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On curved finite element and straight beam element approximations for vibration problems of circular arch structures

By

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1. Introduction

In this paper, we shall consider finite element approximations for the vibration problem of circular arches with the clamped boundary conditions:

$$(1)\qquad \begin{cases} -EA\Big(\dfrac{d^2u}{ds^2}+\dfrac{1}{R}\dfrac{dw}{ds^3}\Big)+\dfrac{EI}{R}\Big(\dfrac{d^3w}{ds^3}-\dfrac{1}{R}\dfrac{d^2u}{ds^2}\Big)=\lambda\rho u & \text{in } \Omega\,, \\ \dfrac{EA}{R}\Big(\dfrac{du}{ds}+\dfrac{1}{R}w\Big)+EI\Big(\dfrac{d^4w}{ds^4}-\dfrac{1}{R}\dfrac{d^3u}{ds^3}\Big)=\lambda\rho w & \text{in } \Omega\,, \\ u=w=\dfrac{dw}{ds}=0 \quad \text{on } s=0, \quad s=L\,. \end{cases}$$

Figure 1 illustrates a circular arch. The above differential equations are derived by applying the Timoshenko shell theory to the structural arch theory ([18]). Here R is the radius of the arch, s is the length along the arch, L is the total arch length, Ω is the interval $(0, L)$, u and w represent the tangential and radial displacements, respectively. E is Young's modulus, ρ is the mass density, A and I are the area and the moment of inertia of the cross section of the arch, respectively. We assume that E, A, I, R and ρ are positive constants. The vibration problem (1) is to find the eigenvalue λ and the corresponding eigenfunction $\{u, w\}$ which is different from identically $\{0, 0\}$. The natural frequency and the mode shape of vibration are related to λ and $\{u, w\}$, respectively.

For the finite element approximations of the static boundary value problem of circular arches:

$$(2)\qquad \begin{cases} -EA\Big(\dfrac{d^2u}{ds^2}+\dfrac{1}{R}\dfrac{dw}{ds}\Big)+\dfrac{EI}{R}\Big(\dfrac{d^3w}{ds^3}-\dfrac{1}{R}\dfrac{d^2u}{ds^2}\Big)=f_1 & \text{in } \Omega\,, \\ \dfrac{EA}{R}\Big(\dfrac{du}{ds}+\dfrac{1}{R}w\Big)+EI\Big(\dfrac{d^4w}{ds^4}-\dfrac{1}{R}\dfrac{d^3u}{ds^3}\Big)=f_2 & \text{in } \Omega\,, \\ u=w=\dfrac{dw}{ds}=0 \quad \text{on } s=0, \quad s=L\,, \end{cases}$$

numerical results have been published by various authors, using curved finite elements or straight beam elements (see for instance, Dawe [4], Moan [11], Murray [12] and the references given there), and convergence proof is given by Kikuchi [7]. Here f_1 and f_2 are given functions which denote the applied forces in the tangential and radial directions, respectively. However it seems that convergence proof of the finite element schemes for the vibration problem (1) of the circular arch is not yet established.

In the present paper, we shall derive error estimates for four approximations based on piecewise linear and cubic Hermite polynomials, i.e., consistent approximation, partial approximation with semi-consistent mass scheme, straight beam element approximation and consistent approximation with semi-consistent mass scheme. They assert that the approximate eigenvalues and eigenfunctions converge to the exact ones. Furthermore, we shall give some numerical results in order to demonstrate the validity of our mathematical results. For numerical studies of some other finite element models on vibration problems of arches, we refer to Petyt and Fleischer [13] and Sabir and Ashwell [15].

Throughout this paper, by $C, C_1, C_2, \cdots$, we shall denote generic positive constants, independent of h, which are not necessarily the same at each occurrence. Here h is the discretization parameter which denotes the mesh size.

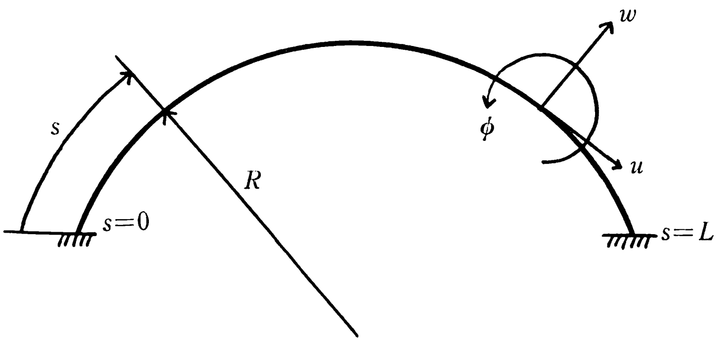


Figure 1. Circular arch.

2. Notations and variational formulation

We shall use the following notations. Let $\boldsymbol{R}^1$ be the space of real numbers. Let $L_2(\Omega)$ be the real space of square integrable functions on $\Omega=(0, L)$. The inner product and the norm in $L_2(\Omega)$ are given by

$$(u, v)=\int_0^L uv\,ds,$$

$$\|u\|=(u, u)^{1/2}, \qquad \text{for } u, v\in L_2(\Omega).$$

For a natural number n, let $H^n(\Omega)$ be the usual real Sobolev space supplied with the norm

$$\|u\|_n=\left(\sum_{i=0}^{n}\left\|\frac{\mathrm{d}^i u}{\mathrm{d}s^i}\right\|^2\right)^{1/2}, \qquad \text{for } u\in H^n(\Omega).$$

Define

$$H_0^n(\Omega)=\left\{u\,;\ u\in H^n(\Omega),\quad \frac{\mathrm{d}^i u(0)}{\mathrm{d}s^i}=\frac{\mathrm{d}^i u(L)}{\mathrm{d}s^i}=0,\quad i=0,1,\cdots,n-1\right\},$$

$$\mathscr{H}=H^1(\Omega)\times H^2(\Omega),$$

$$\mathscr{H}_0=H_0^1(\Omega)\times H_0^2(\Omega).$$

For the variational formulation of the problem (1), let us introduce symmetric bilinear forms on $\mathscr{H}\times\mathscr{H}$ as follows:

$$B(u,w\,;\bar{u},\bar{w})=EA\left(\frac{\mathrm{d}u}{\mathrm{d}s}+\frac{w}{R},\ \frac{\mathrm{d}\bar{u}}{\mathrm{d}s}+\frac{\bar{w}}{R}\right)+EI\left(\frac{\mathrm{d}^2w}{\mathrm{d}s^2}-\frac{1}{R}\frac{\mathrm{d}u}{\mathrm{d}s},\ \frac{\mathrm{d}^2\bar{w}}{\mathrm{d}s^2}-\frac{1}{R}\frac{\mathrm{d}\bar{u}}{\mathrm{d}s}\right),$$

$$G(u,w\,;\bar{u},\bar{w})=\rho\{(u,\bar{u})+(w,\bar{w})\},\qquad \{u,w\},\{\bar{u},\bar{w}\}\in\mathscr{H}.$$

The following quantities are also well defined:

$$N(u,w)=[B(u,w\,;u,w)]^{1/2},$$

$$M(u,w)=[G(u,w\,;u,w)]^{1/2}.$$

Then, we note that Schwarz's inequality for N, B, M and G, and the triangle inequality for N and M hold. The strain energy and the kinetic energy of the arch are associated with $1/2[N(u,w)]^2$ and $1/2[M(u,w)]^2$, respectively.

For the vibration problem (1) of the clamped arch, we introduce the following variational formulation:

$$\textit{Find } \{\lambda,u,w\}\in \boldsymbol{R}^1\times\mathscr{H}_0 \quad \textit{such that}$$

$$B(u,w\,;\bar{u},\bar{w})=\lambda G(u,w\,;\bar{u},\bar{w}) \quad \textit{for each } \{\bar{u},\bar{w}\}\in\mathscr{H}_0. \tag{3}$$

From the theory of positive definite and compact operators (Ciarlet [3], Kikuchi [7]), it is well known that all the eigenvalues $\{\lambda_i\}$ of (3) are arranged as

$$0<\lambda_1\leqq\lambda_2\leqq\cdots<\infty,$$

and the multiplicity of each eigenvalue is always finite and $\lambda_i\to\infty$ as $i\to\infty$. Here the eigenvalues are repeated according to their multiplicity. The corresponding eigenfunctions $\{u_i, w_i\}$ can be normalized as

$$G(u_i,w_i\,;u_j,w_j)=\delta_{ij},$$

where δ_{ij} is Kronecker's delta. It is also well known that $u_i\in H_0^1(\Omega)\cap C^\infty(\bar{\Omega})$ and $w_i\in H_0^2(\Omega)\cap C^\infty(\bar{\Omega})$. Here $C^\infty(\bar{\Omega})$ denotes the real space of infinitely differentiable functions on $\bar{\Omega}$. From the Rayleigh principle, we have

$$\lambda_i=\min_{\substack{\{u,w\}\in\mathscr{H}_0-\{0,0\}\\ G(u,w;u_j,w_j)=0\\ j=1,\cdots,i-1}}\frac{B(u,w\,;u,w)}{G(u,w\,;u,w)},\qquad i=1,2,\cdots, \tag{4}$$

and the minimum is attained by $\{u_i, w_i\}$.

On the other hand, the rotation ϕ of the arch is given by

$$\phi=\frac{\mathrm{d}w}{\mathrm{d}s}-\frac{u}{R},$$

as shown in Figure 1.

3. Finite element schemes

In order to construct finite element schemes, we divide the interval $\Omega=(0, L)$ into a finite number of subintervals $\{\Omega_i\}$ $(i=1, \cdots, m)$ in such a way that

$$0=s_0<s_1<\cdots<s_{i-1}<s_i<\cdots<s_m=L, \qquad \Omega_i=(s_{i-1}, s_i),$$

$$\bar{\Omega}=\bigcup_{i=1}^{m}\bar{\Omega}_i, \qquad \Omega_i\cap\Omega_j=\varnothing \quad (i\neq j),$$

as shown in Figure 2. Let

$$L_i=s_i-s_{i-1}, \qquad h=\max_{1\leq i\leq m} L_i, \qquad \bar{h}=\min_{1\leq i\leq m} L_i.$$

We assume that the finite element decomposition satisfies the following condition

$$0<C\leqq\bar{h}/h\leqq 1,$$

where C is a positive constant. As the basis functions, we shall use piecewise linear and Hermite interpolations $\{b_{h,i}^{(1)}, b_{h,i}^{(2)}, b_{h,i}^{(3)}\}$ $(i=0, 1, \cdots, m)$ which are defined in each finite element Ω_i as follows:

$$(5) \qquad \begin{cases} b_{h,i-1}^{(1)}(s)=1-\bar{s}_i, & b_{h,i}^{(1)}(s)=\bar{s}_i, \\ b_{h,i-1}^{(2)}(s)=(1-\bar{s}_i)^2(1+2\bar{s}_i), & b_{h,i}^{(2)}(s)=(3-2\bar{s}_i)\bar{s}_i^2, \\ b_{h,i-1}^{(3)}(s)=L_i\bar{s}_i(1-\bar{s}_i)^2, & b_{h,i}^{(3)}(s)=L_i\bar{s}_i^2(\bar{s}_i-1), \end{cases}$$

where

$$\bar{s}_i=(s-s_{i-1})/L_i.$$

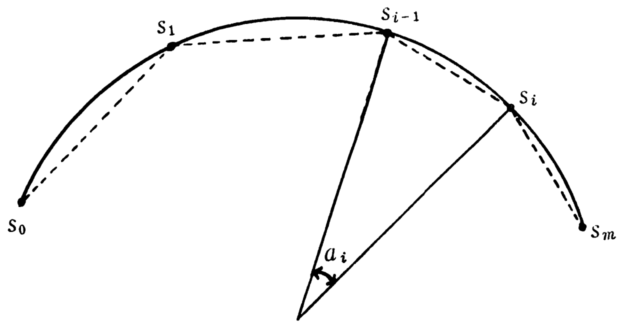


Figure 2. Finite element mesh.

Define finite dimensional spaces U^h, W^h and $\mathcal{S}^h$ by

$$U^h=\left\{u_h\,;\,u_h=\sum_{i=0}^{m}u_{h,i}b_{h,i}^{(1)},\ u_h=0 \text{ on } s=0,\ s=L\right\},$$

$$W^h=\left\{w^h\,;\,w_h=\sum_{i=0}^{m}(w_{h,i}b_{h,i}^{(2)}+\theta_{h,i}b_{h,i}^{(3)}),\ w_h=\frac{\mathrm{d}w_h}{\mathrm{d}s}=0 \text{ on } s=0,\ s=L\right\},$$

$$\mathcal{S}^h=U^h\times W^h\,,$$

where $u_{h,i}$, $w_{h,i}$, $\theta_{h,i}$ are nodal parameteres and

$$\theta_{h,i}=\frac{\mathrm{d}w_h(s_i)}{\mathrm{d}s}\,.$$

For $w_h\in W^h$, we also define $w_h^{(1)}$ and $w_h^{(2)}$ by

$$w_h^{(1)}=\sum_{i=0}^{m}w_{h,i}b_{h,i}^{(2)}\,,$$

$$w_h^{(2)}=\sum_{i=0}^{m}\theta_{h,i}b_{h,i}^{(3)}\,,$$

respectively. Let

$$\phi_{h,i}=\theta_{h,i}-u_{h,i}/R\,,$$

$$\boldsymbol{d}_i=(u_{h,i-1},\ u_{h,i},\ w_{h,i-1},\ w_{h,i},\ \phi_{h,i-1},\ \phi_{h,i})^t\,,$$

where t denotes the transpose. It is noted that $\mathcal{S}^h\subset\mathcal{H}_0$.

