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On codimension one foliations defined by closed one forms with singularities

By

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§ 0. Introduction

Let M be a closed manifold, ω a closed one form of class C^1 on M with singularities $\Sigma=\{x|\omega_x=0\}$ and $\mathscr{F}$ a codimension one foliation on $M-\Sigma$ defined by $\omega=0$. Let Per (ω) be the set of periods of ω i.e. Per $(\omega)=\left\{\int_c \omega|c\text{: closed curve in } M\right\}$. This is a Z-module and we define rank ω by the rank of Per (ω). If $\Sigma=\phi$ the properties of $\mathscr{F}$ are well known (see [2], [4], [5], [6]) in particular all leaves of $\mathscr{F}$ are compact or everywhere dense according to rank $\omega=1$ or $\geqq 2$. The purpose of this note is to generalize this property to the case of $\Sigma\neq\phi$.

To state the theorem we make some definitions. A leaf L of $\mathscr{F}$ is called *singular* if there exists $p\in\Sigma$ such that, for any neighborhood U of p and a function f on U such that $\omega|U=df$, we have $L\cap U\cap f^{-1}(f(p))\neq\phi$. A leaf L is called *compact* if $L\cup\Sigma$ is compact. We say that ω has *generalized isolated singularities* if there exist a neighborhood U of Σ and a function g on U such that $\omega|U=dg$ and the set of singular values of g is isolated. In this case we suppose, by choosing U and g suitably, that Σ is contained in $g^{-1}(0)$.

Our result is as follows.

Theorem 1. *If M is closed and ω has generalized isolated singularities then any non-compact leaf of $\mathscr{F}$ is locally dense and if $\mathscr{F}$ has a locally dense leaf then rank $\omega\geqq 2$.*

If M is an orientable closed surface, let Ω be a volume form on M. It is easy to see that the corespondence $X\leftrightarrow i_X\Omega$, where i_X is the inner product, between vector fields and one forms on M is one to one and X preserves the volume form Ω if and only if $i_X\Omega$ is closed. Moreover the orbits of X are the leaves of the foliation defined by $i_X\Omega=0$. So as a corollary of Theorem 1 we have the following (well known for the case of non-degenerate singularities) result.

Theorem 2. *Let X be a volume preserving vector field on a closed orientable two manifold and suppose that $i_X\Omega$ has generalized isolated singularities. Then any non-trivial orbit of X is periodic or a separatrix joining singular points or locally dense.*

For the proof of Theorem 1 we take a vector field X on $M-\Sigma$ satisfying $\omega(X)\equiv 1$ and consider the local one parameter transformation $\varphi(x, t)$ generated by X. Then $\varphi(\ , t)$ preserves the foliation $\mathscr{F}$ locally, but X is not complete and we see that, for any non-compact leaf L, $\bar{L}$ contains a non-compact singular leaf (Propositions 1.8. and 1.10.). So for the proof of Theorem 1. it is sufficient to show that a non-compact singular leaf is locally dense. This is done in § 2.

If we assume that $\pi_1(M)$ is abelian, then the proof of Theorem 1 becomes very simple and a more accurate description of $\mathscr{F}$ is possible. This will be done in a subsequent paper [3] under more general situations.

§ 1. Preliminaries

In this section we assume M is closed and ω has generalized isolated singularities.

Definition 1.1. A compact codimension zero submanifold D of M is called a *regular neighborhood* of Σ if D satisfies the following conditions.

(i) $U\supset D\supset \mathrm{Int}\, D\supset\Sigma$.

(ii) ∂D is transverse to $g^{-1}(0)$.

(iii) For a connected component A of $D\cap g^{-1}(0)$ we have $\bar{A}\cap\Sigma\neq\phi$.

It is easy to see that for any neighborhood U of Σ there exists a regular neighborhood contained in U. In the sequel we fix a regular neighborhood D.

Definition 1.2. We call $D(\varepsilon)=D\cap g^{-1}([-\varepsilon, \varepsilon])$ a *regular ε-neighborhood* of Σ if $\mathscr{F}$ is transverse to $\partial D\cap g^{-1}(-\varepsilon', \varepsilon')$ for some $\varepsilon'>\varepsilon$. We call each connected component of $W(\varepsilon)=\partial D\cap g^{-1}([-\varepsilon, \varepsilon])$ an *ε-wall* of $D(\varepsilon)$ and we define $W^+(\varepsilon)=\partial D\cap g^{-1}([0, \varepsilon])$, $W^-(\varepsilon)=\partial D\cap g^{-1}([-\varepsilon, 0])$ and $W(0)=\partial D\cap g^{-1}(0)$. $D^{\pm}(\varepsilon)$ are defined similarly.

