\in\Gamma$. Then, (4.5) and (4.6) show that for $k\geqslant 1$,

$$\left|\int e^{i<y,\eta>}\Big(\sum_{\substack{N>l+k>N-m\\ N-m>k,\, m>l}} P_l Q_k v_N\Big)u\mathrm{d}y\right| \leqslant CA^{2N}|\eta|^{-\nu-1}|\xi|^{2m+1-N}N^{sN}. \tag{4.7}$$

On the other hand, if we set $r=\lambda^\mu+s$, then s is a polynomial in λ of order $\mu-1$. Since $D_t^j u(x, 0)=0$ on the support of $v_N(x, 0)$ $(0\leqslant j\leqslant \mu-1)$, the Fourier inversion formula gives

$$\int \mathrm{d}\lambda \int e^{i<y,\eta>}\Big(\sum_{j=0}^{N-m} S_j v_N\Big)u\mathrm{d}y=0.$$

This shows that

$$\int \mathrm{d}\lambda \int e^{i<y,\eta>}\Big(\sum_{j=0}^{N-m} R_j v_N\Big)u\mathrm{d}y=(-1)^\mu \int e^{i<x,\xi>}v_N(x, 0)D_t^\mu u(x, 0)\mathrm{d}x. \tag{4.8}$$

Here we integrate (4.4) by λ, and combining this identity with (4.7), we have for $\xi\in\Gamma$

$$\left|\int e^{i<x,\xi>}D_t^\mu u(x, 0)v_N(x, 0)\mathrm{d}x\right| \leqslant CA^{2N}|\xi|^{2m+1-N}N^{sN}, \tag{4.9}$$

where A depends only on $p(y, D)$ and the size of definition domain of u. Inequality (4.9) means that $(0, \hat{\xi})$ is in the complement of $WF_s(D_t^\mu u(x, 0))$.

Multiplying (4.4) by λ and using the fact that $(0, \hat{\xi})\notin WF_s(D_t^\mu u(x, 0))$, we can show $(0, \hat{\xi})\notin WF_s(D_t^{\mu+1}u(x, 0))$ by the same process. Thus, the theorem follows from the finite number of iterations of this argument.

In the statement of the lemma 3.2 and 3.3, we can replace real roots (resp. non-real roots) by roots with non-positive imaginary parts (resp. roots with positive imaginary parts). Then, there exist a neighborhood W of 0 in R^{n+1}, a conic neighborhood Γ of $-\hat{\xi}$ in $R^n\backslash 0$ and symbols q, r on $W\times(\Gamma\times R)$ which satisfy $p\circ q=r$. Then, noting that $q_k(y;\xi,\lambda)$ is a rational function of λ with poles only in the upper half plane for fixed $y\in W$, $\xi\in\Gamma$, the same reasoning gives also the proof in the case when u is defined in a half neighborhood of the origin.

5. Proof of the theorem 2.2

Suppose that the Cauchy problem $(P)_m$ is $\gamma^{(s)}$-well posed in a neighborhood of 0 in R^{n+1}, and that the characteristic equation $p_0(0, 0; \xi, \lambda)=0$ $(|\xi| \neq 0)$ has at least one non-real root.

Without loss of generality we may assume that $p_0(0, 0; 1, 0, \ldots, 0, \lambda)=0$ has μ real roots and v non-real roots $(v \geqslant 1)$. Then we can easily find the sequence of initial data which is not compatible with (4.9).

