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Renewal theory-A Markov process approach

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§ 1. Introduction.

Renewal Theory is concerned with, among other things, the point process generated by a random walk and the behaviour of solutions of the so-called renewal equation. To be precise, let $\{X_i\}_{i=1}^{\infty}$ be i. i. d. nonnegative random variables with a common distribution $F(\cdot)$. Let X_0 be a nonnegative random variable independent of $\{X_i\}_1^{\infty}$. Set $S_n=\sum_{i=0}^{\infty} X_i$ for $n\geqq 0$. Let $\xi(\cdot)$ be a measurable function from R^+ to R^+ and be bounded a. e. on finite intervals. The equation

(1) $$m(t)=\xi(t)+\int_{(0,t]} m(t-u)dF(u) \qquad \text{for} \quad t\geqq 0$$

is called the *renewal equation.*

The objects of interest are: a) the asymptotic behaviour of the point process $\{S_n\}_0^{\infty}$ and b) the asymptotic behaviour of the solution $m(\cdot)$ to (1). The following results are well-known. Let

(2) $$U(t)=E\{\# n:\ S_n\leqq t\}$$
$$=\sum_{0}^{\infty} P(S_n\leqq t)$$

be the socalled *renewal function.* Assume from now on that $F(\cdot)$ is non-lattice, $0<\lambda^{-1}=\int_0^{\infty} u\ dF(u)<\infty$.

Theorem 1. (*Blackwell*) *For all* $0<h<\infty$

(3) $$U(t+h)-U(t) \longrightarrow \lambda h \quad \text{as} \quad t \longrightarrow \infty$$

Theorem 2. (*Feller*) *If* $\xi(\cdot)$ *is directly Riemann integrable then the solution* $m(\cdot)$ *of* (1) *satisfies*

(4) $$m(t) \longrightarrow \lambda\int_0^{\infty}\xi(u)du\ .$$

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Feller [4] has shown that these two theorems are equivalent.

The object of this note is to present a Markov process formulation that is equivalent to the above. This is done in terms of the Markov Process of the socalled backward *recurrence time* or what we shall call as the *age process.* We shall show that Blackwell's and Feller's theorems are equivalent to the weak convergence of this age process. If one assumes a mild smoothness on $F(\cdot)$, then one can strengthen this weak convergence to that in variation norm. This in turn leads to the following strengthening of (3) and (4):

$$(3)' \qquad U(t+B) \longrightarrow \lambda m(B)$$

for all bounded Borel sets, where we mean, by abuse of notation, $U(B)=\sum_0^\infty P(S_n \in B)$, $m(\cdot)$ is Lebesgue measure and

$$(4)' \qquad m(t) \longrightarrow \lambda \int_0^\infty \xi(u)du$$

for all bounded measurable $\xi(\cdot)$ that are dominated by a multiple of the tail of $F(\cdot)$. It turns out that the smoothness of $F(\cdot)$ is necessary as well for these stronger conclusions. Thus, in the renewal equation (1), to get the convergence of $m(t)$ as $t\to\infty$, one needs either a smoothness condition on $\xi(\cdot)$ like d. r. i. or on $F(\cdot)$ like non-strongly singularity. The Markov process approach besides bringing out this balance between $\xi(\cdot)$ and $F(\cdot)$ into sharper focus, also suggests that when studying the limit behaviour of Markov Chains on general state spaces that do have a topological structure on them it is perhaps worthwhile to prove the weak convergence first as this may hold under fairly mild recurrence conditions rather than try to use the Doeblin-Harris theory as this needs stronger recurrence conditions (although, these yield stronger convergence, such as in variation norm).

The Markov Process approach has been mentioned in Doob [3]. We have learnt after this work was completed that Arjas et al [1] have also obtained results similar to ours.

§ 2. Statement of results.

