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Simply connected compact simple Lie group $E_{7(-133)}$ of type E_7

By

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It is known that there exist four simple Lie groups of type E_7 up to local isomorphism, one of them is compact and the others are non-compact. One of the non-compact Lie groups of type E_7 is considered by H. Freudenthal [1]. He defined a space $\mathfrak{M}$ in $\mathfrak{P}=\mathfrak{J}\oplus\mathfrak{J}\oplus\boldsymbol{R}\oplus\boldsymbol{R}$ (where $\mathfrak{J}$ is the exceptional Jordan algebra over the field of real numbers $\boldsymbol{R}$) by

$$\mathfrak{M}=\{L\in\mathfrak{P}\,|\,L\dot{\times}L=0\}$$

and showed that the Lie algebra of the group

$$\{\alpha\in\mathrm{Iso}_{R}(\mathfrak{P},\ \mathfrak{P})\,|\,\alpha\mathfrak{M}=\mathfrak{M},\ \det\alpha=1\}$$

is a simple Lie algebra of type E_7. In this paper, we consider the compact case. Our result is as follows. The simply connected compact simple Lie group of type E_7 is explicitly given by

$$E_7=\left\{\alpha\in\mathrm{Iso}_C(\mathfrak{P}^C,\ \mathfrak{P}^C)\ \middle|\ \begin{array}{c}\alpha\mathfrak{M}^C=\mathfrak{M}^C,\ \{\alpha\dot{1},\ \alpha\dot{1}\}=1\\ \langle\alpha P,\ \alpha Q\rangle=\langle P,\ Q\rangle\end{array}\right\}$$

where $\mathfrak{P}^C$, $\mathfrak{M}^C$ are the complexifications of $\mathfrak{P}$, $\mathfrak{M}$ respectively and $\{P,\ Q\}$, $\langle P,\ Q\rangle$ inner products defined in $\mathfrak{P}^C$.

1. Preliminaries.

Let $\mathfrak{C}^C$ denote the Cayley algebra over the field of complex numbers $\boldsymbol{C}$ and $\mathfrak{J}^C$ the exceptional Jordan algebra over $\boldsymbol{C}$. This $\mathfrak{J}^C$ is the Jordan algebra consisting of all 3×3 Hermitian matrices X with entries in $\mathfrak{C}^C$

$$X=\begin{pmatrix}\xi_1 & x_3 & \bar{x}_2\\ \bar{x}_3 & \xi_2 & x_1\\ x_2 & \bar{x}_1 & \xi_3\end{pmatrix},\qquad \xi_i\in\boldsymbol{C},\ x_i\in\mathfrak{C}^C$$

with respect to the multiplication $X\circ Y=\dfrac{1}{2}(XY+YX)$. In $\mathfrak{J}^C$, the symmetric inner product $\langle X,\ Y\rangle$, the crossed product $X\times Y$, the cubic form $(X,\ Y,\ Z)$ and the determinant $\det X$ are defined respectively by

$$(X, Y)=\mathrm{tr}(X\circ Y),$$
$$\langle X, Y\rangle=(\tau X, Y)=(\bar{X}, Y),$$
$$X\times Y=\frac{1}{2}(2X\circ Y-\mathrm{tr}(X)Y-\mathrm{tr}(Y)X+(\mathrm{tr}(X)\,\mathrm{tr}(Y)-(X, Y))E),$$
$$(X, Y, Z)=(X, Y\times Z)=(X\times Y, Z),$$
$$\det X=\frac{1}{3}(X, X, X)$$

where $\tau:\mathfrak{J}^C\to\mathfrak{J}^C$ is the complex conjugation with repect to the basic field $\boldsymbol{C}$ (τX is also denoted by $\bar{X}$) and E the 3×3 unit matrix.

Now we define a 56 dimensional vector space $\mathfrak{P}^C$ by

$$\mathfrak{P}^C=\mathfrak{J}^C\oplus\mathfrak{J}^C\oplus\boldsymbol{C}\oplus\boldsymbol{C}.$$

An element $P=\begin{pmatrix} X \\ Y \\ \xi \\ \eta \end{pmatrix}$ of $\mathfrak{P}^C$ is often denoted by $P=X+\dot{Y}+\xi+\dot{\eta}$ briefly.

We define a bilinear symmetric mapping $\dot{\times}:\mathfrak{P}^C\times\mathfrak{P}^C\to\mathfrak{J}^C\oplus\mathfrak{J}^C\oplus\boldsymbol{C}$ by

$$P\dot{\times}Q=\begin{pmatrix} X \\ Y \\ \xi \\ \eta \end{pmatrix}\dot{\times}\begin{pmatrix} Z \\ W \\ \zeta \\ \omega \end{pmatrix}=\begin{pmatrix} 2X\times Z-\eta W-\omega Y \\ 2Y\times W-\xi Z-\zeta X \\ (X, W)+(Y, Z)-3(\xi\omega+\eta\zeta) \end{pmatrix}$$

and a space $\mathfrak{M}^C$ by

$$\mathfrak{M}^C=\{L\in\mathfrak{P}^C \mid L\dot{\times}L=0\}$$
$$=\left\{\begin{pmatrix} M \\ N \\ \mu \\ \nu \end{pmatrix}\in\mathfrak{P}^C \;\middle|\; \begin{array}{l} M\times M=\nu N \\ N\times N=\mu M \\ (M, N)=3\mu\nu \end{array}\right\}.$$