For instance, we set

$$g_{s,\varepsilon}(x)=\int_0^\infty e^{ixt}e^{-\varepsilon t^{1/s}}dt \qquad \varepsilon>0,\ x\in R, \tag{5.1}$$

and consider the initial conditions

$$\begin{cases} D_t^j u(x, 0)=0 & \text{for} \quad 0 \leqslant j \leqslant m-1, \quad j \neq \mu \\ D_t^\mu u(x, 0)=g_{s,\varepsilon}(x_1). \end{cases} \tag{5.2}$$

It is clear that $g_{s,\varepsilon}(x_1) \in \gamma^{(s)}(R^n)$, and by the theorem 2.1, the Fourier transform of $g_{s,\varepsilon}(x_1)v_N(x, 0)$ is estimated such as (4.9) in $\pm\Gamma$. On the other hand, we see easily that

$$|\widehat{(g_{s,\varepsilon}\tilde{v}_N)}(\xi)| \leqslant C_\varepsilon(BN^s)^N|\xi'|^{-N} \tag{5.3}$$

where $\tilde{v}_N(x)=v_N(x, 0)$, $\xi'=(0, \xi_2, \ldots, \xi_n)$ and B is independent of ε. By (4.9) (in $\pm\Gamma$) and (5.3) (in the complement of $\pm\Gamma$) we have

$$|\widehat{(g_{s,\varepsilon}\tilde{v}_N)}(\xi)| \leqslant C_\varepsilon(AN^s)^N|\xi|^{-N} \tag{5.4}$$

for any $\xi(|\xi| \geqslant N^s)$, any integer N. Where A is independent of ε.

The estimate (5.4) shows, with another constant A which also independent of ε, that

$$\left|\left(\frac{\partial}{\partial x_1}\right)^k (g_{s,\varepsilon}\tilde{v}_N)(x)\right| \leqslant C_\varepsilon A^k k!^s, \tag{5.5}$$

for any integer k. In (5.1), setting $x=0$, we obtain, the other hand,

$$\left(\frac{\partial}{\partial x_1}\right)^k g_{s,\varepsilon}(x_1)_{|x_1=0}=s\left(\frac{1}{\varepsilon}\right)^{s(k+1)}\Gamma(s(k+1))$$

Then, taking into account of the Stirling's formula, this leads us to a contradiction.

6. Proof of the lemma 2.2

Let Γ be an open conic set in $R^n\backslash 0$, and W be an open neighborhood of the origin in R^{n+1}. In this section we only consider the symbols on $W\times(\Gamma\times R)$ which are independent of λ.

Let $p=\sum_{k=0}^{\infty} p_k(y, \xi)$ be a symbol, then we shall denote by $p_{[v]}$ the symbol which is defined as follows

$$p_{[v]}=\sum_{k=0}^{\infty}(D_t^v p_k)(y, \xi). \tag{6.1}$$

By the definition (3.1), $p_{[v]}$ is also the symbol with the same class and order those of p.

For the simplicity, we introduce the following notation:

$$C_{k,v,\beta}^{\alpha}=\frac{2(2n)^{-k}k!}{(k+|\alpha|)!(k+v+|\beta|)!^s}, \tag{6.2}$$

we also denote $C_{k,0,\beta}^{\alpha}=C_{k,\beta}^{\alpha}$, then we have

Proposition 6.1. *Let p, q be symbols of class s, and set ${}^v r=p\circ q_{[v]}$ then, there are constants $C_{v,j}$ $(0\leqslant j\leqslant v)$ such that*

$$\sum_{k,\alpha,\beta} C_{k,v,\beta}^{\alpha}\|({}^v r)_{k(\beta)}^{(\alpha)}\| T^{2k+v+|\alpha+\beta|} \ll \sum_{j=0}^{v} C_{v,j}T^j N(p_{[v]}, T)N(q, T), \tag{6.3}$$

where $C_{v,j}$ is independent of p, q and T.

Proof. By virtue of the lemma 3.1, this is easily shown by induction on v.