We now formulate the consistent approximation for the problem (3) as follows:

Find $\{\tilde{\lambda}_h,\ \tilde{u}_h,\ \tilde{w}_h\}\in\boldsymbol{R}^1\times\mathcal{S}^h$ *such that*

$$B(\tilde{u}_h,\ \tilde{w}_h\,;\,\bar{u}_h,\ \bar{w}_h)=\tilde{\lambda}_hG(\tilde{u}_h,\ \tilde{w}_h\,;\,\bar{u}_h,\ \bar{w}_h)\quad \textit{for each } \{\bar{u}_h,\ \bar{w}_h\}\in\mathcal{S}^h\,. \tag{6}$$

From (6), the stiffness and mass matrices $\tilde{\boldsymbol{k}}_i$, $\tilde{\boldsymbol{m}}_i$ for the curved element $\widehat{i-1,\,i}$ corresponding to the nodal displacement vector $\boldsymbol{d}_i$ are given by

$$\tilde{\boldsymbol{k}}_i=\tilde{\boldsymbol{k}}_i^{(1)}+\tilde{\boldsymbol{k}}_i^{(2)}\,,\qquad \tilde{\boldsymbol{m}}_i=\tilde{\boldsymbol{m}}_i^{(1)}+\tilde{\boldsymbol{m}}_i^{(2)}\qquad (\text{order } 6\times 6)\,,$$

where

$$\tilde{\boldsymbol{k}}_i^{(1)}=EA\begin{pmatrix} \frac{1}{L_i}-\frac{L_i}{6R^2}+\frac{L_i^3}{105R^4} & -\frac{1}{L_i}+\frac{L_i}{6R^2}-\frac{L_i^3}{140R^4} & -\frac{1}{2R}+\frac{11L_i^2}{210R^3} & -\frac{1}{2R}+\frac{13L_i^2}{420R^3} & -\frac{L_i}{12R}+\frac{L_i^3}{105R^3} & \frac{L_i}{12R}-\frac{L_i^3}{140R^3} \\ & \frac{1}{L_i}-\frac{L_i}{6R^2}+\frac{L_i^3}{105R^4} & \frac{1}{2R}-\frac{13L_i^2}{420R^3} & \frac{1}{2R}-\frac{11L_i^2}{210R^3} & \frac{L_i}{12R}-\frac{L_i^3}{140R^3} & -\frac{L_i}{12R}+\frac{L_i^3}{105R^3} \\ & & \frac{13L_i}{35R^2} & \frac{9L_i}{70R^2} & \frac{11L_i^2}{210R^2} & -\frac{13L_i^2}{420R^2} \\ & & & \frac{13L_i}{35R^2} & \frac{13L_i^2}{420R^2} & -\frac{11L_i^2}{210R^2} \\ & \text{symmetric} & & & \frac{L_i^3}{105R^2} & -\frac{L_i^3}{140R^2} \\ & & & & & \frac{L_i^3}{105R^2} \end{pmatrix},$$

$$\tilde{\boldsymbol{k}}_i^{(2)}=EI\begin{pmatrix} \frac{3}{R^2L_i} & \frac{3}{R^2L_i} & \frac{6}{RL_i^2} & -\frac{6}{RL_i^2} & \frac{3}{RL_i} & \frac{3}{RL_i} \\ & \frac{3}{R^2L_i} & \frac{6}{RL_i^2} & -\frac{6}{RL_i^2} & \frac{3}{RL_i} & \frac{3}{RL_i} \\ & & \frac{12}{L_i^3} & -\frac{12}{L_i^3} & \frac{6}{L_i^2} & \frac{6}{L_i^2} \\ & & & \frac{12}{L_i^3} & -\frac{6}{L_i^2} & -\frac{6}{L_i^2} \\ & \text{symmetric} & & & \frac{4}{L_i} & \frac{2}{L_i} \\ & & & & & \frac{4}{L_i} \end{pmatrix},$$

$$\tilde{\boldsymbol{m}}_i^{(1)}=\rho\begin{pmatrix} \frac{L_i}{3} & \frac{L_i}{6} & 0 & 0 & 0 & 0 \\ & \frac{L_i}{3} & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & \text{symmetric} & & & 0 & 0 \\ & & & & & 0 \end{pmatrix},$$

$$\tilde{m}_i^{(2)}=\rho\begin{pmatrix} \frac{L_i^3}{105R^2} & -\frac{L_i^3}{140R^2} & \frac{11L_i^2}{210R} & \frac{13L_i^2}{420R} & \frac{L_i^3}{105R} & -\frac{L_i^3}{140R} \\ & \frac{L_i^3}{105R^2} & -\frac{13L_i^2}{420R} & -\frac{11L_i^2}{210R} & -\frac{L_i^3}{140R} & \frac{L_i^3}{105R} \\ & & \frac{13L_i}{35} & \frac{9L_i}{70} & \frac{11L_i^2}{210} & -\frac{13L_i^2}{420} \\ & & & \frac{13L_i}{35} & \frac{13L_i^2}{420} & -\frac{11L_i^2}{210} \\ & \text{symmetric} & & & \frac{L_i^3}{105} & -\frac{L_i^3}{140} \\ & & & & & \frac{L_i^3}{105} \end{pmatrix}.$$

In finite element analysis of the circular arch, each curved element $\widehat{i-1,\,i}$ is replaced by the straight beam element $\overline{i-1,\,i}$ which denotes its chord, as the physical model (see Figure 2). The length L_i^* and the nodal displacement $\bar{\boldsymbol{d}}_i$ of the beam $\overline{i-1,\,i}$ are given by

$$L_i^*=2R\sin\frac{a_i}{2}=L_i p_i\,,$$

$$\bar{\boldsymbol{d}}_i=(\bar{u}_{h,i-1},\ \bar{u}_{h,i},\ \bar{w}_{h,i-1},\ \bar{w}_{h,i},\ \bar{\phi}_{h,i-1},\ \bar{\phi}_{h,i})^t\,,$$

where

$$(7)\qquad a_i=\frac{L_i}{R},\qquad p_i=\frac{2\sin\frac{a_i}{2}}{a_i},$$

and $\bar{u}_{h,j}$, $\bar{w}_{h,j}$ and $\bar{\phi}_{h,j}$ are the tangential displacement, lateral displacement and rotation of the beam at nodal point j ($j=i-1,\ i$), and t denotes the transpose. The element stiffness and mass matrices $\bar{\boldsymbol{k}}_i$, $\bar{\boldsymbol{m}}_i$ of the beam $\overline{i-1,\,i}$ are well known ([9], [14]) and given by

$$\bar{\boldsymbol{k}}_i=\bar{\boldsymbol{k}}_i^{(1)}+\bar{\boldsymbol{k}}_i^{(2)}\,,\qquad \bar{\boldsymbol{m}}_i=\bar{\boldsymbol{m}}_i^{(1)}+\bar{\boldsymbol{m}}_i^{(2)}\qquad (\text{order } 6\times 6)\,,$$

where

$$\bar{\boldsymbol{k}}_i^{(1)}=EA\begin{pmatrix} \frac{1}{L_i^*} & -\frac{1}{L_i^*} & 0 & 0 & 0 & 0 \\ & \frac{1}{L_i^*} & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & \text{symmetric} & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix},$$

$$\bar{k}_i^{(2)}=EI\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & & \frac{12}{L_i^{*3}} & -\frac{12}{L_i^{*3}} & \frac{6}{L_i^{*2}} & \frac{6}{L_i^{*2}} \\ & & & \frac{12}{L_i^{*3}} & -\frac{6}{L_i^{*2}} & -\frac{6}{L_i^{*2}} \\ & \text{symmetric} & & & \frac{4}{L_i^*} & \frac{2}{L_i^*} \\ & & & & & \frac{4}{L_i^*} \end{pmatrix},$$

$$\bar{m}_i^{(1)}=\rho\begin{pmatrix} \frac{L_i^*}{3} & \frac{L_i^*}{6} & 0 & 0 & 0 & 0 \\ & \frac{L_i^*}{3} & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & \text{symmetric} & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix},$$

$$\bar{m}_i^{(2)}=\rho\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & & \frac{13L_i^*}{35} & \frac{9L_i^*}{70} & \frac{11L_i^{*2}}{210} & -\frac{13L_i^{*2}}{420} \\ & & & \frac{13L_i^*}{35} & \frac{13L_i^{*2}}{420} & -\frac{11L_i^{*2}}{210} \\ & \text{symmetric} & & & \frac{L_i^{*3}}{105} & -\frac{L_i^{*3}}{140} \\ & & & & & \frac{L_i^{*3}}{105} \end{pmatrix}.$$

Since $\bar{\boldsymbol{d}}_i$, $\bar{\boldsymbol{k}}_i$ and $\bar{\boldsymbol{m}}_i$ are expressed in the local element system, we transform them into $\boldsymbol{d}_i^*$, $\boldsymbol{k}_i^*$ and $\boldsymbol{m}_i^*$ in the global coordinate system, respectively in the following usual manner with the transformation matrix $\boldsymbol{T}_i$ ([9], [14]):

$$(8)\qquad \begin{cases} \boldsymbol{d}_i^*=\boldsymbol{T}_i^t\bar{\boldsymbol{d}}_i\,, \\ \boldsymbol{k}_i^*=\boldsymbol{T}_i^t\bar{\boldsymbol{k}}_i\boldsymbol{T}_i\,, \\ \boldsymbol{m}_i^*=\boldsymbol{T}_i^t\bar{\boldsymbol{m}}_i\boldsymbol{T}_i\,, \end{cases}$$

where

$$\boldsymbol{d}_i^*=(u_{h,i-1},\ u_{h,i},\ w_{h,i-1},\ w_{h,i},\ \phi_{h,i-1},\ \phi_{h,i})^t\,,$$

$$\boldsymbol{T}_i=\begin{pmatrix} q_i & 0 & -r_i & 0 & 0 & 0 \\ 0 & q_i & 0 & r_i & 0 & 0 \\ r_i & 0 & q_i & 0 & 0 & 0 \\ 0 & -r_i & 0 & q_i & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$q_i=\cos\frac{a_i}{2},\qquad r_i=\sin\frac{a_i}{2}. \tag{9}$$

Following Kikuchi [7], [8], we note that there are considerable differences not only between $\tilde{\boldsymbol{k}}_i$ and $\boldsymbol{k}_i^*$ but also between $\tilde{\boldsymbol{m}}_i$ and $\boldsymbol{m}_i^*$.

Now we introduce mappings J_1, J_2 and J_3 defined by

$$J_1:\ W^h \longrightarrow L_2(\Omega),\qquad J_1 w_h(s)=(w_{h,i-1}+w_{h,i})/2,\qquad s_{i-1}\leqq s<s_i,$$

$$J_2:\ W^h \longrightarrow L_2(\Omega),$$

$$J_2 w_h(s)=-\frac{L_i}{2R}w_{h,i-1}b_{h,i-1}^{(1)}+\frac{L_i}{2R}w_{h,i}b_{h,i}^{(1)},\qquad s_{i-1}\leqq s<s_i,$$

$$J_3:\ U^h \longrightarrow L_2(\Omega),$$

$$J_3 u_h(s)=\frac{L_i}{2R}u_{h,i-1}b_{h,i-1}^{(2)}-\frac{L_i}{2R}u_{h,i}b_{h,i}^{(2)}-\frac{1}{R}u_{h,i-1}b_{h,i-1}^{(3)}-\frac{1}{R}u_{h,i}b_{h,i}^{(3)},$$
$$s_{i-1}\leqq s<s_i,$$

where

$$u_h=\sum_{i=0}^{m}u_{h,i}b_{h,i}^{(1)}\in U^h,\qquad w_h=\sum_{i=0}^{m}w_{h,i}b_{h,i}^{(2)}+\sum_{i=0}^{m}\theta_{h,i}b_{h,i}^{(3)}\in W^h.$$

Define bilinear forms B_h and G_h on $\mathcal{S}^h\times\mathcal{S}^h$ as follows:

$$B_h(u_h,\ w_h\,;\ \bar{u}_h,\ \bar{w}_h)=EA\Big(\frac{\mathrm{d}u_h}{\mathrm{d}s}+J_1w_h/R,\ \frac{\mathrm{d}\bar{u}_h}{\mathrm{d}s}+J_1\bar{w}_h/R\Big)$$
$$+EI\Big(\frac{\mathrm{d}^2w_h}{\mathrm{d}s^2}-\frac{\mathrm{d}u_h}{\mathrm{d}s}\frac{1}{R},\ \frac{\mathrm{d}^2\bar{w}_h}{\mathrm{d}s^2}-\frac{\mathrm{d}\bar{u}_h}{\mathrm{d}s}\frac{1}{R}\Big),$$

$$G_h(u_h,\ w_h\,;\ \bar{u}_h,\ \bar{w}_h)$$
$$=\rho\{(u_h+J_2w_h,\ \bar{u}_h+J_2\bar{w}_h)+(J_3u_h+w_h,\ J_3\bar{u}_h+\bar{w}_h)\},$$

for $\{u_h,\ w_h\}$, $\{\bar{u}_h,\ \bar{w}_h\}\in\mathcal{S}^h$. The following quantities are also well defined:

$$N_h(u_h,\ w_h)=[B_h(u_h,\ w_h\,;\ u_h,\ w_h)]^{1/2},$$

$$M_h(u_h,\ w_h)=[G_h(u_h,\ w_h\,;\ u_h,\ w_h)]^{1/2}.$$

It is noted that the triangle inequality for N_h or M_h, and Schwarz's inequality for N_h, B_h or M_h, G_h hold. For $\{u_h,\ w_h\}$, $\{\bar{u}_h,\ \bar{w}_h\}\in\mathcal{S}^h$ and $\{u,\ w\}$, $\{\bar{u},\ \bar{w}\}\in\mathcal{H}$, put

$$\tilde{B}_h(u_h+u,\ w_h+w\ ;\ \bar{u}_h+\bar{u},\ \bar{w}_h+\bar{w})$$

$$=EA\Big(\frac{\mathrm{d}(u_h+u)}{\mathrm{d}s}+(J_1w_h+w)/R,\ \frac{\mathrm{d}(\bar{u}_h+\bar{u})}{\mathrm{d}s}+(J_1\bar{w}_h+\bar{w})/R\Big)$$

$$+EI\Big(\frac{\mathrm{d}^2(w_h+w)}{\mathrm{d}s^2}-\frac{\mathrm{d}(u_h+u)}{\mathrm{d}s}\frac{1}{R},\ \frac{\mathrm{d}^2(\bar{w}_h+\bar{w})}{\mathrm{d}s^2}-\frac{\mathrm{d}(\bar{u}_h+\bar{u})}{\mathrm{d}s}\frac{1}{R}\Big),$$

$$\tilde{B}_h(u,\ w\ ;\ u_h,\ w_h)$$

$$=EA\Big(\frac{\mathrm{d}u}{\mathrm{d}s}+w/R,\ \frac{\mathrm{d}u_h}{\mathrm{d}s}+J_1w_h/R\Big)$$

$$+EI\Big(\frac{\mathrm{d}^2w}{\mathrm{d}s^2}-\frac{\mathrm{d}u}{\mathrm{d}s}\frac{1}{R},\ \frac{\mathrm{d}^2w_h}{\mathrm{d}s^2}-\frac{\mathrm{d}u_h}{\mathrm{d}s}\frac{1}{R}\Big),$$

$$\tilde{N}_h(u_h+u,\ w_h+w)=[\tilde{B}_h(u_h+u,\ w_h+w\ ;\ u_h+u,\ w_h+w)]^{1/2},$$

$$\tilde{G}_h(u_h+u,\ w_h+w\ ;\ \bar{u}_h+\bar{u},\ \bar{w}_h+\bar{w})$$

$$=\rho\{(u_h+J_2w_h+u,\ \bar{u}_h+J_2\bar{w}_h+\bar{u})+(J_3u_h+w_h+w,\ J_3\bar{u}_h+\bar{w}_h+\bar{w})\},$$

$$\tilde{G}_h(u,\ w\ ;\ u_h,\ w_h)=\rho\{(u,\ u_h+J_2w_h)+(w,\ J_3u_h+w_h)\},$$

$$\tilde{M}_h(u_h+u,\ w_h+w)=[\tilde{G}_h(u_h+u,\ w_h+w\ ;\ u_h+u,\ w_h+w)]^{1/2}.$$