For sufficiently small ε, $D(\varepsilon)$ is a regular ε-neighborhood of Σ, and if $D(\varepsilon)$ is regular then for $\varepsilon'<\varepsilon$ $D(\varepsilon')$ is also regular. There is a one to one correspondence between connected components of $D(\varepsilon)$ and $D(\varepsilon')$.

Definition 1.3. A connected component $D_i(\varepsilon)$ of ε-regular neighborhood of $D(\varepsilon)$ is called an *ε-cell.* We say $D_i(\varepsilon)$ a *separating cell* if $D_i(\varepsilon)\cap W(\varepsilon)$ is disconnected. A leaf L of $\mathscr{F}$ is called a *D-separatrice* if $g^{-1}(0)\cap L\cap D_i(\varepsilon)\neq\phi$ for some separating cell $D_i(\varepsilon)$. We denote $\mathscr{S}_D$ the set of D-separatrices and $S_D=\{x\in M\,|\,L_x\in\mathscr{S}_D\}$ where L_x is the leaf containing x. We define $SD(\varepsilon)=\bigcup D_i(\varepsilon)$ where the union is taken for all separating cells. $SD^{\pm}(\varepsilon)$ are defined similarly.

Clearly $\mathscr{S}_D$ is a finite set and this fact plays a crucial role in the proof of Theorem 1.

Definition 1.4. A vector field X of class C^1 on $M-\Sigma$ is called *$D(\varepsilon)$-regular* if $\omega(X)\equiv 1$ on $M-\Sigma$ and X is tangent to $W(\varepsilon)$. Let $D_X\subset(M-\Sigma)\times\boldsymbol{R}$ be the domain of maximal solutions $\varphi(x, t)$ of X under the initial conditions $\varphi(x, 0)=x$. Define $a^+(x)=\sup\{t|(x, t)\in D_X\}$, $a^-(x)=\sup\{t|(x, -t)\in D_X\}$ and for a subset A of $M-\Sigma$, $a^+(A)=\inf\{a^+(x)|x\in A\}$, $a^-(A)=\inf\{a^-(x)|x\in A\}$. For $x, y\in L$ we define $c^+(x, y)=\sup\{t|L_{\varphi(x,t')}=L_{\varphi(y,t')}$ for $0\leqq t'<t\}$. $c^-(x, y)$ is defined similarly.

Any vector field X on A, where A is a compact set of $M-D(\varepsilon)$, satisfying $\omega(X)\equiv 1$ on A can be extended to a $D(\varepsilon)$-regular vector field. The following lemmas are easy to prove.

Lemma 1.1. *Let L be a non-compact leaf of $\mathscr{F}$ then there exists a segment C transverse to $\mathscr{F}$ such that C is contained in $M-D$ and $L\cap C$ is an infinie set.*

Lemma 1.2. *We have the following properties.*

(i) *For $x, y\in L$, $0<c^+(x, y)=c^+(y, x)\leq\min(a^+(x), a^+(y))$.*

(ii) *For $x, y, z\in L$, $c^+(x, y)\geq\min(c^+(x, z), c^+(z, y))$.*

(iii) *For $x, y\in L$ and $a<a^+(x, y)$, set $x'=\varphi(x, a)$ and $y'=\varphi(y, a)$ then $c^+(x, y)=a+c^+(x', y')$.*

(iv) *For any curve l in L from x to y we have $a^+(l)\leq c^+(x, y)$.*

(v) *If A is a subset of $M-\mathrm{Int}\,D^-(\varepsilon)$ then $a^+(A)\geqq\varepsilon$.*

Definition 1.5. For a curve l: $[a, b]\to L$, a sequence $a<t_1<t_2<\cdots<t_{2n}<b$ is called the *$D^-(\varepsilon)$-partition* (or *$SD^-(\varepsilon)$-partition*) if l is transverse to $W(\varepsilon)$ and if $\bigcup_{i=1}^n(t_{2i-1}, t_{2i})=l^{-1}(\mathrm{Int}\,D^-(\varepsilon))$ (or $=l^{-1}(\mathrm{Int}\,SD^-(\varepsilon))=l^{-1}(\mathrm{Int}\,D^-(\varepsilon))$ respectively).