Corollary 6.1. *Let ${}^v r=p\circ q_{[v]}$, $v=v_1+v_2$ $(v_1, v_2, v\in N)$, then we have*

$$\sum_{k,v,\beta}\|({}^v r)_{k(\beta)}^{(\alpha)}\| T^{2k+v+|\alpha+\beta|}\times C_{k,v,\beta}^{\alpha} \tag{6.4}$$

$$\ll T^{v_1}\sum_{j=0}^{v_2} C_{v_2,j}T^j N(p_{[j]}, T)N(q_{[v_1]}, T).$$

By the assumption of the lemma 3.2, we can find the symbols $a_0^j(y, \xi)$, $b_0^i(y, \xi)$ $(1\leqslant j\leqslant\mu,\ 1\leqslant i\leqslant\nu)$ on $W\times(\Gamma\times R)$ of class s, homogeneous degree j, i respectively in ξ, and satisfying the equation

$$p_0(y;\xi,\lambda)=(\lambda^{\mu}+\sum_{j=1}^{\mu} a_0^j\lambda^{\mu-j})\times(\lambda^{\nu}+\sum_{i=1}^{\nu} b_0^i\lambda^{\nu-i}), \tag{6.5}$$

where W is a neighborhood of 0, and Γ is a conic neighborhood of $\hat{\xi}$. Furthermore, we may assume that for any root $\tau(y, \xi)$ of the equation $\lambda^{\nu}+\sum_{i=1}^{\nu} b_0^i(y, \xi)\lambda^{\nu-i}=0$, the inequalities

$$\begin{cases} |\operatorname{Im}\tau(y, \xi)|\geqslant c_1 \quad (>0) \\ |\tau(y, \xi)^{\mu}+\sum_{j=1}^{\mu} a_0^j(y, \xi)\tau(y, \xi)^{\mu-j}|\geqslant c_2 \quad (>0) \end{cases} \tag{6.6}$$

hold, when $y\in W$, $\xi\in\Gamma\cap(|\xi|=1)$.

We write $a^j=a_0^j+A^j$, $b^i=b_0^i+B^i$, and find A^j, B^i in the class s for which the identity (3.8) holds. For this purpose, we introduce the vector-valued symbol

$C={}^t(A^1,\dots, A^\mu, B^1;\dots, B^\nu)$, and say that the symbol C is of class s with order 0 if each A^j, B^i is a symbol of class s with order $(j, 0)$, $(i, 0)$ respectively, and $A_0^j=B_0^i=0$. We set

$$L(C)=\sum_{p=0}^{\infty} L_p(C)=\sum_{p=0}^{\infty} {}^t(L_{1,p}(C),\dots, L_{m,p}(C))$$

$$\begin{aligned}(6.7)\quad L_{n,p}(C) = &\sum_{t=1}^{\min(n,p)} \sum_{\substack{i+j=n-t\\ j>1}} \sum_{\substack{l+|\alpha|=p-t\\ l>1}} \frac{1}{\alpha!} C_t^{\mu-j} a_0^{j(\alpha)} B^i_{l(\alpha,t)}\\ &+\sum_{t=1}^{\min(n,p)} \sum_{\substack{i+j=n-t\\ j>1}} \sum_{\substack{k+|\alpha|=p-t\\ k>1}} \frac{1}{\alpha!} C_t^{\mu-j} A_k^{j(\alpha)} b^i_{0(\alpha,t)}\\ &+\sum_{t=1}^{\min(n,p)} \sum_{\substack{i+j=n-t\\ i,j>1}} \sum_{\substack{k+l+|\alpha|=p-t\\ l,k>1}} \frac{1}{\alpha!} C_t^{\mu-j} A_k^{j(\alpha)} B^i_{l(\alpha,t)}\end{aligned}$$

$$M(C)=\sum_{p=0}^{\infty} M_p(C)=\sum_{p=0}^{\infty} {}^t(M_{1,p}(C),\dots, M_{m,p}(C))$$