We now formulate the partial approximation with the semi-consistent mass seheme for the problem (3) in the following manner:

$$\textit{Find}\ \{\hat{\lambda}_h,\ \hat{u}_h,\ \hat{w}_h\}\in \boldsymbol{R}^1\times\mathcal{S}^h\qquad \textit{such that}$$

(10) $$B_h(\hat{u}_h,\ \hat{w}_h\ ;\ \bar{u}_h,\ \bar{w}_h)=\hat{\lambda}_hG_h(\hat{u}_h,\ \hat{w}_h\ ;\ \bar{u}_h,\ \bar{w}_h)\quad \textit{for each}\ \{\bar{u}_h,\ \bar{w}_h\}\in\mathcal{S}^h.$$

The stiffness and mass matrices $\boldsymbol{k}_i$, $\boldsymbol{m}_i$ for the curved element $\widehat{i-1,\ i}$, induced from (10) are given by

(11) $$\boldsymbol{k}_i=\boldsymbol{k}_i^{(1)}+\boldsymbol{k}_i^{(2)},\qquad \boldsymbol{m}_i=\boldsymbol{m}_i^{(1)}+\boldsymbol{m}_i^{(2)}\quad (\text{order } 6\times 6),$$

where

$$\boldsymbol{k}_i^{(1)}=EA\begin{pmatrix} \frac{1}{L_i} & -\frac{1}{L_i} & -\frac{1}{2R} & -\frac{1}{2R} & 0 & 0 \\ & \frac{1}{L_i} & \frac{1}{2R} & \frac{1}{2R} & 0 & 0 \\ & & \frac{L_i}{4R^2} & \frac{L_i}{4R^2} & 0 & 0 \\ & \text{symmetric} & & \frac{L_i}{4R^2} & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix},$$

$\boldsymbol{k}_i^{(2)}=\tilde{\boldsymbol{k}}_i^{(2)}$,

$$\boldsymbol{m}_i^{(1)}=\rho\begin{pmatrix} \frac{L_i}{3} & \frac{L_i}{6} & -\frac{L_i^2}{6R} & -\frac{L_i^2}{12R} & 0 & 0 \\ & \frac{L_i}{3} & -\frac{L_i^2}{12R} & \frac{L_i^2}{6R} & 0 & 0 \\ & & \frac{L_i^3}{12R^2} & -\frac{L_i^3}{24R^2} & 0 & 0 \\ & \text{symmetric} & & \frac{L_i^3}{12R^2} & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix},$$

$$\boldsymbol{m}_i^{(2)}=\rho\begin{pmatrix} \frac{13L_i^3}{140R^2} & -\frac{9L_i^3}{280R^2} & \frac{13L_i^2}{70R} & \frac{9L_i^2}{140R} & \frac{11L_i^3}{420R} & -\frac{13L_i^3}{840R} \\ & \frac{13L_i^3}{140R^2} & -\frac{9L_i^2}{140R} & -\frac{13L_i^2}{70R} & -\frac{13L_i^3}{840R} & \frac{11L_i^3}{420R} \\ & & \frac{13L_i}{35} & \frac{9L_i}{70} & \frac{11L_i^2}{210} & -\frac{13L_i^2}{420} \\ \text{symmetric} & & & \frac{13L_i}{35} & \frac{13L_i}{420} & -\frac{11L_i^2}{210} \\ & & & & \frac{L_i^3}{105} & -\frac{L_i^3}{140} \\ & & & & & \frac{L_i^3}{105} \end{pmatrix}.$$

Henceforth we shall call $\boldsymbol{m}_i$ the semi-consistent mass matrix for the curved element $\widehat{i-1,\, i}$.

In order to establish the variational formulation of the straight beam element approximation, let us introduce bilinear forms as follows:

$$B_h^*(u_h,\, w_h\,;\, \bar{u}_h,\, \bar{w}_h)$$

$$=\sum_{i=1}^{m}\Big[EA\Big(\frac{q_i}{\sqrt{p_i}}\frac{\mathrm{d}u_h}{\mathrm{d}s}+\sqrt{p_i}J_1w_h/R,\ \frac{q_i}{\sqrt{p_i}}\frac{\mathrm{d}\bar{u}_h}{\mathrm{d}s}+\sqrt{p_i}J_1\bar{w}_h/R\Big)_{\Omega_i}$$

$$+EI\Big(\frac{q_i}{\sqrt{p_i}p_i}\frac{\mathrm{d}^2w_h^{(1)}}{\mathrm{d}s^2}+\frac{1}{\sqrt{p_i}}\frac{\mathrm{d}^2w_h^{(2)}}{\mathrm{d}s^2}-\frac{1}{\sqrt{p_i}}\frac{\mathrm{d}u_h}{\mathrm{d}s}\frac{1}{R},$$

$$\frac{q_i}{\sqrt{p_i}p_i}\frac{\mathrm{d}^2\bar{w}_h^{(1)}}{\mathrm{d}s^2}+\frac{1}{\sqrt{p_i}}\frac{\mathrm{d}^2\bar{w}_h^{(2)}}{\mathrm{d}s^2}-\frac{1}{\sqrt{p_i}}\frac{\mathrm{d}\bar{u}_h}{\mathrm{d}s}\frac{1}{R}\Big)_{\Omega_i}\Big],$$

$$G_h^*(u_h,\, w_h\,;\, \bar{u}_h,\, \bar{w}_h)$$

$$=\sum_{i=1}^{m}\rho[(\sqrt{p_i}q_iu_h+\sqrt{p_i}p_iJ_2w_h,\ \sqrt{p_i}q_i\bar{u}_h+\sqrt{p_i}p_iJ_2\bar{w}_h)_{\Omega_i}$$

$$+(\sqrt{p_i}p_iJ_3u_h+\sqrt{p_i}q_iw_h^{(1)}+\sqrt{p_i}p_iw_h^{(2)},\ \sqrt{p_i}p_iJ_3\bar{u}_h+\sqrt{p_i}q_i\bar{w}_h^{(1)}+\sqrt{p_i}p_i\bar{w}_h^{(2)})_{\Omega_i}],$$

for $\{u_h, w_h\}=\{u_h, w_h^{(1)}+w_h^{(2)}\}$, $\{\bar{u}_h, \bar{w}_h\}=\{\bar{u}_h, \bar{w}_h^{(1)}+\bar{w}_h^{(2)}\}\in\mathcal{S}^h$. Here $(\cdot, \cdot)_{\Omega_i}$ denotes the $L_2(\Omega_i)$ inner product. The following quantities are also well defined:

$$N_h^*(u_h, w_h)=[B_h^*(u_h, w_h; u_h, w_h)]^{1/2},$$

$$M_h^*(u_h, w_h)=[G_h^*(u_h, w_h; u_h, w_h)]^{1/2}.$$

Also we put

$$\begin{aligned}
&\tilde{B}_h^*(u_h+u, w_h+w; \bar{u}_h+\bar{u}, \bar{w}_h+\bar{w})\\
&=\sum_{i=1}^m\Big[EA\Big(\frac{q_i}{\sqrt{p_i}}\frac{\mathrm{d}u_h}{\mathrm{d}s}+\frac{\mathrm{d}u}{\mathrm{d}s}+(\sqrt{p_i}J_1w_h+w)/R,\ \frac{q_i}{\sqrt{p_i}}\frac{\mathrm{d}\bar{u}_h}{\mathrm{d}s}+\frac{\mathrm{d}\bar{u}}{\mathrm{d}s}\\
&\qquad\qquad+(\sqrt{p_i}J_1\bar{w}_h+\bar{w})/R\Big)_{\Omega_i}\\
&+EI\Big(\frac{q_i}{\sqrt{p_ip_i}}\frac{\mathrm{d}^2w_h^{(1)}}{\mathrm{d}s^2}+\frac{1}{\sqrt{p_i}}\frac{\mathrm{d}^2w_h^{(2)}}{\mathrm{d}s^2}+\frac{\mathrm{d}^2w}{\mathrm{d}s^2}-\Big(\frac{1}{\sqrt{p_i}}\frac{\mathrm{d}u_h}{\mathrm{d}s}+\frac{\mathrm{d}u}{\mathrm{d}s}\Big)/R,\\
&\qquad\frac{q_i}{\sqrt{p_ip_i}}\frac{\mathrm{d}^2\bar{w}_h^{(1)}}{\mathrm{d}s^2}+\frac{1}{\sqrt{p_i}}\frac{\mathrm{d}^2\bar{w}_h^{(2)}}{\mathrm{d}s^2}+\frac{\mathrm{d}^2\bar{w}}{\mathrm{d}s^2}-\Big(\frac{1}{\sqrt{p_i}}\frac{\mathrm{d}\bar{u}_h}{\mathrm{d}s}+\frac{\mathrm{d}\bar{u}}{\mathrm{d}s}\Big)/R\Big)_{\Omega_i}\Big],\\
&\tilde{G}_h^*(u_h+u, w_h+w; \bar{u}_h+\bar{u}, \bar{w}_h+\bar{w})\\
&=\sum_{i=1}^m\rho[(\sqrt{p_i}q_iu_h+\sqrt{p_i}p_iJ_2w_h+u,\ \sqrt{p_i}q_i\bar{u}_h+\sqrt{p_i}p_iJ_2\bar{w}_h+\bar{u})_{\Omega_i}\\
&\quad+(\sqrt{p_i}p_iJ_3u_h+\sqrt{p_i}q_iw_h^{(1)}+\sqrt{p_i}p_iw_h^{(2)}+w,\\
&\qquad\sqrt{p_i}p_iJ_3\bar{u}_h+\sqrt{p_i}q_i\bar{w}_h^{(1)}+\sqrt{p_i}p_i\bar{w}_h^{(2)}+\bar{w})_{\Omega_i}],
\end{aligned}$$

for $\{u_h, w_h\}$, $\{\bar{u}_h, \bar{w}_h\}\in\mathcal{S}^h$, $\{u, w\}$, $\{\bar{u}, \bar{w}\}\in\mathcal{H}_0$. Furthermore, we put

$$\tilde{N}_h^*(u_h+u, w_h+w)=[\tilde{B}_h^*(u_h+u, w_h+w; u_h+u, w_h+w)]^{1/2},$$

$$\tilde{M}_h^*(u_h+u, w_h+w)=[\tilde{G}_h^*(u_h+u, w_h+w; u_h+u, w_h+w)]^{1/2}.$$

From direct calculations, we can obtain

$$\sum_{i=1}^m \boldsymbol{d}_i^t\boldsymbol{k}_i^*\boldsymbol{d}_i^*=B_h^*(u_h, w_h; \bar{u}_h, \bar{w}_h),$$

$$\sum_{i=1}^m \boldsymbol{d}_i^t\boldsymbol{m}_i^*\boldsymbol{d}_i^*=G_h^*(u_h, w_h; \bar{u}_h, \bar{w}_h),$$

where $\boldsymbol{d}_i=(u_{h,i-1}, u_{h,i}, w_{h,i-1}, w_{h,i}, \phi_{h,i-1}, \phi_{h,i})^t$ and $\boldsymbol{d}_i^*=(\bar{u}_{h,i-1}, \bar{u}_{h,i}, \bar{w}_{h,i-1}, \bar{w}_{h,i}, \bar{\phi}_{h,i-1}, \bar{\phi}_{h,i})^t$ are the nodal displacements in the global coordinate system. Thus we can formulate the straight beam approximation for the problem (3) as follows:

Find $\{\lambda_h^*, u_h^*, w_h^*\}\in\boldsymbol{R}^1\times\mathcal{S}^h$ *such that*

(12) $$B_h^*(u_h^*, w_h^*; \bar{u}_h, \bar{w}_h)=\lambda_h^*G_h^*(u_h^*, w_h^*; \bar{u}_h, \bar{w}_h)\quad \textit{for each } \{\bar{u}_h, \bar{w}_h\}\in\mathcal{S}^h.$$

Now, by using (11), the matrix expressions (8) can be rewritten as

$$\boldsymbol{k}_i^* = \boldsymbol{T}_i^{(1)} \boldsymbol{k}_i^{(1)} \boldsymbol{T}_i^{(1)} + \boldsymbol{T}_i^{(2)} \boldsymbol{k}_i^{(2)} \boldsymbol{T}_i^{(2)},$$

$$\boldsymbol{m}_i^* = \boldsymbol{T}_i^{(3)} \boldsymbol{m}_i^{(1)} \boldsymbol{T}_i^{(3)} + \boldsymbol{T}_i^{(4)} \boldsymbol{m}_i^{(2)} \boldsymbol{T}_i^{(4)},$$

where $\boldsymbol{T}_i^{(j)}$ ($j=1, 2, 3, 4$) are diagonal matrices given by

$$\boldsymbol{T}_i^{(1)} = \mathrm{diag}\,(q_i/\sqrt{p_i},\ q_i/\sqrt{p_i},\ \sqrt{p_i},\ \sqrt{p_i},\ 1,\ 1),$$

$$\boldsymbol{T}_i^{(2)} = \mathrm{diag}\,(1/\sqrt{p_i},\ 1/\sqrt{p_i},\ q_i/\sqrt{p_i^3},\ q_i/\sqrt{p_i^3},\ 1/\sqrt{p_i},\ 1/\sqrt{p_i}),$$

$$\boldsymbol{T}_i^{(3)} = \mathrm{diag}\,(\sqrt{p_i}q_i,\ \sqrt{p_i}q_i,\ \sqrt{p_i^3},\ \sqrt{p_i^3},\ 1,\ 1),$$

$$\boldsymbol{T}_i^{(4)} = \mathrm{diag}\,(\sqrt{p_i^3},\ \sqrt{p_i^3},\ \sqrt{p_i}q_i,\ \sqrt{p_i}q_i,\ \sqrt{p_i^3},\ \sqrt{p_i^3}).$$

On the other hand, from (7) and (9) it follows that

$$p_i = 1 + O(h^2), \qquad q_i = 1 + O(h^2). \tag{13}$$

Thus if we employ the following approximation in the matrices $\boldsymbol{T}_i^{(j)}$ ($j=1, 2, 3, 4$)

$$p_i \doteqdot 1, \qquad q_i \doteqdot 1,$$

we can obtain

$$\boldsymbol{k}_i^* \doteqdot \boldsymbol{k}_i^{(1)} + \boldsymbol{k}_i^{(2)} = \boldsymbol{k}_i, \qquad \boldsymbol{m}_i^* \doteqdot \boldsymbol{m}_i^{(1)} + \boldsymbol{m}_i^{(2)} = \boldsymbol{m}_i.$$

Therefore, the partial approximation with the semi-consistent mass scheme is similar to the straight beam approximation in the above sense.