Lemma 1.3. *For $x, y\in L\cap(M-\mathrm{Int}\,D^-(\varepsilon))$, there exists a curve l: $[a, b]\to L$ from x to y which has the $SD^-(\varepsilon)$-partition. If l has the $SD^-(\varepsilon)$-partition then $a^+(l)=\min\{-g(l(t_i))|i=1, 2, \cdots, 2n\}$ if $n>0$ and $a^+(l)\geq\varepsilon$ if $n=0$. Moreover let $j=\min\{i|-g(l(t_i))=a^+(l)\}$ and $j'=\max\{i|-g(l(t_i))=a^+(l)\}$ then $a^+(l|[a, t_j])>a^+(l)$ and $a^+(l\,|[t_{j'}, b])>a^+(l)$.*

Proof. Let l be a curve in L from x to y, then by the transversality theorem of Pontrjagin-Thom, we can suppose that l is transverse to $W(\varepsilon)$ so l has the $D^-(\varepsilon)$-partition. If $l([t_{2i-1}, t_{2i}])$ is contained in a non-separating cell $D_i(\varepsilon)$ then we can join $l(t_{2i-1})$ and $l(t_{2i})$ by a curve in $D_i(\varepsilon)\cap W^-(\varepsilon)\cap L$. So we have a curve l' in $L\cap(M-(\mathrm{Int}\,D(\varepsilon)-\mathrm{Int}\,SD^-(\varepsilon)))$ and l' has the $SD^-(\varepsilon)$-partition. The other statements are trivial.

Proposition 1.4. *If $\varphi(x, [0, a])\cap S_D=\phi$ then for any $y\in L_x$ such that $a^+(y)\geqq a$, we have $c^+(x, y)\geqq a$.*

Proof. It is sufficient to prove for the case of $0<a<\varepsilon$. If $x, y\notin\mathrm{Int}\,D(\varepsilon)$, choose a curve l in L_x from x to y with the $SD^-(\varepsilon)$-partition $0<t_1<\cdots<t_{2n}<1$. Then we have $a^+(l)\geqq a$. In fact if $a^+(l)<a$ then take j as in Lemma 1.3. Then $\varphi(l(t_j), a^+(l))\in S_D$ and $\varphi(x, a^+(l))\in S_D$, this is a contradiction. If $x\in\mathrm{Int}\,D(\varepsilon)$ then

we can take $x' \notin \operatorname{Int} D(\varepsilon)$ such that $c^+(x, x') \geqq a$ and $\varphi(x', [0, a]) \cap S_D = \phi$. So the problem is reduced to the above case.

Definition 1.6. For a subset A of $M-\Sigma$, the *saturation* of A is defined by $Q(A) = \{x \mid L_x \cap A \neq \phi\}$.

Lemma 1.5. *Let L be a non-compact leaf then for any $\varepsilon>0$ there exist $x \in L$ and a closed curve C transverse to $\mathscr{F}$ passing through x such that $Q(C) \subset Q(\varphi(x, [0, \varepsilon]))$. Moreover C can be choosen in $M-D(\varepsilon')$ for sufficiently small $\varepsilon'>0$.*

Proof. Choose a segment C as in Lemma 1.1, then there exists $x \in C$ and ε' such that $0<\varepsilon'<\varepsilon$ and $x, \varphi(x, \varepsilon') \in L$. For $\varepsilon''<\min(c^-(x, \varphi(x, \varepsilon')), \varepsilon')$ we have $Q(\varphi(x, [0, \varepsilon'])) = Q(\varphi(x, [-\varepsilon'', \varepsilon']))$ and there exists a curve l in $L_{\varphi(x, -\varepsilon'')}$ from $\varphi(x, -\varepsilon'')$ to $\varphi(x, \varepsilon'-\varepsilon'')$. Then by modification of $\varphi(x, [-\varepsilon'', \varepsilon'-\varepsilon'']) * l$ we obtain a closed transversal curve C' such that $C' \ni x$ and $Q(C) = Q(\varphi(x, [-\varepsilon'', \varepsilon'-\varepsilon''])) \subset Q(\varphi(x, [0, \varepsilon]))$. If we choose ε'' so that $\varphi(x, -\varepsilon'') \notin S_D$ and l has the $SD(\varepsilon)$-partition $0<t_1<\cdots<t_{2n}<1$ then $\min\{|g(l(t_i))| \mid i=1, 2, \cdots, 2n\} \neq 0$. So for sufficiently small ε_0, l is a curve in $M-D(\varepsilon_0)$ and C' can be taken in $M-D(\varepsilon_0)$.