$$\begin{aligned}(6.8)\quad M_{n,p}(C)= &\sum_{\substack{i+j=n\\ i>1}} \sum_{\substack{l+|\alpha|=p\\ 1<l<p-1}} \frac{1}{\alpha!} a_0^{j(\alpha)} B^i_{l(\alpha)}\\ &+\sum_{\substack{i+j=n\\ j>1}} \sum_{\substack{k+|\alpha|=p\\ 1<k<p-1}} \frac{1}{\alpha!} A_k^{j(\alpha)} b^i_{0(\alpha)}+\sum_{\substack{i+j=n\\ i,j>1}} \sum_{\substack{k+l+|\alpha|=p\\ 1<k,l<p-1}} \frac{1}{\alpha!} A_k^{j(\alpha)} B^i_{l(\alpha)}\end{aligned}$$

where $B^i_{l(\alpha,\nu)}=D_x^\alpha D_t^\nu B_l^i$, $n=1, 2,\dots, m$, $p=1, 2,\dots,$ and $\sum_{i+j=k}=0$, if $k<0$, $C_i^j=0$, if $i>j$.

Then the equation (3.8) becomes

$$(6.8)\qquad \sum_{i+j=n} (a_0^j B_p^i + A_p^j b_0^i) = -L_{n,p}(C)-M_{n,p}(C)-F_{n,p}+G_{n,p}$$

where, $n=1, 2,\dots, m$, $p=1, 2,\dots,$

$$\begin{aligned}(6.10)\qquad F_{n,p}= &\sum_{i+j=n} \sum_{|\alpha|=p} \frac{1}{\alpha!} a_0^{j(\alpha)} b^i_{0(\alpha)}\\ &+\sum_{t=1}^{\min(n,p)} \sum_{i+j=n-t} \sum_{|\alpha|=p-t} \frac{1}{\alpha!} C_t^{\mu-j} a_0^{j(\alpha)} b^i_{0(\alpha,t)}\end{aligned}$$

and $G_{n,k}$ is the coefficient of λ^{m-n} in $p_k(y;\xi,\lambda)$ $(k=1, 2,\dots, m)$.

Denote by $H(y,\xi)$ the coefficient matrix of equation (6.9) then, $\det H(y,\xi)$ is the resultant of $\lambda^\mu+a_0^1\lambda^{\mu-1}+\cdots+a_0^\mu$ and $\lambda^\nu+b_0^1\lambda^{\nu-1}+\cdots+b_0^\nu$ as the polynomials of λ, then from (6.6), there is the inverse matrix $D(y,\xi)=(d_{i,j}(y,\xi))_{1<i,j<m}$ of $H(y,\xi)$, and for the entries of $D(y,\xi)$ we can show the followings.

$$(6.11)\quad \begin{cases}\text{For } 1\leqslant i\leqslant \mu,\ d_{ij}(y,\xi) \text{ is a symbol on } W\times(\Gamma\times R) \text{ of class } s \text{ and homogeneous degree } i-j \text{ in } \xi.\\ \text{For } \mu+1\leqslant i\leqslant m,\ d_{ij}(y,\xi) \text{ is a symbol on } W\times(\Gamma\times R) \text{ of class } s \text{ and homogeneous degree } i-j-\mu \text{ in } \xi.\end{cases}$$

We rewrite the equation (6.9) in the matrix form by use of $D(y, \xi)$, namely

$$C=-D(L(C)+M(C))-DF+DG. \tag{6.12}$$

The proof of the lemma 3.2 is achieved in solving the equation (6.12) by successive approximation.

For the vector-valued symbol C, we introduce the following formal norms,

$$N(C, T)=\sum_{j=1}^{\mu} N(A^j, T)+\sum_{i=1}^{\nu} N(B^i, T) \tag{6.13}$$

$$N_m(C, T)=\sum_{j=1}^{\mu}\sum_{v=0}^{m} N(A^j_{[v]}, T)+\sum_{i=1}^{\nu}\sum_{v=0}^{m} N(B^i_{[v]}, T)$$

and at first we shall show that if C is a symbol of class s with order 0, then the right hand side of (6.12) defines a symbol of the same order and class. We consider the most delicate part of $L_{n,p}(C)$. Let us denote