Let $F(\cdot)$ be a probability distribution on $(0, \infty)$. Assume throughout that $F(\cdot)$ is non-lattice and $0<\lambda^{-1}=\int_0^\infty u\, dF(u)<\infty$. For each $x\varepsilon[0, T)$, let $X_0^{(x)}$ be a random variable with

$$(5) \qquad P(X_0^{(x)}\leqq t)=\frac{F(x+t)-F(x)}{1-F(x)} \quad \text{for} \quad t\geqq 0, \quad \text{and} \quad x<T$$

where $T=\sup\{x:\ F(x)<1\}$.

Let $\{X_i\}_1^\infty$ be i. i. d. r. v. with distribution $F(\cdot)$ and independent of $X_0^{(x)}$. Set

(6) $$A(t)=x+t \quad \text{for} \quad 0\leqq t\leqq X_0^{(x)}$$
$$t-X_0^{(x)} \qquad X_0^{(x)}\leqq t\leqq X_0^{(x)}+X_1$$
$$t-S_n^{(x)} \qquad S_n^{(x)}\leqq t<S_{n+1}^{(x)},$$
$$n=0, 1, 2, \cdots$$

where $S_n^{(x)}=X_0^{(x)}+X_1+\cdots+X_n$ for $n=0, 1, 2, \cdots$. From the very definition, we get the following:

Proposition 1. *The stochastic process* $\{A(t);\ t\geqq 0\}$ *is a Markov Process on* $[0, \infty)$ *with stationary transition probabilities.*

The transition function $P(x, t, E)\equiv P(A(t)\in E\mid A(0)=x)$ satisfies the equation

(7) $$P(x, t, E)=\chi_E(x+t)\Big(\frac{1-F(x+t)}{1-F(x)}\Big)$$
$$+\int_{(0,t]}P(0, t-u, E)\frac{dF(x+u)}{(1-F(x))}.$$

One solves (7) for $x=0$ and uses (7) to obtain it for all x. Clearly,

(8) $$P(0, t, E)=P\{\bigcup_{n=0}^{\infty}(S_n^{(0)}\leqq t<S_{n+1}^{(0)},\ t-S_n^{(0)}\in E)\}$$
$$=\int_{(0,t]}\chi_E(t-u)(1-F(t-u))U(du)$$

where $U(t)=\sum_0^\infty P(S_n^{(0)}\leqq t)$ is the renewal function. From now on we assume $F(\cdot)$ is non lattice. $F(0)=0$ and $0<\lambda^{-1}=\int_0^\infty t\ dF(t)<\infty$. Let $\pi(E)=\lambda\int_E(1-F(u))$ du for all Broel sets E in R^+. We now state an equivalent form of Theorems 1 and 2.

Theorem 3. *For all initial condition x, the Markov Process* $A(t)$ *converges weakly to* $\pi(\cdot)$.

The proof of the equivalence is done by showing Theorem 2 ⇒ Theorem 3 ⇒ Theorem 1. In fact, let $f(\cdot)$ be a bounded continuous function on $[0, \infty)$. Then $a_0(t)\equiv E_0(f(A(t)))$ satisfies

$$a_0(t)=f(t)(1-F(t))+\int_{(0,t]}a_0(t-u)d\ F(u).$$

The function $\xi(t)\equiv f(t)(1-F(t))$ is directly Riemann integrable and so by Theorem 2 $a_0(t)\rightarrow\int_0^\infty f(t)(1-F(t))dt$. Since $a_x(t)\equiv E_x f(A(t))$ satisfies

$$a_x(t)=f(x+t)\Big(\frac{1-F(x+t)}{1-F(x)}\Big)+\int_0^t a_0(t-u)\Big(\frac{dF(x+u)}{1-F(x)}\Big)$$

We get by bounded convergence theorem $\lim_t a_x(t)=\lim_t a_0(t)$. Thus, Theorem 2 ⇒ Theorem 3.

For h small and positive, the function $f(t)\equiv(1-F(t))^{-1}\chi_{[0,h)}(t)$ is bounded and continuous a.e. So by Theorem 3, $E_0 f(A(t))\rightarrow\lambda h$. But $E_0 f(A(t))\equiv U(t)-U(t-h)$. Thus, Theorem 3 $\Rightarrow$ Theorem 1. We shall give in § 3 an independent proof of Theorem 3, using a coupling argument.