Proposition 1.6. *Let C be a closed curve in $M-D(\varepsilon)$ transverse to $\mathscr{F}$, if $C \cap S_D = \phi$ then $Q(C) = M-\Sigma$.*

Proof. We can assume that C is an orbit of a $D(\varepsilon)$-regular vector field. If $Q(C) \neq M-\Sigma$ then there exist $x \in Q(C)$ and $t>0$ such that $\varphi(x, t) \notin Q(C)$. Choose $y \in L_x \cap C$ then $\varphi(y, (-\infty, \infty)) \cap S_D = \phi$. By Proposition 1.4. we have $c^+(x, y)>t$. This contradicts to $\varphi(x, t) \notin Q(C)$.

Theorem 1.7. *If $\mathscr{S}_D = \phi$(for example if g is a Morse function without critical points of index* 1 *and* dim $M-1$) *then if rank $\omega \geqq 2$ all leaves are dense in $M-\Sigma$ and if rank $\omega \leqq 1$ all leaves are compact.*

Proof. If there exists a non-compact leaf L then by Lemma 1.5. there exists a closed transverse curve C in $M-D(\varepsilon)$ and, since $Q(C) = M-\Sigma$, we see that $C \cap L$ is an infinite set. By Proposition 1.4. the holonomy pseudogroup acting on C is really a group of rotations of C. Since this group has an infinite orbit $L \cap C$, all orbits are dense in C and all leaves of $\mathscr{F}$ are dense in $M-\Sigma$. Since rank ω is finite this group contains a rotation of irrational angle so rank $\omega \geqq 2$. If all leaves of $\mathscr{F}$ are compact then there are two possibilities. If there is a closed transversal curve C then the holonomy pseudogroup actiong on C contains only rational rotations. So there exists a closed transversal curve C' which intersects with any leaf at exactly one point. So there is a map $\pi: M-\Sigma \to C'$, $\pi(x) = L_x \cap C'$ this map can be extended to $\pi': M \to C'$ and it is easy to see that $\omega = \pi^* dt$ where t is a parameter of C' such that $\omega\left(\frac{\partial}{\partial t}\right) = 1$. So rank $\omega = 1$. If there is no closed transverse curve, choose a leaf L and put $T(L) = \{\varphi(x, t) \mid x \in L, (x, t) \in D_X\}$. Then by Proposition 1.4. $T(L)$ is a connected component of $M-\Sigma$ and we can constract a map f: $M-\Sigma\to\mathrm{R}$ such that $\omega=df$. So rank $\omega=0$.

Proposition 1.8. *The set of non-compact leaves of $\mathscr{F}$ is open in $M-\Sigma$ and its boundary consists of compact separatrices.*

This proposition can be proved by the same method as Theorem 3.4. of Haefliger [1]. But in this case this follows from the following lemma and the finiteness of $\mathscr{S}_D$.

Lemma 1.9. *Let L be a non-singular compact leaf of $\mathscr{F}$ and $x\in L$. Set $t_0=\sup\{t\,|\,L_{\varphi(x,t')}$ is a non-singular compact leaf for $0\leq t'<t\}$. Then $L_{\varphi(x,t_0)}$ is a compact D-separatrice.*

Proof. It is clear that $L'=L_{\varphi(x,t_0)}$ is a D-separatrice. If it is non-compact, then by Lemma 1.1. there exist $y\in L'$ and $\varepsilon<t_0$ such that $\varphi(y,-\varepsilon)\in L'$. But by Proposition 1.4. $c^-(\varphi(x,t_0),y)>t_0$, so $L'=L_{\varphi(x,t_0-\varepsilon)}$ is compact. This is a contradiction.

Proposition 1.10. *Suppose $\mathscr{S}_D\neq\phi$. Let L_x be a non-compact leaf then for any $\varepsilon>0$ there exist $0<\varepsilon',\varepsilon''<\varepsilon$ such that $\varphi(x,\varepsilon')$, $\varphi(x,-\varepsilon'')\in S_D$.*

Proof. Otherwise we have $\varphi(x,(0,\varepsilon))\cap S_D=\phi$. Choose $x'=\varphi(x,\varepsilon_0)$ for sufficiently small ε_0. Then by Proposition 1.8. $L_{x'}$ is non-compact and $\varphi(x',[0,\varepsilon-\varepsilon_0])\cap\mathscr{S}_D=\phi$. By Lemma 1.5. there exist $y\in L_{x'}$ and a closed transversal curve C such that $Q(C)\subset Q(\varphi(y,[0,\varepsilon-\varepsilon_0]))$. But by Proposition 1.4. $c^+(x',y)\geq\varepsilon-\varepsilon_0$. So we have $Q(\varphi(y,[0,\varepsilon-\varepsilon_0]))\cap S_D=\phi$ and $Q(C)\cap S_D=\phi$. Then by Propostion 1.6, we have $Q(C)=M-\Sigma$, this contradicts to $\mathscr{S}_D\neq\phi$.