$$\begin{aligned} L^3_{n,p}(C) &= \sum_{t=1}^{\min(n,p)} \sum_{\substack{i+j=n-t\\ i,j>1}} \sum_{\substack{k+l+|\alpha|=p-t\\ k,l>1}} \frac{1}{\alpha!} C^{\mu-j}_t A^{j(\alpha)}_k B^i_{l(\alpha,t)} \\ &= \sum_{t=1}^{\min(n,p)} \sum_{\substack{i+j=n-t\\ i,j>1}} C^{\mu-j}_t (A^j \circ B^i_{[t]})_{p-t} \end{aligned}$$

then for $v \leqslant m$, we see easily

$$\begin{aligned} &\sum_{p,\alpha,\beta} C^{\alpha}_{p,\beta} \| (D^v_t L^3_{n,p})^{(\alpha)}_{(\beta)} \| T^{2p+|\alpha+\beta|} \\ &\ll \sum_{\varphi=0}^{v} C^{v}_{\varphi} \sum_{t=1}^{n} \sum_{\substack{i+j=n-t\\ i,j>1}} C^{\mu-j}_t (2n)^{-t} T^t \sum_{p>t,\alpha,\beta} C^{\alpha}_{p-t,t,\beta} \| ({}^{\varphi}\gamma)^{(\alpha)}_{p-t(\beta)} \| \times \\ &\qquad\qquad \times T^{2(p-t)+t+|\alpha+\beta|} \\ &\ll \sum_{\varphi=0}^{v} C^{v}_{\varphi} \sum_{t=1}^{n} \sum_{\substack{i+j=n-t\\ i,j>1}} C^{\mu-j}_t (2n)^{-t} T^t \sum_{k,\alpha,\beta} C^{\alpha}_{k,\beta} \| ({}^{\varphi}\gamma)^{(\alpha)}_{k(\beta)} \| T^{2k+t+|\alpha+\beta|}, \end{aligned}$$

where we have set ${}^{\varphi}\gamma = A^j_{[v-\varphi]} \circ B^i_{[t+\varphi]}$ which also depends on t, i, and j. We divide the last term into two parts:

$$\sum_{\substack{t,\varphi\\ t+\varphi>m}} C^{v}_{\varphi} \sum \cdots + \sum_{\substack{t,\varphi\\ t+\varphi<m}} C^{v}_{\varphi} \sum \cdots$$

In the first term, setting $t+\varphi=m+t_1$ $(t \geqslant t_1)$, we apply the corollary 6.1, then we have

$$\begin{aligned} \sum_{\substack{t,\varphi\\ t+\varphi>m}} C^{v}_{\varphi} \cdots \ll \sum_{\substack{t,\varphi\\ t+\varphi>m}} C^{v}_{\varphi} \sum_{\substack{i+j=n-t\\ i,j>1}} C^{\mu-j}_t (2n)^{-t} T^{2t-t_1} \sum_{\theta=0}^{t_1} C_{t_1,\theta} T^{\theta} \times \\ \times N(A^j_{[v-\varphi+\theta]}, T) N(B^i_{[m]}, T). \end{aligned}$$

The second term is estimated by the lemma 3.1 as follows,

$$\sum_{\substack{t,\varphi \\ t+\varphi<m}} C_\varphi^v \cdots \ll \sum_{\substack{t,\varphi \\ t+\varphi<m}} C_\varphi^v \sum_{\substack{i+j=n-t \\ i,j\geq 1}} C_t^{\mu-j}(2n)^{-t}T^{2t}N(A^j_{[v-\varphi]}, T)N(B^i_{[t+\varphi]}, T)\,.$$