Furthermore, we can propose the consistent approximation with the semi-consistent mass scheme in such a way that

$$\textit{Find } \{\bar{\bar{\lambda}}_h,\ \bar{\bar{u}}_h,\ \bar{\bar{w}}_h\} \in \boldsymbol{R}^1 \times \mathcal{S}^h \quad \textit{such that}$$

$$B(\bar{\bar{u}}_h,\ \bar{\bar{w}}_h\,;\ \bar{u}_h,\ \bar{w}_h) = \bar{\bar{\lambda}}_h G_h(\bar{\bar{u}}_h,\ \bar{\bar{w}}_h\,;\ \bar{u}_h,\ \bar{w}_h) \qquad \textit{for each } \{\bar{u}_h,\ \bar{w}_h\} \in \mathcal{S}^h. \tag{14}$$

The stiffness and mass matrices $\bar{\bar{\boldsymbol{k}}}_i$, $\bar{\bar{\boldsymbol{m}}}_i$ for the curved element $\widehat{i-1, i}$, induced from (14) are given by

$$\bar{\bar{\boldsymbol{k}}}_i = \tilde{\boldsymbol{k}}_i, \qquad \bar{\bar{\boldsymbol{m}}}_i = \boldsymbol{m}_i \quad (\text{order } 6\times 6).$$

4. Rate of convergence

In this section, we shall obtain error estimates for the approximations. By $\{\tilde{\lambda}_{h,i}\}$, $\{\hat{\lambda}_{h,i}\}$, $\{\lambda_{h,i}^*\}$ and $\{\bar{\bar{\lambda}}_{h,i}\}$ ($i=1, 2, \cdots, \bar{N}$, $\bar{N}=3m-3$), we denote the approximate eigenvalues defined by (6), (10), (12) and (14), respectively. Also $\{\tilde{u}_{h,i}, \tilde{w}_{h,i}\}$, $\{\hat{u}_{h,i}, \hat{w}_{h,i}\}$, $\{u_{h,i}^*, w_{h,i}^*\}$ and $\{\bar{\bar{u}}_{h,i}, \bar{\bar{w}}_{h,i}\}$ ($i=1, 2, \cdots, \bar{N}$) represent the eigenfunctions corresponding to $\{\tilde{\lambda}_{h,i}\}$, $\{\hat{\lambda}_{h,i}\}$, $\{\lambda_{h,i}^*\}$ and $\{\bar{\bar{\lambda}}_{h,i}\}$, respectively. By using Lemma 2 below, for sufficiently small h, the eigenvalues are arranged as

$$0 < \tilde{\lambda}_{h,1} \leqq \tilde{\lambda}_{h,2} \leqq \cdots \leqq \tilde{\lambda}_{h,\bar{N}},$$

$$0 < \hat{\lambda}_{h,1} \leqq \hat{\lambda}_{h,2} \leqq \cdots \leqq \hat{\lambda}_{h,\bar{N}},$$

$$0 < \lambda_{h,1}^* \leqq \lambda_{h,2}^* \leqq \cdots \leqq \lambda_{h,\bar{N}}^*,$$

$$0 < \bar{\bar{\lambda}}_{h,1} \leqq \bar{\bar{\lambda}}_{h,2} \leqq \cdots \leqq \bar{\bar{\lambda}}_{h,\bar{N}}.$$

The associated eigenfunctions can be orthonormalized in the sense that

$$G(\tilde{u}_{h,i},\ \tilde{w}_{h,i};\ \tilde{u}_{h,j},\ \tilde{w}_{h,j})=\delta_{ij},\quad G(\tilde{u}_{h,i},\ \tilde{w}_{h,i};\ u_i,\ w_i)\geqq 0,\qquad 1\leqq i,\ j\leqq \bar{N},$$

$$G_h(\hat{u}_{h,i},\ \hat{w}_{h,i};\ \hat{u}_{h,j},\ \hat{w}_{h,j})=\delta_{ij},\quad \tilde{G}_h(\hat{u}_{h,i},\ \hat{w}_{h,i};\ u_i,\ w_i)\geqq 0,\qquad 1\leqq i,\ j\leqq \bar{N},$$

$$G_h^*(u_{h,i}^*,\ w_{h,i}^*;\ u_{h,j}^*,\ w_{h,j}^*)=\delta_{ij},\quad \tilde{G}_h^*(u_{h,i}^*,\ w_{h,i}^*;\ u_i,\ w_i)\geqq 0,\qquad 1\leqq i,\ j\leqq \bar{N},$$

$$G_h(\bar{u}_{h,i},\ \bar{w}_{h,i};\ \bar{u}_{h,j},\ \bar{w}_{h,j})=\delta_{ij},\quad \tilde{G}_h(\bar{u}_{h,i},\ \bar{w}_{h,i};\ u_i,\ w_i)\geqq 0,\qquad 1\leqq i,\ j\leqq \bar{N}.$$

Since $\mathcal{S}^h\subset\mathcal{H}_0$, it is clear from the min-max principle ([17]) that

$$\tilde{\lambda}_{h,i}\geqq\lambda_i,\qquad i=1,\ 2,\ \cdots \bar{N}.$$

First, in order to derive error bounds for the eigenvalues of (10) some results which we shall use are prepared.

Lemma 1. *For* $\{u_h,\ w_h\}=\{u_h,\ w_h^{(1)}+w_h^{(2)}\}\in\mathcal{S}^h$, *there exist positive constants* C_i $(i=1,\ 2,\ 3,\ 4,\ 5)$ *independent of* h *such that*

$$\|J_1w_h-w_h\|\leqq C_1h\|w_h\|,$$

$$\left\|\frac{\mathrm{d}^2w_h^{(j)}}{\mathrm{d}s^2}\right\|\leqq C_2\left\|\frac{\mathrm{d}^2w_h}{\mathrm{d}s^2}\right\|/h,\qquad j=1,\ 2,$$

$$\|w_h^{(j)}\|\leqq C_3\|w_h\|,\qquad j=1,\ 2,$$

$$\|J_2w_h\|\leqq C_4h\|w_h\|,$$

$$\|J_3u_h\|\leqq C_5h\|u_h\|.$$

Proof. The first two inequalities are proved in [7]. Let

$$u_h=\sum_{i=0}^m u_{h,i}b_{h,i}^{(1)},\qquad w_h=w_h^{(1)}+w_h^{(2)}=\sum_{i=0}^m(w_{h,i}b_{h,i}^{(2)}+\theta_{h,i}b_{h,i}^{(3)}).$$

Since the basis functions $\{b_{h,i-1}^{(2)},\ b_{h,i}^{(2)},\ b_{h,i-1}^{(3)},\ b_{h,i}^{(3)}\}$ defined by (5) are linearly independent on Ω_i, there exist positive constants K_1 and K_2 independent of h such that

$$\frac{1}{L_i}\|w_h\|_{\Omega_i}^2\geqq K_1(w_{h,i-1}^2+w_{h,i}^2+L_i^2\theta_{h,i-1}^2+L_i^2\theta_{h,i}^2),$$

$$\|J_3u_h\|_{\Omega_i}^2\leqq K_2\frac{L_i^3}{R^2}(u_{h,i-1}^2+u_{h,i}^2).$$

Simple calculations yield

$$\|w_h^{(1)}\|_{\Omega_i}^2=L_i(13w_{h,i-1}^2+9w_{h,i-1}w_{h,i}+13w_{h,i}^2)/35,$$

$$\|w_h^{(2)}\|_{\Omega_i}^2=L_i^3(2\theta_{h,i-1}^2-3\theta_{h,i-1}\theta_{h,i}+2\theta_{h,i}^2)/210.$$

Thus it follows that

$$\|w_h\|^2_{\Omega_i} \geqq K_1 L_i (w^2_{h,i-1} + w^2_{hh,i}) = K_1 L_i (18 w^2_{h,i-1} + 18 w^2_{h,i})/18$$
$$\geqq K_1 L_i (13 w^2_{h,i-1} + 9 w_{h,i-1} w_{h,i} + 13 w^2_{h,i})/18$$
$$= 35 K_1 \|w^{(1)}_h\|^2_{\Omega_i}/18\,,$$

and

$$\|w_h\|^2_{\Omega_i} \geqq K_1 L^3_i (\theta^2_{h,i-1} + \theta^2_{h,i}) = K_1 L^3_i (4\theta^2_{h,i-1} + 4\theta^2_{h,i})/4$$
$$\geqq K_1 L^3_i (2\theta^2_{h,i-1} - 3\theta_{h,i-1}\theta_{h,i} + 2\theta^2_{h,i})/4$$
$$= 210 K_1 \|w^{(2)}_h\|^2_{\Omega_i}/4\,.$$

Hence we have

$$\|w^{(j)}_h\| \leqq C_3 \|w_h\|\,, \qquad j=1,\,2\,.$$

Similarly, simple calculations yield

$$\|J_2 w_h\|^2_{\Omega_i} = \frac{L^3_i}{12R^2}(w^2_{h,i-1} - w_{h,i-1}w_{h,i} + w^2_{h,i}) \leqq \frac{h^2}{12R^2} L_i (2w^2_{h,i-1} + 2w^2_{h,i})\,,$$
$$\|u_h\|^2_{\Omega_i} = L_i (u^2_{h,i-1} + u_{h,i-1}u_{h,i} + u^2_{h,i})/3 \geqq L_i (u^2_{h,i-1} + u^2_{h,i})/6\,.$$

Thus we have

$$h^2 \|w_h\|^2_{\Omega_i} \geqq h^2 K_1 L_i (w^2_{h,i-1} + w^2_{h,i}) = 6R^2 K_1 \frac{h^2}{12R^2} L_i (2w^2_{h,i-1} + 2w^2_{h,i})$$
$$\geqq 6R^2 K_1 \|J_2 w_h\|^2_{\Omega_i}\,,$$
$$\|J_3 u_h\|^2_{\Omega_i} \leqq K_2 \frac{h^2 L_i}{R^2}(u^2_{h,i-1} + u^2_{h,i}) = \frac{6K_2 h^2}{R^2} L_i (u^2_{h,i-1} + u^2_{h,i})/6$$
$$\leqq \frac{6K_2 h^2}{R^2} \|u_h\|^2_{\Omega_i}\,.$$

Therefore, we obtain

$$\|J_2 w_h\| \leqq C_4 h \|w_h\|\,,$$
$$\|J_3 u_h\| \leqq C_5 h \|u_h\|\,.$$

The proof is complete.

Lemma 2. *Let* $\{u,\,w\} \in \mathcal{H}_0$. *Then we have*

$$C_1 N(u,\,w) \leqq \|u\|_1 + \|w\|_2 \leqq C_2 N(u,\,w)\,.$$

where C_1 *and* C_2 *are positive constants independent of* $\{u,\,w\}$. *Let* $\{u_h,\,w_h\} = \{u_h,\,w^{(1)}_h + w^{(2)}_h\} \in \mathcal{S}^h$. *Then for sufficiently small h, we have*

$$C_3 N(u_h,\,w_h) \leqq N_h(u_h,\,w_h) \leqq C_4 N(u_h,\,w_h)\,,$$
$$C_5 N_h(u_h,\,w_h) \leqq N^*_h(u_h,\,w_h) \leqq C_6 N_h(u_h,\,w_h)\,,$$
$$C_7 M(u_h,\,w_h) \leqq M_h(u_h,\,w_h) \leqq C_8 M(u_h,\,w_h)\,,$$
$$C_9 M_h(u_h,\,w_h) \leqq M^*_h(u_h,\,w_h) \leqq C_{10} M_h(u_h,\,w_h)\,,$$

where C_i $(i=3,\,4,\,\cdots,\,10)$ *are positive constants independent of h.*

Proof. The first three inequalities are proved in [7]. From the definition of M_h, it follows that

$$[M_h(u_h, w_h)]^2=\rho(\|u_h+J_2w_h\|^2+\|J_3u_h+w_h\|^2)$$
$$=[M(u_h, w_h)]^2+2\rho(u_h, J_2w_h)+\rho\|J_2w_h\|^2+2\rho(J_3u_h, w_h)+\rho\|J_3u_h\|^2.$$

Thus applying Schwarz's inequality and Lemma 1, we have

$$|[M_h(u_h, w_h)]^2-[M(u_h, w_h)]^2|$$
$$\leqq C(2\rho\|u_h\|h\|w_h\|+\rho h^2\|w_h\|^2+2\rho h\|u_h\|\cdot\|w_h\|+\rho h^2\|u_h\|^2)$$
$$\leqq C\{\rho h^2(\|u_h\|^2+\|w_h\|^2)+2\rho h(\|u_h\|^2+\|w_h\|^2)\}$$
$$=Ch(h+2)[M(u_h, w_h)]^2.$$

Therefore for sufficiently small h, there exist positive constants C_7 and C_8 independent of h such that

$$C_7M(u_h, w_h)\leqq M_h(u_h, w_h)\leqq C_8M(u_h, w_h).$$

Similarly, from the definitions of M_h and M_h^* and (13), Schwarz's inequality and Lemma 1, we have

$$|[M_h(u_h, w_h)]^2-[M_h^*(u_h, w_h)]^2|$$
$$\leqq Ch^2\rho[(\|u_h\|+\|J_2w_h\|)^2+(\|J_3u_h\|+\|w_h^{(1)}\|+\|w_h^{(2)}\|)^2]$$
$$\leqq \bar{\bar{C}}h^2\rho(\|u_h\|+\|w_h\|)^2\leqq \tilde{C}h^2[M(u_h, w_h)]^2\leqq \bar{C}h^2[M_h(u_h, w_h)]^2.$$

Hence for sufficiently small h, we have the desired inequality. This completes the proof.

Lemma 3 (Schultz [16], Kikuchi [7]). *For a given* $\{f_1, f_2\}\in L_2(\Omega)\times L_2(\Omega)$, *define* $\{u, w\}\in\mathcal{H}_0$ *by*

$$B(u, w; \bar{u}, \bar{w})=(f_1, \bar{u})+(f_2, \bar{w}) \qquad \textit{for each } \{\bar{u}, \bar{w}\}\in\mathcal{H}_0.$$

Let $\{\hat{u}_h, \hat{w}_h\}\in\mathcal{S}^h$ *be an interpolating element such that*

$$\hat{u}_h=\sum_{i=0}^{m}u(s_i)b_{h,i}^{(1)},$$

$$\hat{w}_h=\sum_{i=0}^{m}\left(w(s_i)b_{h,i}^{(2)}+\frac{dw(s_i)}{ds}b_{h,i}^{(3)}\right).$$

Then, $u\in H_0^1(\Omega)\cap H^2(\Omega)$, $w\in H_0^2(\Omega)\cap H^3(\Omega)$ *and*

$$\tilde{N}_h(\hat{u}_h-u, \hat{w}_h-w)\leqq C_1h(\|u\|_2+\|w\|_3)\leqq C_2h(\|f_1\|+\|f_2\|),$$

where C_1 *and* C_2 *are positive constants independent of* h.