To strengthen the above convergence of $A(t)$ to π from weak to variation norm, we need a mild smoothness condition on $F(\cdot)$. With this in mind we introduce a

Definition. A distribution function $F(\cdot)$ on R is *strongly singular* if for all n, the n fold convolution $F^{(n)}(\cdot)$ is singular with respect to Lebesgue measure.

Clearly, $F(\cdot)$ is *strongly singular* iff $U(\cdot)$ is *singular* with respect to Lebesgue measure.

Theorem 4. *Let $F(\cdot)$ be not strongly singular. Then, for all x,*

$$\lim_t \|P_x(A(t)\in\cdot)-\pi(\cdot)\|=0$$

where $\|\cdot\|$ is variation norm.

It turns out that the converse is true as well.

Theorem 5. *Let $F(\cdot)$ be strongly singular. Then*

$$\lim \|P_x(A(n)\in\cdot)-\pi(\cdot)\|=2.$$

The proofs of these two theorems are in § 4.

§ 3. A coupling proof of the weak convergence of $A(t)$ to $\pi(\cdot)$

Here is the plan of the proof. First we show that $\pi(\cdot)$ is stationary for $A(\cdot)$. Next, we construct two processes $A_1(\cdot)$, $A_2(\cdot)$ such that both are age processes with $A_1(0)$ distributed according to $\pi(\cdot)$ and $A_2(0)=0$ w.p. 1. This construction is done in such a way that for every $\varepsilon>0$, there exists a non-anticipating random time T such that $0\leqq(A_1(T)-A_2(T))<\varepsilon$. This forces the limit behaviour of the distribution of $A_1(\cdot)$ and $A_2(\cdot)$ to be the same. But since $A_1(\cdot)$ is stationary with distribution $\pi(\cdot)$, it follows that $A_2(\cdot)$ converges weakly to $\pi(\cdot)$. Now the details. We begin by establishing the stationarity of $\pi(\cdot)$.

Theorem 6. *The measure $\pi(\cdot)$ is stationary for $A(\cdot)$.*

Proof. Let $f(\cdot)$ be bounded measurable on $[0, \infty)$. Let

$$m(x, t)\equiv E_x(f(A(t)))$$

(9)
$$m(t)=\int_0^\infty m(x, t)\pi(dx).$$

We need to show that $m(t)\equiv m(0)$. Now $m(x, t)$ satisfies

(10) $$m(x,\, t)=f(x+t)\Big(\frac{1-F(x+t)}{1-F(x)}\Big)+\int_0^t m(0,\, t-u)\Big(\frac{dF(x+u)}{1-F(x)}\Big)$$

and we have

(11) $$\lambda^{-1}m(t)=\int_0^\infty f(x+t)(1-F(x+t))dx+\int_0^\infty\Big(\int_0^t m(0),\, t-u)dF(x+u)\Big)dx$$

But, since $m(0,\, t)$ satisfies the renewal equation

$$m(0,\, t)=f(t)(1-F(t))+\int_0^t m(0,\, t-u)dF(u)\,,$$

$$m(0,\, t)=\int_0^t \xi(t-u)U(du) \quad \text{where} \quad \xi(t)=f(t)(1-F(t)))\,.$$

Thus

$$\int_0^\infty\Big(\int_0^t m(0,\, t-u)dF(x+u))dx=\int_0^t m(0,\, t-u)(1-F(u))du$$

$$=\int_0^t \xi(u)du\,,$$

since $\int_0^t (1-F(t-u))U(du)\equiv 1$.
Now (11) shows that $\lambda^{-1}m(t)\equiv\int_0^\infty f(u)(1-F(u))du$. q.e.d.