On the other hand, $N(M_n(C)_{[v]}, T)\,(v\leqslant m)$ is majorized by $TE(T)N_m(C, T)$ $+EN_m(C, T)N_m(C, T)$ in virtue of the lemma 3.1 and (3.7). Therefore, taking account of $F_{n,0}=G_{n,0}=0$ $(1\leqslant n\leqslant m)$, we can show that the right side of (6.12) is estimated by

$$TE_1(T)N_m(C, T)+E_2(T)N_m(C, T)N_m(C, T)+T^2E_3(T) \tag{6.14}$$

where $E_i(T)$ is the power series of T which converges for small $T>0$, does not depend on C. This shows that the right hand side of (6.12) defines a symbol of class s with order 0. Moreover, the estimate (6.14) means the followings,

$$\left\{\begin{array}{l}\text{For sufficiently small } \varepsilon>0, \text{ there is a } \delta>0, \text{ such that } N_m({}^\lambda C, T)\leqslant\varepsilon \text{ for} \\ \text{any } 0\leqslant T\leqslant\delta \text{ and any } \lambda=0, 1, 2,\ldots,\end{array}\right. \tag{6.15}$$

where ${}^\lambda C$ is defined successively by ${}^{\lambda+1}C=-D(L({}^\lambda C)+M({}^\lambda C))-DF+DG$ ${}^0C=0$, $\lambda=1, 2, 3,\ldots.$

By the analogous estimate, we can majorize $N_m({}^{\lambda+2}C-{}^{\lambda+1}C, T)$ by $T\tilde{E}_1(T)\cdot$ $N_m({}^{\lambda+1}C-{}^\lambda C, T)+\tilde{E}_2(T)(N_m({}^\lambda C, T)+N_m({}^{\lambda+1}C, T))N_m({}^{\lambda+1}C-{}^\lambda C, T)$, where $\tilde{E}_i(T)$ does not depend on C. Therefore this estimate and (6.15) show that the successive approximation converges.

7. Proof of the lemma 3.3

From the lemma 3.2, we can decompose p in the form (3.8), then the proof of the lemma 3.3 is carried out by construct the inverse symbol of $r=\lambda^v+\sum_{i=1}^{v} b^i\lambda^{v-i}$. By (6.8) and the homogeneity, we may suppose that

$$|r_0(y;\, \xi,\, \lambda)|\geqslant c_0^{-1}|\eta|^v, \tag{7.1}$$

for any $y\in W$, any $\eta\in\Gamma\times R$, where c_0 is a positive constant.

Proposition 7.1. *If we set* $q_0(y;\eta)=1/r_0(y;\eta)$, *then* q_0 *is a symbol on* $W\times(\Gamma\times R)$ *of class s with order* $(0, -v)$. *Furthermore, there are constants* C, A *such that*

$$|q^{(\alpha)}_{0(\beta)}(y\,;\,\eta)|\leqslant CA^{|\alpha+\beta|}|\eta|^{-v-1}|\xi|^{1-|\alpha|}|\beta|!^s\alpha! \tag{7.2}$$

is valid when $|\alpha+\beta|\geqslant 1$, *for any* $(y;\eta)\in W\times(\Gamma\times R)$.

Proof. At first, we remark that the symbol with order $(j, v-j)$ is also the symbol with order $(0, v)$ for any non-negative integer j. Therefore, by this note and the obvious inequality

$$|(\lambda^v)^{(\alpha)}_{(\beta)}|\leqslant CA^{|\alpha+\beta|}|\eta|^{v-1}|\xi|^{1-|\alpha|}|\beta|!^s\alpha!, \qquad \text{for} \quad |\alpha+\beta|\geqslant 1,$$

we have for $|\alpha+\beta|\geqslant 1$, (7.3) $$|r_{0(\beta)}^{(\alpha)}(y;\eta)|\leqslant C\left(\frac{A}{3}\right)^{|\alpha+\beta|}|\eta|^{\nu-1}|\xi|^{1-|\alpha|}|\beta|!^s\alpha!.$$