We now have the following proposition.

Proposition 1. *Let* $\{f_1, f_2\}\in L_2(\Omega)\times L_2(\Omega)$. *Define* $\{u, w\}\in\mathcal{H}_0$, $\{u_h, w_h\}\in\mathcal{S}^h$,

$\{\tilde{\bar{u}}_h, \tilde{\bar{w}}_h\} \in \mathcal{S}^h$, $\{u_h^*, w_h^*\} \in \mathcal{S}^h$ *and* $\{\bar{\bar{u}}_h, \bar{\bar{w}}_h\} \in \mathcal{S}^h$ *by*

(15) $$B(u, w\,; \bar{u}, \bar{w}) = (f_1, \bar{u}) + (f_2, \bar{w}) \qquad \text{for each } \{\bar{u}, \bar{w}\} \in \mathcal{H}_0\,,$$

(16) $$B_h(u_h, w_h\,; \bar{u}_h, \bar{w}_h) = (f_1, \bar{u}_h + J_2\bar{w}_h) + (f_2, J_3\bar{u}_h + \bar{w}_h) \qquad \text{for each } \{\bar{u}_h, \bar{w}_h\} \in \mathcal{S}^h\,,$$

$$B(\tilde{\bar{u}}_h, \tilde{\bar{w}}_h\,; \bar{u}_h, \bar{w}_h) = (f_1, \bar{u}_h) + (f_2, \bar{w}_h) \qquad \text{for each } \{\bar{u}_h, \bar{w}_h\} \in \mathcal{S}^h\,,$$

$$B_h^*(u_h^*, w_h^*\,; \bar{u}_h, \bar{w}_h) = \sum_{i=1}^{m} [(f_1, \sqrt{p_i}q_i\bar{u}_h + \sqrt{p_i}p_iJ_2\bar{w}_h)_{\Omega_i} + (f_2, \sqrt{p_i}p_i\bar{u}_h + \sqrt{p_i}q_i\bar{w}_h^{(1)} + \sqrt{p_i}p_i\bar{w}_h^{(2)})_{\Omega_i}] \qquad \text{for each } \{\bar{u}_h, \bar{w}_h\} \in \mathcal{S}^h\,,$$

$$B(\bar{\bar{u}}_h, \bar{\bar{w}}_h\,; \bar{u}_h, \bar{w}_h) = (f_1, \bar{u}_h + J_2\bar{w}_h) + (f_2, J_3\bar{u}_h + \bar{w}_h) \qquad \text{for each } \{\bar{u}_h, \bar{w}_h\} \in \mathcal{S}^h\,,$$

respectively. Then, for sufficiently small h*, there exist positive constants* C_i $(i=1, \cdots, 8)$ *independent of* h *such that*

$$N_h(u_h, w_h) \leqq C_1(\|f_1\| + \|f_2\|)\,,$$
$$\tilde{N}_h(u-u_h, w-w_h) \leqq C_2(\|f_1\| + \|f_2\|)h\,,$$
$$N(\tilde{\bar{u}}_h, \tilde{\bar{w}}_h) \leqq C_3(\|f_1\| + \|f_2\|)\,,$$
$$N(u-\tilde{\bar{u}}_h, w-\tilde{\bar{w}}_h) \leqq C_4(\|f_1\| + \|f_2\|)h\,,$$
$$N_h^*(u_h^*, w_h^*) \leqq C_5(\|f_1\| + \|f_2\|)\,,$$
$$\tilde{N}_h^*(u-u_h^*, w-w_h^*) \leqq C_6(\|f_1\| + \|f_2\|)h\,,$$
$$N(\bar{\bar{u}}_h, \bar{\bar{w}}_h) \leqq C_7(\|f_1\| + \|f_2\|)\,,$$
$$N(u-\bar{\bar{u}}_h, w-\bar{\bar{w}}) \leqq C_8(\|f_1\| + \|f_2\|)h\,.$$

Proof. From (16) and Lemmas 1, 2, it follows that

$$\begin{aligned}
[N_h(u_h, w_h)]^2 &= (f_1, u_h + J_2w_h) + (f_2, J_3u_h + w_h) \\
&\leqq \|f_1\|(\|u_h\| + \|J_2w_h\|) + \|f_2\|(\|J_3u_h\| + \|w_h\|) \\
&\leqq (\|f_1\| + \|f_2\|)(\|u_h\| + C_4h\|w_h\| + C_5h\|u_h\| + \|w_h\|) \\
&\leqq C(\|f_1\| + \|f_2\|)(\|u_h\| + \|w_h\|) \\
&\leqq C(\|f_1\| + \|f_2\|)(\|u_h\|_1 + \|w_h\|_2) \\
&\leqq C(\|f_1\| + \|f_2\|)N_h(u_h, w_h)\,.
\end{aligned}$$

Thus, we have

$$N_h(u_h, w_h) \leqq C_1(\|f_1\| + \|f_2\|)\,.$$

Furthermore by (15) and the definition of $\tilde{B}_h$, we have

$$\tilde{B}_h(u, w; \bar{u}_h, \bar{w}_h)= EA\Big(\frac{\mathrm{d}u}{\mathrm{d}s}+w/R, \frac{\mathrm{d}\bar{u}_h}{\mathrm{d}s}+\bar{w}_h/R\Big)$$

$$+EI\Big(\frac{\mathrm{d}^2 w}{\mathrm{d}s^2}-\frac{1}{R}\frac{\mathrm{d}u}{\mathrm{d}s}, \frac{\mathrm{d}^2\bar{w}_h}{\mathrm{d}s^2}-\frac{1}{R}\frac{\mathrm{d}\bar{u}_h}{\mathrm{d}s}\Big)+EA\Big(\frac{\mathrm{d}u}{\mathrm{d}s}+w/R, (J_1\bar{w}_h-\bar{w}_h)/R\Big)$$

$$=(f_1, \bar{u}_h)+(f_2, \bar{w}_h)+EA\Big(\frac{\mathrm{d}u}{\mathrm{d}s}+w/R, (J_1\bar{w}_h-\bar{w}_h)/R\Big).$$

Thus, from (16) and Lemmas 1, 2, 3, we have

$$|\tilde{B}_h(u_h-u, w_h-w; \bar{u}_h, \bar{w}_h)|=|\tilde{B}_h(u, w; \bar{u}_h, \bar{w}_h)-B_h(u_h, w_h; \bar{u}_h, \bar{w}_h)|$$

$$=|\tilde{B}_h(u, w; \bar{u}_h, \bar{w}_h)-(f_1, J_2\bar{w}_h+\bar{u}_h)-(f_2, J_3\bar{u}_h+\bar{w}_h)|$$

$$=\Big|-(f_1, J_2\bar{w}_h)-(f_2, J_3\bar{u}_h)+EA\Big(\frac{\mathrm{d}u}{\mathrm{d}s}+w/R, (J_1\bar{w}_h-\bar{w}_h)/R\Big)\Big|$$

$$\leqq \|f_1\|\cdot\|J_2\bar{w}_h\|+\|f_2\|\cdot\|J_3\bar{u}_h\|+C(\|f_1\|+\|f_2\|)\|J_1\bar{w}_h-\bar{w}_h\|$$

$$\leqq Ch\|f_1\|\cdot\|\bar{w}_h\|+Ch\|f_2\|\cdot\|\bar{u}_h\|+C(\|f_1\|+\|f_2\|)h\|\bar{w}_h\|_2$$

$$\leqq Ch(\|f_1\|+\|f_2\|)(\|\bar{u}_h\|_1+\|\bar{w}_h\|_2)\leqq Ch(\|f_1\|+\|f_2\|)N_h(\bar{u}_h, \bar{w}_h).$$

Therefore, by equating $\{\bar{u}_h, \bar{w}_h\}$ to $\{u_h-\hat{u}_h, w_h-\hat{w}_h\}$, we obtain

$$|B_h(u_h-\hat{u}_h, w_h-\hat{w}_h; u_h-\hat{u}_h, w_h-\hat{w}_h)-\tilde{B}_h(u-\hat{u}_h, w-\hat{w}_h; u_h-\hat{u}_h, w_h-\hat{w}_h)|$$
$$\leqq Ch(\|f_1\|+\|f_2\|)N_h(u_h-\hat{u}_h, w_h-\hat{w}_h),$$

where $\{\hat{u}_h, \hat{w}_h\}\in\mathcal{S}^h$ is the interpolating element of $\{u, w\}$. Applying Schwarz's inequality and Lemma 3 gives

$$[N_h(u_h-\hat{u}_h, w-\hat{w}_h)]^2\leqq Ch(\|f_1\|+\|f_2\|)N_h(u_h-\hat{u}_h, w_h-\hat{w}_h)$$
$$+\tilde{B}_h(u-\hat{u}_h, w-\hat{w}_h; u_h-\hat{u}_h, w_h-\hat{w}_h)$$
$$\leqq Ch(\|f_1\|+\|f_2\|)N_h(u_h-\hat{u}_h, w_h-\hat{w}_h)$$
$$+\tilde{N}_h(u-\hat{u}_h, w-\hat{w}_h)N_h(u_h-\hat{u}_h, w_h-\hat{w}_h)$$
$$\leqq C_1 h(\|f_1\|+\|f_2\|)N_h(u_h-\hat{u}_h, w_h-\hat{w}_h).$$

Thus

$$N_h(u_h-\hat{u}_h, w_h-\hat{w}_h)\leqq C_1 h(\|f_1\|+\|f_2\|).$$

Therefore, using the triangle inequality yields the desired inequality

$$\tilde{N}_h(u-u_h, w-w_h)\leqq C_1 h(\|f_1\|+\|f_2\|).$$

By the same technique as described in the above, we can obtain the others, applying (13) and Lemmas 1, 2, 3. This completes the proof.

Lemma 4. *Let*

$$\mathcal{E}_i=\{\{\bar{u}_i,\ \bar{w}_i\}\in\text{span}[\{u_1,\ w_1\},\ \cdots,\ \{u_i,\ w_i\}],\ M(\bar{u}_i,\ \bar{w}_i)=1\}.$$

For $\{\bar{u}_i,\ \bar{w}_i\}=\sum_{j=1}^{j}\alpha_j\{u_j,\ w_j\}\in\mathcal{E}_i$, *define* $\{\tilde{u}_{hi},\ \tilde{w}_{hi}\}\in\mathcal{S}^h$ *by*

$$B_h(\tilde{u}_{hi},\ \tilde{w}_{hi}\,;\ \bar{u}_h,\ \bar{w}_h)=\rho(\sum_{j=1}^{i}\lambda_j\alpha_ju_j,\ \bar{u}_h+J_2\bar{w}_h)+\rho(\sum_{j=1}^{i}\lambda_j\alpha_jw_j,\ J_3\bar{u}_h+\bar{w}_h)\tag{17}$$

$$\text{for each } \{\bar{u}_h,\ \bar{w}_h\}\in\mathcal{S}^h.$$

Then, for sufficiently small h, we have

$$\tilde{M}_h(\tilde{u}_{hi}-\bar{u}_i,\ \tilde{w}_{hi}-\bar{w}_i)\leqq Ch\lambda_i\,,\tag{18}$$

where C is a positive constant independent of h.

Proof. Since $\{u_j,\ w_j\}$ $(j=1,\ \cdots,\ i)$ are the solutions of (3), we have

$$B(\bar{u}_i,\ \bar{w}_i\,;\ \bar{u},\ \bar{w})=\rho(\sum_{j=1}^{i}\lambda_j\alpha_ju_j,\ \bar{u})+\rho(\sum_{j=1}^{i}\lambda_j\alpha_jw_j,\ \bar{w})$$

$$\text{for each } \{\bar{u},\ \bar{w}\}\in\mathcal{H}_0.$$

Thus from Lemma 1 and Proposition 1, we have

$$\begin{aligned}\tilde{M}_h(\tilde{u}_{hi}-\bar{u}_i,\ \tilde{w}_{hi}-\bar{w}_i)&\leqq C(\|\tilde{u}_{hi}-\bar{u}_i\|^2+\|J_2\tilde{w}_{hi}\|^2+\|\tilde{w}_{hi}-\bar{w}_i\|^2+\|J_3\tilde{u}_h\|^2)^{1/2}\\&\leqq C[\|\tilde{u}_{hi}-\bar{u}_i\|^2+\|\tilde{w}_{hi}-\bar{w}_i\|^2+Ch^2(\|\tilde{u}_{hi}\|^2+\|\tilde{w}_{hi}\|^2)]^{1/2}\\&\leqq Ch(\|\sum_{j=1}^{i}\lambda_j\alpha_ju_j\|+\|\sum_{j=1}^{i}\lambda_j\alpha_jw_j\|)\leqq Ch\lambda_iM(\bar{u}_i,\ \bar{w}_i)=Ch\lambda_i\,,\end{aligned}$$

for sufficiently small h. The proof is complete.