Now we construct the two processes $A_1(\cdot)$, $A_2(\cdot)$. Recall the set up in the beginning of the section 2. Since $A_1(0)$ is distributed according to $\pi(\cdot)$ so is X_0. Let $\{X_i^1\}_1^\infty$ and $\{X_i^2\}_0^\infty$ be two independent sequences of i.i.d. r.v. with distribution $F(\cdot)$. The sequence $Y_i=X_i^1-X_i^2$ for $i=1, 2, \cdots$, being i.i.d. mean zero non-lattice random variables, given any $\varepsilon>0$, there exists a random variable $N(\varepsilon)$ such that

$$0<\sum_{i=0}^{N(\varepsilon)} X_i^1-\sum_{i=0}^{N(\varepsilon)} X_i^2\equiv\Delta<\varepsilon\,.$$

Let $\{X_i^3\}_1^\infty$ be a sequence of random variables defined by

$$X_i^3=\begin{cases} X_i^1 & 0\leqq i\leqq N(\varepsilon) \\ X_i^2 & i>N(\varepsilon)\end{cases}$$

Let $\{A_3(t)\,;\ t\geqq 0\}$ be the age process associated with $\{X_i^3\}_0^\infty$. As stochastic processes $A_3(\cdot)$ and $A_1(\cdot)$ are clearly equivalent. Also the processes $\{A_2(\cdot)\}$ and $\{A_3(\cdot)\}$ are coupled in the sense that

$$A_2(t)=A_3(t-\Delta) \quad \text{for} \quad t\geqq T\equiv\sum_0^{N(\varepsilon)} X_i^2 \quad \text{and if}$$

$$A^3(t)>\Delta\,, \quad \text{then} \quad A_2(t)=A_3(t)-\Delta\,.$$

Proof of Theorem 3. Now let $f(\cdot)$ be bounded and uniformly continuous on $(0,\, \infty)$. Then,

$$Ef(A_2(t))=E\{f(A_2(t));\ T\leqq t\}$$
$$+E\{f(A_2(t));\ T>t,\ A_3(t)<\varepsilon\}$$
$$+E\{f(A_2(t));\ T>t,\ A_3(t)\geqq\varepsilon\}$$

and doing a similiar decomposition of $Ef(A_3(t))$, we see that

$$|Ef(A_2(t))-Ef(A_3(t))|\leqq 2\|f\|(P(T>t)+P(A_3(t)<\varepsilon))+\eta(\varepsilon)$$

where

$$\eta(\varepsilon)\equiv\sup_{|x-y|\leq\varepsilon}|f(x)-f(y)|$$

Since $A_3(t)$ has distribution $\pi(\cdot)$ and since $f(\cdot)$ is uniformly continuous, given a $\delta>0$, we can choose $\varepsilon>0$ such that

$$P(A_3(t)<\varepsilon)<\delta \quad \text{for all} \quad t$$

and

$$\eta(\varepsilon)<\delta\,.$$

Now $P(T>t)\rightarrow 0$ as $t\uparrow\infty$. Thus,

$$\overline{\lim_{t}}\,|Ef(A_2(t))-Ef(A_3(t))|\leqq(2\|f\|+1)\delta\,.$$

Since δ is arbitrary we are done.

§4. Proofs of Theorems 4 and 5.

Our proofs are based on the following ergodic theorem for Markov chains on general state spaces formulated and proved in a manner suitable to us by Athreya and Ney [2].

Theorem 7. *Let $\{X_n\}_0^\infty$ be a Markov Chain on a measurable space $(S, \mathcal{S})$ with transition function $P(x, E)$. Assume*

(1) $\exists A\in\mathcal{S}$ *such that*

$$P(X_n\in A \quad \text{for some} \quad n\geqq 1\,|\,x_0=x)\equiv 1$$

(2) $\exists n_0, \lambda$ *and a probability measure $\varphi(\cdot)$ on A*

$$P(X_{n_0}\in E\,|\,X_0=x)\geqq\lambda\varphi(E) \qquad \text{for all } x \text{ in } A,\ E\subset A$$

(3) *g.c.d. of such n_0 as A varies is one.*

Then, there exists a σ finite measure $\pi(\cdot)$ that is invariant for $P(\cdot,\cdot)$ and is unique upto a multiplicate constant. Further, when that $\pi(\cdot)$ is finite and normalized to be a probability measure, $\overline{\lim_{n}}\,\|P_x(X_n\in\cdot)-\pi(\cdot)\|=0$ for all x.