Now, we shall show by induction on $|\alpha+\beta|$ that

(7.4) $$|q_{0(\beta)}^{(\alpha)}(y;\eta)|\leqslant C_1{}^{|\alpha+\beta|+1}A^{|\alpha+\beta|}|\eta|^{-\nu-1}|\xi|^{1-|\alpha|}|\beta|!^s|\alpha|!$$

holds on $W\times(\Gamma\times R)$ for $|\alpha+\beta|\geqslant 1$. Suppose that (7.4) is valid when $1\leqslant|\alpha+\beta|\leqslant p$, and prove it when $|\alpha+\beta|=p+1$. From the identity

$$0=(r_0q_0)_{(\beta)}^{(\alpha)}=\sum C_\gamma^\alpha C_\nu^\beta r_{0(\beta-\nu)}^{(\alpha-\gamma)}q_{0(\nu)}^{(\gamma)},$$

we obtain

$$q_{0(\beta)}^{(\alpha)}=-\sum C_\gamma^\alpha C_\nu^\beta r_0^{-1}r_{(\beta-\nu)}^{(\alpha-\gamma)}q_{0(\nu)}^{(\gamma)},$$

then by the induction hypothesis show

$$\begin{aligned}|q_{0(\beta)}^{(\alpha)}|&\leqslant c_0CC_1^{|\alpha+\beta|}|\eta|^{-\nu-1}|\xi|^{1-|\alpha|}A^{|\alpha+\beta|}\times\\&\quad\times\sum_{p=0}^{|\alpha|}\sum_{q=0}^{|\beta|}C_p^{|\alpha|}(|\alpha|-p)!\left(\frac{1}{3}\right)^{|\alpha|-p}C_q^{|\beta|}(|\beta|-q)!^sq!^s\left(\frac{1}{3}\right)^{|\beta|-q}\\&\leqslant(4c_0C)C_1^{|\alpha+\beta|}A^{|\alpha+\beta|}|\eta|^{-\nu-1}|\xi|^{1-|\alpha|}|\beta|!^s|\alpha|!.\end{aligned}$$

Therefore, if we choose $C_1\geqslant 4c_0C$, the proof is complete. (c.f. lemma 3.1 in [3])

Proposition 7.2 *If we set* $r\circ q_0=1-h$ $(q_0=1/r_0)$, *then* h *is a symbol of class* s *with order* $(1,-1)$ *such that*

(7.5) $$N(h,T)\ll T^2C(T)$$

holds.

Proof. If we set $r=\lambda^\nu+s$, and rewrite $r\circ q_0-1=(\lambda^\nu\circ q_0-\lambda^\nu q_0)+(s\circ q_0-s_0q_0)$, then from the preceding proposition, the second term is of order $(1,-1)$, furthermore, by the definition (3.2) we have

$$N(s\circ q_0-s_0q_0,T)=\sum_{k>1,\alpha,\beta}C_{k,\beta}^\alpha\|(s\circ q_0)_{k(\beta)}^{(\alpha)}\|T^{2k+|\alpha+\beta|}\ll T^2C(T),$$

On the other hand, if we denote $\tau=\lambda^\nu\circ q_0-\lambda^\nu q_0$, then the proof of the lemma 3.1 shows that

$$\sum_{k,\alpha,\beta}C_{k,\beta}^\alpha\sup\{|\eta||\xi|^{-1+k+|\alpha|}|\tau_{k(\beta)}^{(\alpha)}|\}\ll N'(\lambda^\nu,T)N'(q_0,T)\ll T^2C(T).$$

Proof of the lemma 3.3. We write $h^1=h,\ldots$

$$h^p=\underbrace{h\circ h\circ h\cdots h\circ h}_{p\text{-times}}$$

then by the proposition 7.2 and the lemma 3.1, the series

$$\tau=h+h^2+\cdots+h^p+\cdots$$

defines a symbol of class s with order $(1, -1)$. Noting that the formal composition is associative, we have the lemma 3.3 by setting $q = q_0 \circ (1+\tau) = q_0 + q_0 \circ \tau$. The estimate (3.9) follows from (7.2) and the fact that τ is a symbol with order $(1, -1)$.

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