Lemma 5. *Let* $\mathcal{S}_i^h$ *be an arbitrary i-dimensional subspace of* $\mathcal{S}^h$. *Let* $\{u_h,\ w_h\}$ $(\neq\{0,\ 0\})$ *be an arbitrary element of* $\mathcal{S}_i^h$ *such that*

$$\tilde{G}_h(u_h,\ w_h\,;\ u_j,\ w_j)=0\,,\qquad j=1,\ 2,\ \cdots,\ i-1\,.\tag{19}$$

For $\{u_h,\ w_h\}$, *define* $\{u',\ w'\}\in\mathcal{H}_0$ *and* $\{u'_h,\ w'_h\}\in\mathcal{S}^h$ *by*

$$B(u',\ w'\,;\ \bar{u},\ \bar{w})=\rho(u_h+J_2w_h,\ \bar{u})+\rho(w_h+J_3u_h,\ \bar{w})\tag{20}$$

$$\text{for each } \{\bar{u},\ \bar{w}\}\in\mathcal{H}_0\,,$$

$$B_h(u'_h,\ w'_h\,;\ \bar{u}_h,\ \bar{w}_h)=\rho(u_h+J_2w_h,\ \bar{u}_h+J_2\bar{w}_h)+\rho(w_h+J_3u_h,\ \bar{w}_h+J_3\bar{u}_h)\tag{21}$$

$$\text{for each } \{\bar{u}_h,\ \bar{w}_h\}\in\mathcal{S}^h\,,$$

respectively. Then, $G(u',\ w'\,;\ u_j,\ w_j)=0$, $j=1,\ 2,\ \cdots,\ i-1$, *and for sufficiently small h, there exists a positive constant C independent of h such that*

$$B_h(u'_h,\ w'_h\,;\ u'_h,\ w'_h)\leqq\lambda_i^{-1}G_h(u_h,\ w_h\,;\ u_h,\ w_h)(1+C\lambda_ih)\,.$$

Proof. By (3), (20) and (19), we have

$$G(u_j,\ w_j\,;\ u',\ w')=\lambda_j^{-1}B(u_j,\ w_j\,;\ u',\ w') \tag{22}$$
$$=\rho\lambda_j^{-1}\{(u_h+J_2w_h,\ u_j)+(w_h+J_3u_h,\ w_j)\}=\lambda_j^{-1}\tilde{G}_h(u_h,\ w_h\,;\ u_j,\ w_j)=0$$
$$j=1,\ 2,\ \cdots,\ i-1\,.$$

Applying Proposition 1 and Lemma 1, we obtain

$$\tilde{M}_h(u'-u'_h,\ w'-w'_h)\leqq Ch(\|u_h+J_2w_h\|+\|w_h+J_3u_h\|)\leqq ChM_h(u_h,\ w_h)\,. \tag{23}$$

Using (20), we have

$$B(u',\ w'\,;\ u',\ w')=\rho(u_h+J_2w_h,\ u')+\rho(w_h+J_3u_h,\ w')$$
$$=\tilde{G}_h(u_h,\ w_h\,;\ u',\ w')\leqq M_h(u_h,\ w_h)\cdot M(u',\ w')\,.$$

Hence, applying (4) and (22) yields

$$\lambda_i[M(u',\ w')]^2\leqq B(u',\ w'\,;\ u',\ w')\leqq M_h(u_h,\ w_h)\cdot M(u',\ w')\,.$$

Thus we have

$$M(u',\ w')\leqq\lambda_i^{-1}M_h(u_h,\ w_h)\,.$$

From (21), it follows that

$$\begin{aligned}B_h(u'_h,\ w'_h\,;\ u'_h,\ w'_h)&=G_h(u_h,\ w_h\,;\ u'_h,\ w'_h)\\&\leqq M_h(u_h,\ w_h)\cdot M_h(u'_h,\ w'_h)\\&\leqq M_h(u_h,\ w_h)[\tilde{M}_h(u'_h-u',\ w'_h-w')+M(u',\ w')]\\&\leqq M_h(u_h,\ w_h)[\tilde{M}_h(u'_h-u',\ w'_h-w')+\lambda_i^{-1}M_h(u_h,\ w_h)]\\&\leqq M_h(u_h,\ w_h)[ChM_h(u_h,\ w_h)+\lambda_i^{-1}M_h(u_h,\ w_h)]\\&=\lambda_i^{-1}G_h(u_h,\ w_h\,;\ u_h,\ w_h)(1+C\lambda_ih)\,.\end{aligned}$$

This completes the proof.

We are now in a position to prove the following theorem for the eigenvalues of the partial approximation with the semi-consistent mass scheme.

Theorem 1. *Let λ_i and $\hat{\lambda}_{h,i}$ be the eigenvalues defined by* (3) *and* (10), *respectively. Then there exists a positive constant C independent of h such that*

$$|\lambda_i-\hat{\lambda}_{h,i}|\leqq C\lambda_i^2h\,,$$

for sufficiently small h.

Proof. We define a mapping $P:\mathcal{E}_i\to\mathcal{S}^h$ given by

$$P(\bar{u}_i,\ \bar{w}_i)=\{\tilde{u}_{hi},\ \tilde{w}_{hi}\},\qquad\{\bar{u}_i,\ \bar{w}_i\}\in\mathcal{E}_i\,,$$

where $\{\tilde{u}_{hi},\ \tilde{w}_{hi}\}\in\mathcal{S}^h$ is defined by (17). Using (17) and (18) yields

$$B_h(\tilde{u}_{hi},\ \tilde{w}_{hi}\,;\ \tilde{u}_{hi},\ \tilde{w}_{hi})=\tilde{G}_h\Big(\sum_{j=1}^{i}\lambda_j\alpha_ju_j,\ \sum_{j=1}^{i}\lambda_j\alpha_jw_j\,;\ \tilde{u}_{hi},\ \tilde{w}_{hi}\Big)$$

$$\leqq \lambda_i \tilde{M}_h(\bar{u}_i, \bar{w}_i)\cdot M_h(\tilde{u}_{hi}, \tilde{w}_{hi})$$

$$\leqq \lambda_i[\tilde{M}_h(\bar{u}_i-\tilde{u}_{hi}, \bar{w}_i-\tilde{w}_{hi})+M_h(\tilde{u}_{hi}, \tilde{w}_{hi})]M_h(\tilde{u}_{hi}, \tilde{w}_{hi})$$

$$\leqq \lambda_i[M_h(\tilde{u}_{hi}, \tilde{w}_{hi})]^2(1+C\lambda_i h)=\lambda_i G_h(\tilde{u}_{hi}, \tilde{w}_{hi}; \tilde{u}_{hi}, \tilde{w}_{hi})(1+C\lambda_i h).$$

Thus we have

$$\frac{B_h(P(\bar{u}_i, \bar{w}_i); P(\bar{u}_i, \bar{w}_i))}{G_h(P(\bar{u}_i, \bar{w}_i); P(\bar{u}_i, \bar{w}_i))}=\frac{B_h(\tilde{u}_{hi}, \tilde{w}_{hi}; \tilde{u}_{hi}, \tilde{w}_{hi})}{G_h(\tilde{u}_{hi}, \tilde{w}_{hi}; \tilde{u}_{hi}, \tilde{w}_{hi})}\leqq \lambda_i(1+C\lambda_i h).$$

Since $\mathcal{S}_i^h=P\mathcal{E}_i$ is i-dimensional, we obtain

(24) $$\hat{\lambda}_{h,i}\leqq \lambda_i(1+C\lambda_i h),$$

for sufficiently small h, from the min-max principle.

On the other hand, by (21) we have

$$G_h(u_h, w_h; u_h, w_h)=B_h(u_h', w_h'; u_h, w_h)\leqq N_h(u_h', w_h')N_h(u_h, w_h).$$

Thus an application of Lemma 5 leads to

$$[G_h(u_h, w_h; u_h, w_h)]^2\leqq B_h(u_h', w_h'; u_h', w_h')\cdot B_h(u_h, w_h; u_h, w_h)$$

$$\leqq \lambda_i^{-1}G_h(u_h, w_h; u_h, w_h)(1+\bar{C}\lambda_i h)B_h(u_h, w_h; u_h, w_h),$$

from which follows

$$\frac{B_h(u_h, w_h; u_h, w_h)}{G_h(u_h, w_h; u_h, w_h)}\geqq \lambda_i(1+\bar{C}\lambda_i h)^{-1}\geqq \lambda_i(1-C\lambda_i h).$$

Since $\{u_h, w_h\}\neq\{0, 0\}$ is an arbitrary element of $\mathcal{S}_i^h$, using the min-max principle yields

(25) $$\hat{\lambda}_{h,i}\geqq \lambda_i(1-C\lambda_i h),$$

for sufficiently small h. Combining (24) and (25) completes the proof.

Secondly we shall estimate the error bounds for the eigenfunctions of (10). For the exact solution $\{\lambda_i, u_i, w_i\}$ of (3), we define $\{\hat{u}_{hi}, \hat{u}_{hi}\}\in\mathcal{S}^h$ by

(26) $$B_h(\hat{u}_{hi}, \hat{w}_{hi}; \bar{u}_h, \bar{w}_h)=\lambda_i\tilde{G}_h(u_i, w_i; \bar{u}_h, \bar{w}_h) \quad \text{for each } \{\bar{u}_h, \bar{w}_h\}\in\mathcal{S}^h.$$

Then, applying Lemma 4 leads to

(27) $$\tilde{M}_h(\hat{u}_{hi}-u_i, \hat{w}_{hi}-w_i)\leqq C\lambda_i h=C_1 h.$$

Moreover, using the system $\{\{\hat{u}_{h,1}, \hat{w}_{h,1}\}, \cdots, \{\hat{u}_{h,i}, \hat{w}_{h,i}\}\}$ which is orthonormal with respect to G_h, we expand

(28) $$\{\hat{u}_{hi}, \hat{w}_{hi}\}=\sum_{k=1}^{i-1}G_h(\hat{u}_{hi}, \hat{w}_{hi}; \hat{u}_{h,k}, \hat{w}_{h,k})\{\hat{u}_{h,k}, \hat{w}_{h,k}\}$$

$$+G_h(\hat{u}_{hi}, \hat{w}_{hi}; \hat{u}_{h,i}, \hat{w}_{h,i})\{\hat{u}_{h,i}, \hat{w}_{h,i}\}+\{\hat{f}_{1i}'', \hat{f}_{2i}''\}.$$

We put

$$\{\hat{\hat{f}}'_{1i},\ \hat{\hat{f}}'_{2i}\}=\sum_{k=1}^{i-1}G_h(\hat{\hat{u}}_{hi},\ \hat{\hat{w}}_{hi}\,;\ \hat{u}_{h,k},\ \hat{w}_{h,k})\{\hat{u}_{h,k},\ \hat{w}_{h,k}\}. \tag{29}$$

In order to derive error bounds for the eigenfunctions, we obtain the following lemmas for the estimates of $M_h(\hat{\hat{f}}''_{1i},\ \hat{\hat{f}}''_{2i})$ and $M_h(\hat{\hat{f}}'_{1i},\ \hat{\hat{f}}'_{2i})$.

Lemma 6. *If $\lambda_i<\lambda_{i+1}$, then for sufficiently small h, there exists a positive constant C independent of h such that*

$$M_h(\hat{\hat{f}}''_{1i},\ \hat{\hat{f}}''_{2i})\leqq Ch\,.$$

Proof. From (28) it follows that

$$G_h(\hat{\hat{f}}''_{1i},\ \hat{\hat{f}}''_{2i}\,;\ \hat{u}_{h,k},\ \hat{w}_{h,k})=0\,,\qquad k=1,\ 2,\ \cdots,\ i\,. \tag{30}$$

By (10) we have

$$B_h(\hat{u}_{h,k},\ \hat{w}_{h,k}\,;\ \bar{u}_h,\ \bar{w}_h)=\hat{\lambda}_{h,k}G_h(\hat{u}_{h,k},\ \hat{w}_{h,k}\,;\ \bar{u}_h,\ \bar{w}_h) \tag{31}$$

$$\text{for each } \{\bar{u}_h,\ \bar{w}_h\}\in\mathcal{S}^h\,.$$

Moreover, combining (31) and (30) yields

$$B_h(\hat{u}_{h,k},\ \hat{w}_{h,k}\,;\hat{\hat{f}}''_{1i},\ \hat{\hat{f}}''_{2i})=\hat{\lambda}_{h,k}G_h(\hat{u}_{h,k},\ \hat{w}_{h,k}\,;\hat{\hat{f}}''_{1i},\ \hat{\hat{f}}''_{2i})=0\,. \tag{32}$$

$$k=1,\ 2,\ \cdots,\ i\,.$$

From (28) and (30), it follows that

$$G_h(\hat{\hat{u}}_{hi},\ \hat{\hat{w}}_{hi}\,;\hat{\hat{f}}''_{1i},\ \hat{\hat{f}}''_{2i})=\sum_{k=1}^{i}G_h(\hat{\hat{u}}_{hi},\ \hat{\hat{w}}_{hi}\,;\ \hat{u}_{h,k},\ \hat{w}_{h,k})G_h(\hat{u}_{h,k},\ \hat{w}_{h,k}\,;\hat{\hat{f}}''_{1i},\ \hat{\hat{f}}''_{2i}) \tag{33}$$

$$+G_h(\hat{\hat{f}}''_{1i},\ \hat{\hat{f}}''_{2i}\,;\hat{\hat{f}}''_{1i},\ \hat{\hat{f}}''_{2i})=G_h(\hat{\hat{f}}''_{1i},\ \hat{\hat{f}}''_{2i}\,;\hat{\hat{f}}''_{1i},\ \hat{\hat{f}}''_{2i})\,.$$

By (28) and (32), we obtain

$$B_h(\hat{\hat{u}}_{hi},\ \hat{\hat{w}}_{hi}\,;\hat{\hat{f}}''_{1i},\ \hat{\hat{f}}''_{2i})=\sum_{k=1}^{i}G_h(\hat{\hat{u}}_{hi},\ \hat{\hat{w}}_{hi}\,;\ \hat{u}_{h,k},\ \hat{w}_{h,k})B_h(\hat{u}_{h,k},\ \hat{w}_{h,k}\,;\hat{\hat{f}}''_{1i},\ \hat{\hat{f}}''_{2i}) \tag{34}$$

$$+B_h(\hat{\hat{f}}''_{1i},\ \hat{\hat{f}}''_{2i}\,;\hat{\hat{f}}''_{1i},\ \hat{\hat{f}}''_{2i})=B_h(\hat{\hat{f}}''_{1i},\ \hat{\hat{f}}''_{2i}\,;\hat{\hat{f}}''_{1i},\ \hat{\hat{f}}''_{2i})\,.$$

Therefore, using (34), (33) and (26) leads to

$$\begin{aligned} &B_h(\hat{\hat{f}}''_{1i},\ \hat{\hat{f}}''_{2i}\,;\hat{\hat{f}}''_{1i},\ \hat{\hat{f}}''_{2i})-\lambda_iG_h(\hat{\hat{f}}''_{1i},\ \hat{\hat{f}}''_{2i}\,;\hat{\hat{f}}''_{1i},\ \hat{\hat{f}}''_{2i}) \\ &\quad=B_h(\hat{\hat{u}}_{hi},\ \hat{\hat{w}}_{hi}\,;\hat{\hat{f}}''_{1i},\ \hat{\hat{f}}''_{2i})-\lambda_iG_h(\hat{\hat{u}}_{hi},\ \hat{\hat{w}}_{hi}\,;\hat{\hat{f}}''_{1i},\ \hat{\hat{f}}''_{2i}) \\ &\quad=\lambda_i\tilde{G}_h(u_i,\ w_i\,;\hat{\hat{f}}''_{1i},\ \hat{\hat{f}}''_{2i})-\lambda_iG_h(\hat{\hat{u}}_{hi},\ \hat{\hat{w}}_{hi}\,;\hat{\hat{f}}''_{1i},\ \hat{\hat{f}}''_{2i}) \\ &\quad=\lambda_i\tilde{G}_h(u_i-\hat{\hat{u}}_{hi},\ w_i-\hat{\hat{w}}_{hi}\,;\hat{\hat{f}}''_{1i},\ \hat{\hat{f}}''_{2i}) \\ &\quad\leqq\lambda_i\tilde{M}_h(u_i-\hat{\hat{u}}_{hi},\ w_i-\hat{\hat{w}}_{hi})\cdot M_h(\hat{\hat{f}}''_{1i},\ \hat{\hat{f}}''_{2i})\,. \end{aligned} \tag{35}$$