§ 2. Proof of Theorem 1

By Proposition 1.8. the set of noncompact leaves is open in $M-\Sigma$. Let Ω be a connected component of the open set. We show that any leaf L in Ω is dense in Ω. If $\mathscr{S}_D\neq\phi$ then the theorem is reduced to Theorem 1.7. If $\mathscr{S}_D\neq\phi$ then by Proposition 1.10. and the finiteness of separating cells, we see that, for any leaf L in Ω, $\bar{L}$ contains a non-compact separatrice. So it is sufficient to show that any non-compact separatrice in Ω is dense in Ω. We use the following notations.

$\mathscr{S}_A=\{L\,|\,L\in\mathscr{S}_D$ is dense in $\Omega\}$, $S_A=\{x\,|\,L_x\in\mathscr{S}_A\}$
$\mathscr{S}_B=\{L\,|\,L\in\mathscr{S}_D$ is contained in Ω and non-compact but not dense in $\Omega\}$
$\mathscr{S}_C=\{L\,|\,L\in\mathscr{S}_D$ is contained in $\bar{\Omega}$ and is compact$\}$
$D_A(\varepsilon)=\{D_i(\varepsilon)\,|\,L\cap g^{-1}(0)\cap D_i(\varepsilon)\neq\phi$ for some $L\in\mathscr{S}_A$ and $D_i(\varepsilon)\subset SD(\varepsilon)\}$
$W_A(\varepsilon)=\{W_i(\varepsilon)\,|\,W_i(0)\subset L\in\mathscr{S}_A\}$

S_*, $D_*(\varepsilon)$, $D_*^\pm(\varepsilon)$, $W_*(\varepsilon)$, $W_*^\pm(\varepsilon)$, $W_*(0)$ $(*=A, B$ or $C)$ are defined similarly. Our purpose is to show that $\mathscr{S}_B=\phi$.

Proposition 2.1. *$\mathscr{S}_A\neq\phi$ and a leaf of $\mathscr{S}_B$ are not locally dense.*

Proof. By Proposition 1.10. $S_D \cap \Omega$ is dense in Ω. Since $\mathscr{S}_D$ is finite, there exist $L_1, \cdots, L_k \in \mathscr{S}_D$ such that L_i is dense in an open set $\Omega_i \subset \Omega$. But if $\Omega_i \cap \Omega_j \neq \phi$ then L_i is dense in $\Omega_i \cup \Omega_j$. Since Ω is connected, each L_i is dense in Ω.

The next lemma follows easily from the finiteness of $\mathscr{S}_B$ and $\mathscr{S}_C$.

Lemma 2.2. *There exists* $\varepsilon_0 > 0$ *such that* $W_A(\varepsilon_0) \cap (S_B \cup S_C) = \phi$ *and* $W_B(\varepsilon_0) \cap S_C = \phi$.

Lemma 2.3. *For any* $x \in W_B(0)$ *and* $\varepsilon > 0$, *there exists* $0 < \varepsilon', \varepsilon'' < \varepsilon$, *such that* $\varphi(x, -\varepsilon') \in S_B$ *and* $\varphi(x, \varepsilon'') \in S_B$.

Proof. By Lemma 1.1. there exist $y \in L_x$ and $\varepsilon' < \min(\varepsilon_0, \varepsilon)$ such that $\varphi(y, \varepsilon') \in L_x$. Choose a curve l in L_x from x to y with the $SD^-(\varepsilon'')$-partition $0 < t_1 < \cdots < t_{2n} < 1$ where $\varepsilon' < \varepsilon'' < \min(\varepsilon_0, \varepsilon)$. If $n = 0$ then $\varphi(x, \varepsilon') \in L_{\varphi(y, \varepsilon')} = L_x \in \mathscr{S}_B$. If $n > 0$, put $\varepsilon_1 = -g(l(t_1))$ then $0 < \varepsilon_1 < \min(\varepsilon_0, \varepsilon)$ and $\varphi(x, \varepsilon_1) \in L = L_{(l(t_1)), \varepsilon_1)}$. Clearly $L \in \mathscr{S}_D$, if $L \in \mathscr{S}_A$ then $l(t_1) \in W_A(\varepsilon_0) \cap S_B$, this contradicts to Lemma 2.2. If $L \in \mathscr{S}_C$ then $\varphi(x, \varepsilon_1) \in W_B(\varepsilon_0) \cap S_C$ and this is also a contradiction. So $L \in \mathscr{S}_B$. Thus we proved the existence of $0 < \varepsilon'' < \varepsilon$ such that $\varphi(x, \varepsilon'') \in S_B$. $\varphi(x, -\varepsilon') \in S_B$ is similar.