Their proof is based on a simple regeneration lemma and the discrete renewal equation. We refer to [2] for details.

The following result is a key step in the proof of Theorem 4.

Theorem 8. *Let $F(\cdot)$ be not strongly singular. Then, for all sufficiently small and positive δ the sequence $\{X_n \equiv A(n\delta)\}_0^\infty$ satisfies the hypotheses of Theorem 7.*

Assuming the validity of Theorem 7 and 8, we now present the proof of Theorem 4.

Proof of Theorem 4. Since $\pi(\cdot)$ is invariant for the process $A(\cdot)$, we have for $n\delta \leqq t < (n+1)\delta$,

$$P_x(A(t)\in E)-\pi(E)=\int_0^\infty P(y,\ t-n\delta,\ E)P_x(A(n\delta)\in dy)$$

$$-\int_0^\infty P(y,\ t-n\delta,\ E)\pi(dy)\,,$$

and hence,

$$\|P_x(A(t)\in\cdot)-\pi(\cdot)\|\leqq\|P_x(A(n\delta)\in\cdot)-\pi(\cdot)\|$$

which goes to zero as $n\to\infty$ by Theorem 6 and 7.

Proof of Theorem 5. If $F(\cdot)$ is strongly singular then the renewal measure $U(\cdot)$ is singular and hence there exists a set B such that $U(B)=0$ and $m(B^c)=0$. Let $B_0=\bigcap_{n=1}^\infty(n-B)$. Then, $m(B_0^c)=0$ and $U(n-B_0)\leqq U(n-(n-B))=0$ for each n. Now, since

$$P_0(A(t)\in E)=\int\chi_E(u)(1-F(u))U(t-du)\,,$$

and since $U(n-B_0)=0$,

$$P_0(A(n)\in B_0)=0\,.$$

On the other hand, $\pi(\cdot)$ being absolutely continuous with respect to $m(\cdot)$, $\pi(B_0^c)=0$. Hence, $\|P(A(n)\in\cdot)-\pi(\cdot)\|=2$, for all n thus proving Theorem 4.

All that remains now is to prove Theorem 8. We need the following lemma and its corollaries in the proof of Theorem 8.

Lemma. *Let $f(\cdot)$ be a non-negative measurable function on R that is not zero a.e. Then $\exists a<b$, $\delta>0$ such that*

$$h(x)\equiv\int f(x-y)f(y)dy>\delta \qquad \textit{for all} \quad a<x<b\,.$$

Proof. Since $h(x)\geqq h_k(x)\equiv\int_{|y|\leq k}f_k(x-y)f_k(y)dy$ for all $k>0$, where $f_k(u)=\min(f(u),\,k)$, it is enough to prove the assertion for $h_k(\cdot)$ for some k. It is easily verified that the convolution $(f_1*f_2)(\cdot)$ is continuous if $f_1\in L_1$ and $f_2\in L_\infty$. Thus, $h_k(\cdot)$ is continuous. Finally, since $f\neq 0$ a.e. $\exists x_0$ and k such that $h_k(x_0)>0$.

Corollary 1. *Let $F(\cdot)$ be a probability distribution on R that is not singular with respect to Lebesgue measure. Then $(F*F)(\cdot)$ has an absolutely continuous component whose density is bounded away from zero in a non-degenerate interval.*

Corollary 2. *Since $F(\cdot)$ is not strongly singular there exists n_0 such that $F^{(n_0)}(\cdot)$ is not singular with respect to Lebesgue measure. By corollary 1, $F^{(2n_0)}(\cdot)$ has a density that is bounded below in a non-degenerate interval. But for any $a<b$, $U(b)-U(a)\geqq F^{(2n_0)}(b)-F^{(2n_0)}(a)\geqq \text{const.}(b-a)$ if a, b lie in such an interval.*