The eigenvalues are characterized by

$$\hat{\lambda}_{h,k}=\min_{\substack{(\hat{u}_h,\hat{w}_h)\in S^h-(0,0)\\ G_h(\hat{u}_h,\hat{w}_h;\,\hat{u}_{h,j},\hat{w}_{h,j})=0\\ j=1,\cdots,k-1}}\frac{B_h(\hat{u}_h,\,\hat{w}_h\,;\,\hat{u}_h,\,\hat{w}_h)}{G_h(\hat{u}_h,\,\hat{w}_h\,;\,\hat{u}_h,\,\hat{w}_h)},\qquad k=1,\cdots,\bar{N}.$$

Thus, from (30) we have

$$B_h(\hat{f}''_{1i},\,\hat{f}''_{2i}\,;\,\hat{f}''_{1i},\,\hat{f}''_{2i})\geqq\hat{\lambda}_{h,i+1}G_h(\hat{f}''_{1i},\,\hat{f}''_{2i}\,;\,\hat{f}''_{1i},\,\hat{f}''_{2i}). \tag{36}$$

Hence, from (35) and (36) it follows that

$$(\hat{\lambda}_{h,i+1}-\lambda_i)G_h(\hat{f}''_{1i},\,\hat{f}''_{2i}\,;\,\hat{f}''_{1i},\,\hat{f}''_{2i})\leqq\lambda_i\tilde{M}_h(u_i-\hat{u}_{hi},\,w_i-\hat{w}_{hi})M_h(\hat{f}''_{1i},\,f''_{2i}).$$

Thus

$$(\hat{\lambda}_{h,i+1}-\lambda_i)M_h(\hat{f}''_{1i},\,\hat{f}''_{2i})\leqq\lambda_i\tilde{M}_h(u_i-\hat{u}_{hi},\,w_i-\hat{w}_{hi}).$$

On the other hand, an application of Theorem 1 yields

$$\hat{\lambda}_{h,i+1}-\lambda_i\geqq(\lambda_{i+1}-\lambda_i)/2>0,$$

for sufficiently small h. Therefore, using (27) we have

$$M_h(\hat{f}''_{1i},\,\hat{f}''_{2i})\leqq\frac{2\lambda_i}{\lambda_{i+1}-\lambda_i}\tilde{M}_h(u_i-\hat{u}_{hi},\,w_i-\hat{u}_{hi})\leqq Ch,$$

for sufficiently small h. Hence we have the desired estimate. The proof is complete.

For the estimate of $M_h(\hat{f}'_{1i},\,\hat{f}'_{2i})$, the following lemma is presented.

Lemma 7. *If $\lambda_i>\lambda_{i-1}$, then for sufficiently small h, there exists a positive constant C independent of h such that*

$$M_h(\hat{f}'_{1i},\,\hat{f}'_{2i})\leqq Ch.$$

Proof. From (29) and (31), it follows that

$$\begin{aligned}B_h(\hat{u}_{hi},\,\hat{w}_{hi}\,;\,\hat{f}'_{1i},\,\hat{f}'_{2i})&=\sum_{k=1}^{i-1}\hat{\lambda}_{h,k}[G_h(\hat{u}_{hi},\,\hat{w}_{hi}\,;\,\hat{u}_{h,k},\,\hat{w}_{h,k})]^2\\&=B_h(\hat{f}'_{1i},\,\hat{f}'_{2i}\,;\,\hat{f}'_{1i},\,\hat{f}'_{2i}),\end{aligned}$$

and that

$$\begin{aligned}G_h(\hat{u}_{hi},\,\hat{w}_{hi}\,;\,\hat{f}'_{1i},\,\hat{f}'_{2i})&=\sum_{k=1}^{i-1}[G_h(\hat{u}_{hi},\,\hat{w}_{hi}\,;\,\hat{u}_{h,k},\,\hat{w}_{h,k})]^2\\&=G_h(\hat{f}'_{1i},\,\hat{f}'_{2i}\,;\,\hat{f}'_{1i},\,\hat{f}'_{2i}).\end{aligned}$$

Thus, by (26) we have

$$\begin{aligned}&-B_h(\hat{f}'_{1i},\,\hat{f}'_{2i}\,;\,\hat{f}'_{1i},\,\hat{f}'_{2i})+\lambda_iG_h(\hat{f}'_{1i},\,\hat{f}'_{2i}\,;\,\hat{f}'_{1i},\,\hat{f}'_{2i})\\&=-B_h(\hat{u}_{hi},\,\hat{w}_{hi}\,;\,\hat{f}'_{1i},\,\hat{f}'_{2i})+\lambda_iG_h(\hat{u}_{hi},\,\hat{w}_{hi}\,;\,\hat{f}'_{1i},\,\hat{f}'_{2i})\\&=-\lambda_i\tilde{G}_h(u_i,\,w_i\,;\,\hat{f}'_{1i},\,\hat{f}'_{2i})+\lambda_iG_h(\hat{u}_{hi},\,\hat{w}_{hi}\,;\,\hat{f}'_{1i},\,\hat{f}'_{2i})\end{aligned}$$

$$=\lambda_i\tilde{G}_h(\hat{u}_{hi}-u_i,\ \hat{w}_{hi}-w_i\,;\ \hat{f}'_{1i},\ \hat{f}'_{2i})$$
$$\leqq\lambda_i\tilde{M}_h(\hat{u}_{hi}-u_i,\ \hat{w}_{hi}-w_i)\cdot M_h(\hat{f}'_{1i},\ \hat{f}'_{2i})\,.$$

From the property of the eigenvalue and (29), we obtain

$$B_h(\hat{f}'_{1i},\ \hat{f}'_{2i}\,;\ \hat{f}'_{1i},\ \hat{f}'_{2i})\leqq\hat{\lambda}_{h,i-1}G_h(\hat{f}'_{1i},\ \hat{f}'_{2i}\,;\ \hat{f}'_{1i},\ \hat{f}'_{2i})\,.$$

Therefore, we have

$$(\lambda_i-\hat{\lambda}_{h,i-1})M_h(\hat{f}'_{1i},\ \hat{f}'_{2i})\leqq\lambda_i\tilde{M}_h(\hat{u}_{hi}-u_i,\ \hat{w}_{hi}-w_i)\,.$$

On the other hand, an application of Theorem 1 yields

$$\lambda_i-\hat{\lambda}_{h,i-1}\geqq(\lambda_i-\lambda_{i-1})/2>0\,,$$

for sufficiently small h. Hence, using (27) we have

$$M_h(\hat{f}'_{1i},\ \hat{f}'_{2i})\leqq\frac{2\lambda_i}{\lambda_i-\lambda_{i-1}}\tilde{M}_h(\hat{u}_{hi}-u_i,\ \hat{w}_{hi}-w_i)\leqq Ch\,.$$

for sufficiently small h. This completes the proof.

We now state the following theorem relating to the convergence for the eigenfunctions of (10).

Theorem 2. *Let λ_i be an eigenvalue of multiplicity $p+1$ ($p\geqq0$, $\lambda_{i-1}<\lambda_i=\lambda_{i+1}=\cdots=\lambda_{i+p}<\lambda_{i+p+1}$) of* (3) *and $\{u_i,\ w_i\},\ \cdots,\ \{u_{i+p},\ w_{i+p}\}$ be the associated eigenfunctions. Let $\{\hat{\lambda}_{h,k},\ \hat{u}_{h,k},\ \hat{w}_{h,k}\}$ ($k=1,\ \cdots,\ \bar{N}$) be the solution of* (10). *Then for sufficiently small h, there exist positive constants C_1 and C_2 independent of h such that*

$$\mathrm{dist}\,[\{u_j,\ w_j\},\ \hat{\mathscr{B}}_{h,p}]_1\leqq C_1h\,,\qquad \mathrm{dist}\,[\{u_j,\ w_j\},\ \hat{\mathscr{B}}_{h,p}]\leqq C_2h\,,$$
$$j=i,\ i+1,\ \cdots,\ i+p\,,$$

where

$$\hat{\mathscr{B}}_{h,p}=\{\{\hat{u},\ \hat{w}\}\in\mathrm{span}\,[\{\hat{u}_{h,i},\ \hat{w}_{h,i}\},\ \cdots,\ \{\hat{u}_{h,i+p},\ \hat{w}_{h,i+p}\}]\,;\ M_h(\hat{u},\ \hat{w})=1\},$$
$$\mathrm{dist}\,[\{u,\ w\},\ \hat{\mathscr{B}}_{h,p}]_1=\inf_{\{\hat{u},\hat{w}\}\in\hat{\mathscr{B}}_{h,p}}\tilde{M}_h(u-\hat{u},\ w-\hat{w})\,,$$
$$\mathrm{dist}\,[\{u,\ w\},\ \hat{\mathscr{B}}_{h,p}]=\inf_{\{\hat{u},\hat{w}\}\in\hat{\mathscr{B}}_{h,p}}M(u-\hat{u},\ w-\hat{w})\,.$$

Moreover, if λ_i is a simple eigenvalue, then there exist positive constants C_3 and C_4 independent of h such that

$$\tilde{M}_h(u_i-\hat{u}_{h,i},\ w_i-\hat{w}_{h,i})\leqq C_3h\,,\qquad M(u_i-\hat{u}_{h,i},\ w_i-\hat{w}_{h,i})\leqq C_4h\,,$$

for sufficiently small h.

Proof. We shall prove only the first estimate. The others follow in the same fashion. Define $\{\hat{u}_j^*,\ \hat{w}_j^*\}$ by

$$\{\hat{u}_j^*,\ \hat{w}_j^*\}=\sum_{k=i}^{i+p}\bar{a}_k\{\hat{u}_{h,k},\ \hat{w}_{h,k}\},\qquad j=i,\ \cdots,\ i+p\,,$$

where

$$\bar{a}_k=\tilde{G}_h(u_j,\ w_j\,;\ \hat{u}_{h,k},\ \hat{w}_{h,k})\,.$$

For $\{\lambda_j,\ u_j,\ w_j\}$, we define $\{\hat{\hat{u}}_{hj},\ \hat{\hat{w}}_{hj}\}\in\mathcal{S}^h$ by

$$B_h(\hat{\hat{u}}_{hj},\ \hat{\hat{w}}_{hj}\,;\ \bar{u}_h,\ \bar{w}_h)=\lambda_j\tilde{G}_h(u_j,\ w_j\,;\ \bar{u}_h,\ \bar{w}_h)\qquad\text{for each } \{\bar{u}_h,\ \bar{w}_h\}\in\mathcal{S}^h\,.$$

Then using Lemma 4, we have

$$\tilde{M}_h(\hat{\hat{u}}_{hj}-u_j,\ \hat{\hat{w}}_{hj}-w_j)\leqq Ch\,,\qquad j=i,\ \cdots,\ i+p\,,$$

for sufficiently small h. Write $\{\hat{\hat{u}}_{hj},\ \hat{\hat{w}}_{hj}\}$ as follows:

$$\{\hat{\hat{u}}_{hj},\ \hat{\hat{w}}_{hj}\}=\sum_{k=1}^{i-1}G_h(\hat{\hat{u}}_{hj},\ \hat{\hat{w}}_{hj}\,;\ \hat{u}_{h,k},\ \hat{w}_{h,k})\{\hat{u}_{h,k},\ \hat{w}_{h,k}\}$$

$$+\sum_{k=i}^{i+p}G_h(\hat{\hat{u}}_{hj},\ \hat{\hat{w}}_{hj}\,;\ \hat{u}_{h,k},\ \hat{w}_{h,k})\{\hat{u}_{h,k},\ \hat{w}_{h,k}\}+\{\hat{\hat{f}}_{1j},\ \hat{\hat{f}}_{2j}\}.$$

Since $\lambda_{i+p}<\lambda_{i+p+1}$ and since $\{u_j,\ w_j\}$ is the eigenfunction corresponding to λ_{i+p}, using Lemma 6 leads to

$$M_h(\hat{\hat{f}}_{1j},\ \hat{\hat{f}}_{2j})\leqq\hat{C}h\,,$$

for sufficiently small h. We set

$$\{\hat{\hat{f}}'_{1j},\ f'_{2j}\}=\sum_{k=1}^{i-1}G_h(\hat{\hat{u}}_{hj},\ \hat{\hat{w}}_{hj}\,;\ \hat{u}_{h,k},\ \hat{w}_{h,k})\{\hat{u}_{h,k},\ \hat{w}_{h,k}\}.$$

Since $\lambda_{i-1}<\lambda_i$ and since $\{u_j,\ w_j\}$ is the eigenfunction corresponding to λ_i, applying Lemma 7 leads to

$$M_h(\hat{\hat{f}}'_{1j},\ \hat{\hat{f}}'_{2j})\leqq\hat{C}h\,,$$

for sufficiently small h. Hence, using Schwarz's inequality we have

$$\begin{aligned}&M_h(\hat{u}_j^*-\hat{\hat{u}}_{hj},\ \hat{w}_j^*-\hat{\hat{w}}_{hj})\\&\leqq M_h(\hat{\hat{f}}'_{1j},\ \hat{\hat{f}}'_{2j})+\sum_{k=i}^{i+p}|\tilde{G}_h(u_j-\hat{\hat{u}}_{hj},\ w_j-\hat{\hat{w}}_{hj}\,;\ \hat{u}_{h,k},\ \hat{w}_{h,k})|M_h(\hat{u}_{h,k},\ \hat{w}_{h,k})\\&\quad+M_h(\hat{\hat{f}}_{1j},\ \hat{\hat{f}}_{2j})\leqq\hat{C}h+(p+1)\tilde{M}_h(u_j-\hat{\hat{u}}_{hj},\ w_j-\hat{\hat{w}}_{hj})+\hat{C}h=Ch\,.\end{aligned}$$

Thus, for sufficiently small h,

$$\tilde{M}_h(u_j-\hat{u}_j^*,\ w_j-\hat{w}_j^*)\leqq\tilde{M}_h(u_j-\hat{\hat{u}}_{hj},\ w_j-\hat{\hat{w}}_{hj})+M_h(\hat{\hat{u}}_{hj}-\hat{u}_j^*,\ \hat{\hat{w}}_{hj}-\hat{w}_j^*)\leqq Ch\,,$$

and

$$\begin{aligned}&[\tilde{M}_h(u_j-\hat{u}_j^*,\ w_j-\hat{w}_j^*)]^2=\tilde{G}_h(u_j-\hat{u}_j^*,\ w_j-\hat{w}_j^*\,;\ u_j-\hat{u}_j^*,\ w_j-\hat{w}_j^*)\\&\quad=G(u_j,\ w_j\,;\ u_j,\ w_j)-2\tilde{G}_h(u_j,\ w_j\,;\ \hat{u}_j^*,\ \hat{w}_j^*)+G_h(\hat{u}_j^*,\ \hat{w}_j^*\,;\ \hat{u}_j^*,\ \hat{w}_j^*)\\&\quad=1-2\sum_{k=i}^{i+p}\bar{a}_k^2+\sum_{k=i}^{i+p}\bar{a}_k^2=1-\sum_{k=i}^{i+p}\bar{a}_k^2\end{aligned}$$

$$=\left(1+\sqrt{\sum_{k=i}^{i+p}\bar{a}_k^2}\right)\left(1-\sqrt{\sum_{k=i}^{i+p}\bar{a}_k^2}\right).$$