Definition 2.1. We say two leaves L_0 and L_1 of $\mathscr{F}$ are *negatively related* if for any $x \in L_0$ and $y \in L_1$ there exists $\varepsilon = \varepsilon(x, y) > 0$ such that $L_{\varphi(x, -t)} = L_{\varphi(y, -t)}$ for any $0 < t < \varepsilon$. Clearly this is an equivalence relation.

Lemma 2.4. *If* $L_0 \in \mathscr{S}_A$ *and* $L_1 \in \mathscr{S}_B$ *then* L_0 *and* L_1 *are not negatively related.*

Proof. Otherwise take $x \in L_0 \cap W_A(0)$ and $y \in L_1 \cap W_B(0)$. By Lemma 2.2. $\varphi(x, -t) \notin S_B$ for $0 < t < \varepsilon_0$ but by Lemma 2.3. there exists $0 < \varepsilon' < \min(\varepsilon_0, \varepsilon(x, y))$ such that $\varphi(y, -\varepsilon') \in S_B$. This contradicts to $L_{\varphi(x, -\varepsilon')} = L_{\varphi(y, -\varepsilon')}$.

Proposition 2.5. $\mathscr{S}_B = \phi$.

Proof. Suppose that $\mathscr{S}_B \neq \phi$ and take $x \in W_A(0)$ and $y \in L_x \cap W_B^-(\varepsilon_0)$. Choose a curve l in L_x from x to y with the $SD^-(\varepsilon_0)$-partion $0 < t_1 < \cdots < t_{2n} < t_{2n+1} = 1$. If $n = 0$ then $\varphi(x, -g(y)) \in W_A(\varepsilon_0) \cap S_B$, this contradicts to Lemma 2.2. So $n > 0$. Put $\varepsilon_i = -g(l(t_i))$, $L_i = L_{\varphi(l(t_i), \varepsilon_i)} \in \mathscr{S}_D$ then $\varepsilon_{2i-1} = \varepsilon_{2i}$ and L_{2i-1} is negatively related to L_{2i}. Consider $k = \max\{i \mid l(t_i) \in W_A(\varepsilon_0)\}$. If k is even then, by considering $l \mid [t_k, t_{k+1}]$, we have $\varphi(l(t_k), \varepsilon_{k+1}) \in W_A(\varepsilon_0) \cap (S_B \cup S_C)$. This contradicts to Lemma 2.2. If k is odd then $\varepsilon_k = \varepsilon_{k+1}$ and L_k is negatively related to L_{k+1}. So by Lemma 2.4. $L_{k+1} \in \mathscr{S}_C$ and by Lemma 2.2. $k < 2n - 2$. We consider the curve $l \mid [t_{k+1}, 1]$ with the $SD^-(\varepsilon_0)$-partition and put $\varepsilon = a^+(l \mid [t_{k+1}, 1])$. If $\varepsilon < \varepsilon_k$ then $\varphi(l(t_k), \varepsilon) \in W_A(\varepsilon_0) \cap (S_B \cup S_C)$, this is a contradiction. If $\varepsilon > \varepsilon_k$ then $\varphi(l(1), \varepsilon_k) \in W_B(\varepsilon_0) \cap S_C$, this is a contradiction. If $\varepsilon = \varepsilon_k$, put $j = \max\{i \mid \varepsilon_i = \varepsilon, i > k+1\}$ then $L_k \in \mathscr{S}_A$ is negatively related to L_j. By Lemma 2.4. we have $L_j \in \mathscr{S}_C$ and, by considering $l \mid [t_j, 1]$, we have $\varphi(l(1), \varepsilon_j) \in W_B(\varepsilon_0) \cap \mathscr{S}_C$. This contradicts to Lemma 2.2. Thus we proved Theorem 1.

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