Proof of Theorem 8. Let $I_n=[(n-1)h, nh]$, $n=1, 2, \cdots$, where $h>0$ will be chosen later. Let E be a Borel set in $I_1=[0, h)$ and n_0 a fixed integer. Then

$$P(x_1, n_0h, E)\equiv P(A(n_0h)\in E\,|\,A(0)=x)$$

$$=\text{Prob}\bigcup_{n=0}^{\infty}\{(n_0-1)h\leqq S_n<n_0h,\ X_{n+1}>n_0h-S_n$$

$$\text{and}\quad n_0h-S_n\in E\}$$

where $S_n=X_0+X_1+\cdots+X_n$, for $n=0, 1, 2, \cdots$, with X_i being independent and

$$P(X_0\leqq t)\equiv\frac{F(x+t)-F(x)}{1-F(x)}$$

$$P(X_i\leqq t)=F(t)\qquad\text{for}\quad i\geqq 1.$$

$$P(A(n_0h)\in E\,|\,A(0)=x)=\sum_{n=0}^{\infty}P(S_n\in I_{n_0},\ X_{n+1}>n_0h-S_n,\ n_0h-S_n\in E)$$

$$=\sum_{n=0}^{\infty}\int_{I_{n_0}}(1-F(n_0h-y))\chi_E(n_0h-y)P(S_n\in dy)$$

$$=\int_{E-\{0\}}(1-F(u))U_{n_0}^{x}(du)$$

where $U_{n_0}^{x}(B)=\sum_0^{\infty}P(S_n\in n_0h-B)$ for any Borel set B in $I_1=[0, h)$. Now,

$$U_{n_0}^{x}(B)=\int_0^{\infty}U(n_0h-z-B)F_x(dz)$$

where $U(\cdot)$ is the usual renewal measure (i.e., with $x=0$). Since

$$F_x(z)=\frac{F(x+z)-F(x)}{1-F(x)},$$

$$U_{n_0}^{x}(B)\geqq\int_x^{\infty}U(n_0h+x-v-B)F(dv)$$

$$\geqq\int_h^{\infty}\Big(\int_{n_0h+x-v-B}k(w)dw\Big)F(dv)$$

where $k(\cdot)$ is the density of the absolutely continuous component of $U(\cdot)$ with respect to Lebesgue measure. Making a change of variables yields

$$U_{n_0}^{x}(B)\geqq\int_h^{\infty}\Big(\int_B k(n_0h+x-v-s)ds\Big)F(dv).$$

By corollary 2 $\exists$, $0<a<b<\infty$ and $\delta>0$, such that $k(x)>\delta$ for $a<x<b$. Now choose h small so that $(b-a)>10h$. Since $F(0)=0$, $\exists$, $I_r\equiv[(r-1)h, rh)$ such that $F(I_r)>0$. As x and s vary in $[0, h)$, and v varies in $[rh, (r+1)h)$, $n_0h+x-v-s$

has to lie between $(n_0-r-2)h$ and $(n_0+1-r)h$. Since $(b-a)>10h$ we can find integers n_1 and n_2 such that $(n_2-n_1)>10$ and $a<n_1h<n_2h<b$. Thus for $n_1+r+2\leqq n_0\leqq n_2+r-1$:

$$U^x_{n_0}(B)\geqq\delta F(I_r)m(B)$$

where $m(\cdot)$ is Lebesgue measure. Hence, there is an $\alpha>0$ such that for all $0\leqq x<h$, Borel sets E in $[0, h)$:

$$P\{A(n_0h)\in E\mid A(0)=x\}\geqq\alpha\int_E(1-F(u))du$$

for all $n_1+r+2\leqq n_0\leqq n_2+r-1$. Also by the definition of the age process we do get the recurrence condition

$$P(A(nh)\in[0, h) \quad \textit{for some} \quad n\geqq 1\mid A(0)=x)\equiv 1$$

for all x in R^+ and all $h>0$. Thus, for the Markov Chain $\{Z_n\equiv A(nh)\}$ we have produced a set $A\equiv[0, h)$, an integer N such that

$$P(Z_n\in E\mid Z_0=x)\geqq\alpha\int_E(1-F(u))du$$

for all x in A, $E\subset A$, $n=N, N+1, N+2$. This is precisely what makes $\{Z_n\}_0^\infty$ satisfy the hypothesis of Theorem 7.

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