Thus, for sufficiently small h,

$$\sum_{k=i}^{i+p}\bar{a}_k^2 \neq 0, \qquad 1-\sqrt{\sum_{k=i}^{i+p}\bar{a}_k^2} \geqq 0.$$

Hence, the following is well defined:

$$\{\bar{u}_j^*, \bar{w}_j^*\}=\{\hat{u}_j^*, \hat{w}_j^*\}/\sqrt{\sum_{k=i}^{i+p}\bar{a}_k^2} \in \hat{\mathscr{B}}_{h\ p}.$$

Moreover, we have

$$\begin{aligned}
[M_h(\hat{u}_j^*-\bar{u}_j^*, \hat{w}_j^*-\bar{w}_j^*)]^2 &= G_h(\hat{u}_j^*-\bar{u}_j^*, \hat{w}_j^*-\bar{w}_j^*;\, \hat{u}_j^*-\bar{u}_j^*, \hat{w}_j^*-\bar{w}_j^*) \\
&= G_h(\hat{u}_j^*, \hat{w}_j^*;\, \hat{u}_j^*, \hat{w}_j^*)-2G_h(\hat{u}_j^*, \hat{w}_j^*;\, \bar{u}_j^*, \bar{w}_j^*)+G_h(\bar{u}_j^*, \bar{w}_j^*;\, \bar{u}_j^*, \bar{w}_j^*) \\
&= \sum_{k=i}^{i+p}\bar{a}_k^2-2\sqrt{\sum_{k=i}^{i+p}\bar{a}_k^2}+1=\left(1-\sqrt{\sum_{k=i}^{i+p}\bar{a}_k^2}\right)^2.
\end{aligned}$$

Using the triangle inequality, we have

$$\begin{aligned}
\tilde{M}_h(u_j-\bar{u}_j^*, w_j-\bar{w}_j^*) &\leqq \tilde{M}_h(u_j-\hat{u}_j^*, w_j-\hat{w}_j^*)+M_h(\hat{u}_j^*-\bar{u}_j^*, \hat{w}_j^*-\bar{w}_j^*) \\
&= \tilde{M}_h(u_j-\hat{u}_j^*, w_j-\hat{w}_j^*)+\left(1-\sqrt{\sum_{k=i}^{i+p}\bar{a}_k^2}\right) \\
&= \tilde{M}_h(u_j-\hat{u}_j^*, w_j-\hat{w}_j^*)+[\tilde{M}_h(u_j-\hat{u}_j^*, w_j-\hat{w}_j^*)]^2\Big/\left(1+\sqrt{\sum_{k=i}^{i+p}\bar{a}_k^2}\right) \leqq Ch,
\end{aligned}$$

for sufficiently small h. Therefore, we have the desired estimate. This completes the proof.

We can now show the following theorems for the consistent approximation, the straight beam approximation and the consistent approximation with the semi-consistent mass scheme. For the consistent approximation, we can use Nitsche's trick ([17], [7]). Since these results are easily obtained by the same arguments as used for the partial approximation with the semi-consistent mass scheme, we omit the proofs.

Theorem 3. *Let λ_i, $\tilde{\lambda}_{h,i}$, $\lambda_{h,i}^*$ and $\bar{\bar{\lambda}}_{h,i}$ be the eigenvalues defined by* (3), (6), (12) *and* (14), *respectively. Then for sufficiently small h, there exist positive constants C_1, C_2 and C_3 independent of h such that*

$$0 \leqq \tilde{\lambda}_{h,i}-\lambda_i \leqq C_1\lambda_i^2 h^2,$$

$$|\lambda_{h,i}^*-\lambda_i| \leqq C_2\lambda_i^2 h,$$

$$|\bar{\bar{\lambda}}_{h,i}-\lambda_i| \leqq C_3\lambda_i^2 h.$$

Theorem 4. *Let λ_i be an eigenvalue of multiplicity* $p+1$ ($p \geqq 0$, $\lambda_{i-1}<\lambda_i=\cdots$

$=\lambda_{i+p}<\lambda_{i+p+1})$ *of* (3) *and* $\{u_i, w_i\}, \cdots, \{u_{i+p}, w_{i+p}\}$ *be the associated eigenfunctions. Let* $\{\tilde{\lambda}_{h,k}, \tilde{u}_{h,k}, \tilde{w}_{h,k}\}, \{\lambda^*_{h,k}, u^*_{h,k}, w^*_{h,k}\}$ *and* $\{\bar{\bar{\lambda}}_{h,k}, \bar{\bar{u}}_{h,k}, \bar{\bar{w}}_{h,k}\}$ $(k=1, \cdots, \bar{N})$ *be the solutions of* (6), (12) *and* (14), *respectively. Then, for sufficiently small h, there exist positive constants* C_q $(q=1, \cdots, 5)$ *independent of h such that*

$$\mathrm{dist}\,[\{u_j, w_j\}, \tilde{\mathscr{B}}_{h,p}]\leqq C_1 h^2,$$

$$\mathrm{dist}\,[\{u_j, w_j\}, \mathscr{B}^*_{h,p}]_2\leqq C_2 h, \qquad \mathrm{dist}\,[\{u_j, w_j\}, \mathscr{B}^*_{h,p}]\leqq C_3 h,$$

$$\mathrm{dist}\,[\{u_j, w_j\}, \bar{\bar{\mathscr{B}}}_{h,p}]_1\leqq C_4 h, \qquad \mathrm{dist}\,[\{u_j, w_j\}, \bar{\bar{\mathscr{B}}}_{h,p}]\leqq C_5 h,$$

$j=i, \cdots, i+p$, *where*

$$\tilde{\mathscr{B}}_{h,p}=\{\{\hat{u}, \hat{w}\}\in \mathrm{span}\,[\{\tilde{u}_{h,i}, \tilde{w}_{h,i}\}, \cdots, \{\tilde{u}_{h,i+p}, \tilde{w}_{h,i+p}\}]\,;\, M(\hat{u}, \hat{w})=1\},$$

$$\mathscr{B}^*_{h,p}=\{\{\hat{u}, \hat{w}\}\in \mathrm{span}\,[\{u^*_{h,i}, w^*_{h,i}\}, \cdots, \{u^*_{h,i+p}, w^*_{h,i+p}\}]\,;\, M^*_h(\hat{u}, \hat{w})=1\},$$

$$\bar{\bar{\mathscr{B}}}_{h,p}=\{\{\hat{u}, \hat{w}\}\in \mathrm{span}\,[\{\bar{\bar{u}}_{h,i}, \bar{\bar{w}}_{h,i}\}, \cdots, \{\bar{\bar{u}}_{h,i+p}, \bar{\bar{w}}_{h,i+p}\}]\,;\, M_h(\hat{u}, \hat{w})=1\},$$

$$\mathrm{dist}\,[\{u, w\}, \mathscr{B}^*_{h,p}]_2=\inf_{\{\hat{u}, \hat{w}\}\in\mathscr{B}^*_{h,p}} \tilde{M}^*_h(u-\hat{u}, w-\hat{w}).$$

Moreover, if λ_i *is a simple eigenvalue, then there exist positive constants* C_q $(q=6, \cdots, 10)$ *independent of h such that*

$$M(u_i-\tilde{u}_{h,i}, w_i-\tilde{w}_{h,i})\leqq C_6 h^2,$$

$$\tilde{M}^*_h(u_i-u^*_{h,i}, w_i-w^*_{h,i})\leqq C_7 h, \qquad M(u_i-u^*_{h,i}, w_i-w^*_{h,i})\leqq C_8 h,$$

$$\tilde{M}_h(u_i-\bar{\bar{u}}_{h,i}, w_i-\bar{\bar{w}}_{h,i})\leqq C_9 h, \qquad M(u_i-\bar{\bar{u}}_{h,i}, w_i-\bar{\bar{w}}_{h,i})\leqq C_{10} h,$$

for sufficiently small h.

5. Numerical experiments

In this section we shall give some numerical examples to illustrate the results obtained in the preceeding section. The following two typical examples of circular clamped arches are dealt as shown in Figure 3.

Example 1. $R=20$, $L=20\pi/3$, $E=A=I=\rho=1$.

Example 2. $R=10$, $L=10\pi$, $E=A=I=\rho=1$.

The exact eigenvalues and eigenfunctions of the above examples are not known. The eigenvalue λ is equal to ω^2, where ω is the natural frequency of vibration. For the first eigenvalue of Example 1, Den Hartog [5] and Volterra and Morell [19] have presented the following approximate numerical solutions by using the Rayleigh-Ritz method

$$\lambda^{(1)}_{1,D}=\frac{675EI}{\rho R^4}\doteqdot 4.219\times 10^{-3},$$

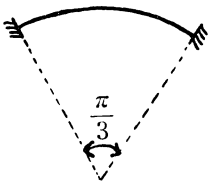
$$\lambda^{(1)}_{1,VM}=\frac{(1.351)^2 E}{\rho L^2}\doteqdot 4.161\times 10^{-3},$$

respectively. Also for the first eigenvalue of Example 2, Den Hartog [5] and Archer [1] have presented the following numerical solutions

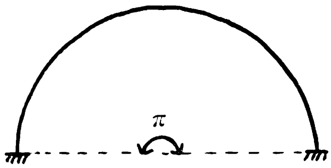
$$\lambda_{1,D}^{(2)}=\frac{(4.38)^2 EI}{\rho R^4}\doteqdot 1.918\times 10^{-3},$$

$$\lambda_{1,A}^{(2)}=\frac{19.22EI}{\rho R^4}=1.922\times 10^{-3}.$$

Now the finite element solutions are obtained by employing the four approximations, i.e., consistent approximation, partial approximation with the semi-consistent mass scheme, beam approximation and consistent approximation with the semi-consistent mass scheme. The arches are devided into uniform finite elements with equal length. Tables 1 and 2 show the numerical results of the finite ele-



(a) Example 1.



(b) Example 2.

Figure 3. Examples.

Table 1. The results for Example 1 (first eigenvalues).

Number of elements	Partial semi-consistent	Beam	Consistent	Consistent semi-consistent
4	4.149×10^{-3}	4.205×10^{-3}	4.331×10^{-3}	4.260×10^{-3}
6	4.148×10^{-3}	4.176×10^{-3}	4.232×19^{-3}	4.199×10^{-3}
8	4.135×10^{-3}	4.135×10^{-3}	4.167×10^{-3}	4.166×10^{-3}
$\lambda_{1,D}^{(1)}$	4.219×10^{-3}			
$\lambda_{1,VM}^{(1)}$	4.161×10^{-3}			

Table 2. The results for Example 2 (first eigenvalues).

Number of elements	Partial semi-consistent	Beam	Consistent	Consistent semi-consistent
4	1.905×10^{-3}	1.987×10^{-3}	3.285×10^{-3}	2.667×10^{-3}
6	1.840×10^{-3}	1.874×10^{-3}	2.437×10^{-3}	2.206×10^{-3}
8	1.826×10^{-3}	1.846×10^{-3}	2.163×10^{-3}	2.043×10^{-3}
10	1.826×10^{-3}	1.835×10^{-3}	2.050×10^{-3}	1.979×10^{-3}
12	1.816×10^{-3}	1.816×10^{-3}	1.968×10^{-3}	1.912×10^{-3}
$\lambda_{1,D}^{(2)}$	1.918×10^{-3}			
$\lambda_{1,A}^{(2)}$	1.922×10^{-3}			

ment methods in comparision with those obtained by Den Hartog, Volterra and Morell, and Archer. Figure 4 illustrates the usefulness of expressing convergence. The first mode shape for Example 2 is shown in Figure 5, which is in good agreement with the result obtained by Archer.

All the computations were done in single-precision arithmetic on the FACOM 230-28 computer at Ehime University.

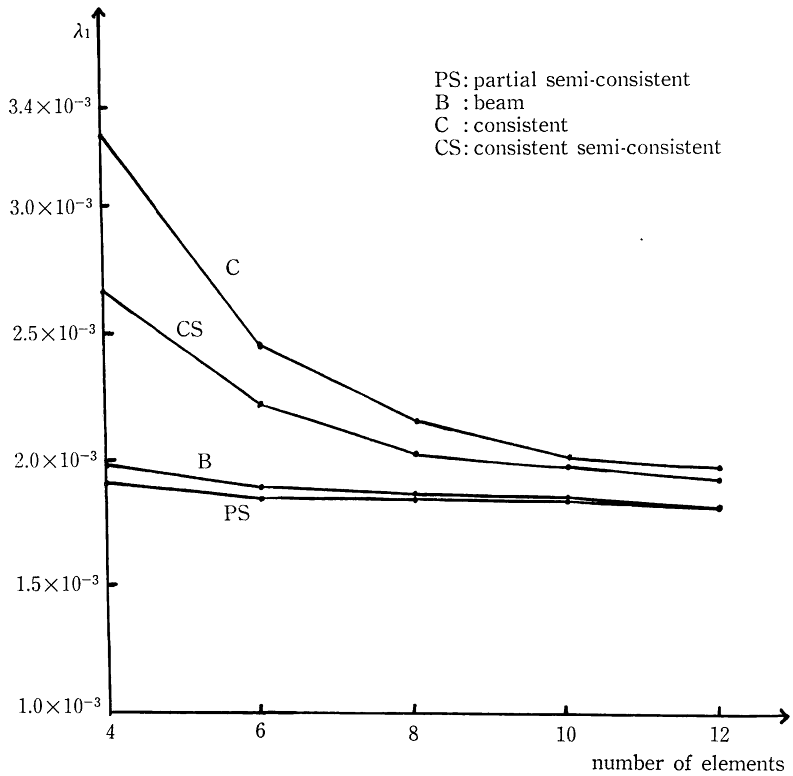


Figure 4. Convergence of eigenvalues for Example 2.

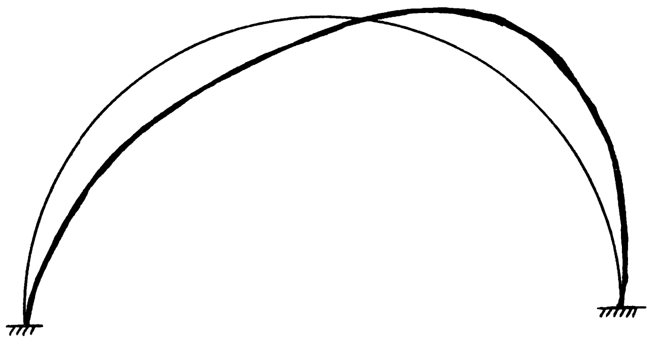


Figure 5. First mode shape for Example 2 (12 finite elements).

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