For example, the following elements of $\mathfrak{P}^C$

$$\begin{pmatrix} X \\ \frac{1}{\eta}(X\times X) \\ \frac{1}{\eta^2}\det X \\ \eta \end{pmatrix},\quad \begin{pmatrix} \frac{1}{\xi}(Y\times Y) \\ Y \\ \xi \\ \frac{1}{\xi^2}\det Y \end{pmatrix},\quad \dot{1}=\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},\quad \dot{1}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

where $\eta \neq 0$, $\xi \neq 0$, belong to $\mathfrak{M}^C$. Finally, in $\mathfrak{P}^C$, we define the skew-symmetric inner product $\{P, Q\}$ and the positive definite Hermitian inner product $\langle P, Q\rangle$ respectively by

$$\{P, Q\} = (X, W) - (Z, Y) + \xi\omega - \zeta\eta\,,$$

$$\langle P, Q\rangle = \langle X, Z\rangle + \langle Y, W\rangle + \bar{\xi}\zeta + \bar{\eta}\omega$$

for $P = X + \dot{Y} + \xi + \dot{\eta}$, $Q = Z + \dot{W} + \zeta + \dot{\omega} \in \mathfrak{P}^C$.

2. Group E_7 and subgroups E_6, $U(1)$.

The group E_7 is defined to be the group of linear isomorphisms of $\mathfrak{P}^C$ leaving the space $\mathfrak{M}^C$, the skew-symmetric inner product $\{P, Q\}$ and the Hermitian inner product $\langle P, Q\rangle$ invariant:

$$E_7 = \left\{ \alpha \in \mathrm{Iso}_C(\mathfrak{P}^C, \mathfrak{P}^C) \;\middle|\; \begin{array}{c} \alpha\mathfrak{M}^C = \mathfrak{M}^C, \ \{\alpha 1, \alpha\dot{1}\} = 1 \\ \langle \alpha P, \alpha Q\rangle = \langle P, Q\rangle \ \text{for } P, Q \in \mathfrak{P}^C \end{array} \right\}.$$

Remark. In the definition of the group E_7, we may replace the condition $\{\alpha 1, \alpha\dot{1}\} = 1$ by the condition $\{\alpha P, \alpha Q\} = \{P, Q\}$.

Obviously the group E_7 is compact as a closed subgroup of the unitary group $U(56) = U(\mathfrak{P}^C) = \{\alpha \in \mathrm{Iso}_C(\mathfrak{P}^C, \mathfrak{P}^C) \mid \langle \alpha P, \alpha Q\rangle = \langle P, Q\rangle\}$.

We define a subgroup E_6 of E_7 by

$$E_6 = \{\alpha \in E_7 \mid \alpha 1 = 1\}.$$

Lemma 1. *If $\alpha \in E_7$ satisfies $\alpha 1 = 1$, then $\alpha\dot{1} = \dot{1}$.*

Proof. Put $\alpha\dot{1} = M + \dot{N} + \mu + \dot{\nu}$. Then $\langle \alpha 1, \alpha\dot{1}\rangle = 0$ and $\{\alpha 1, \alpha\dot{1}\} = 1$ imply $\mu = 0$ and $\nu = 1$ respectively. And $\langle \alpha\dot{1}, \alpha\dot{1}\rangle = 1$ implies $\langle M, M\rangle + \langle N, N\rangle + 1 = 1$, hence $M = N = 0$. Thus we have $\alpha\dot{1} = \dot{1}$.

Proposition 2. *The group E_6 is a simply connected compact simple Lie group of type E_6.*

Proof. We define a group $E_{6(-78)}$ by

$$E_{6(-78)} = \{\beta \in \mathrm{Iso}_C(\mathfrak{J}^C, \mathfrak{J}^C \mid \det \beta X = \det X,\ \langle \beta X, \beta Y\rangle = \langle X, Y\rangle\}$$
$$= \{\beta \in \mathrm{Iso}_C(\mathfrak{J}^C, \mathfrak{J}^C) \mid \beta X \times \beta Y = \tau\beta\tau(X \times Y),\ \langle \beta X, \beta Y\rangle = \langle X, Y\rangle\}.$$

Then $E_{6(-78)}$ is a simply connected compact simple Lie group of type E_6 [3]. We shall show that the group E_6 is isomorphic to the group $E_{6(-78)}$. It is easy to verify that, for $\beta \in E_{6(-78)}$, the linear mapping $\alpha : \mathfrak{P}^C \to \mathfrak{P}^C$,

$$\alpha=\begin{pmatrix}\beta & 0 & 0 & 0\\ 0 & \tau\beta\tau & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

belongs to E_7. Conversely, suppose $\alpha\in E_7$ satisfies $\alpha 1=1$ and $\alpha\dot{1}=\dot{1}$ (Lemma 1). Since α leaves the orthogonal complement $\mathfrak{J}^C\oplus\mathfrak{J}^C$ of $\boldsymbol{C}\oplus\boldsymbol{C}$ invariant, α has the following form

$$\alpha=\begin{pmatrix}\beta & \varepsilon & 0 & 0\\ \delta & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

where $\beta, \gamma, \delta, \varepsilon$ are linear transformations of $\mathfrak{J}^C$. Since

$$\alpha\begin{pmatrix} X\\ \dfrac{1}{\eta}(X\times X)\\ \dfrac{1}{\eta^2}\det X\\ \eta\end{pmatrix}=\begin{pmatrix}\beta X+\dfrac{1}{\eta}\varepsilon(X\times X)\\ \delta X+\dfrac{1}{\eta}\gamma(X\times X)\\ \dfrac{1}{\eta^2}\det X\\ \eta\end{pmatrix}\in\mathfrak{M}^C,$$

we have

$$\Big(\beta X+\frac{1}{\eta}\varepsilon(X\times X)\Big)\times\Big(\beta X+\frac{1}{\eta}\varepsilon(X\times X)\Big)=\eta\Big(\delta X+\frac{1}{\eta}\gamma(X\times X)\Big)$$

for all $0\neq\eta\in\boldsymbol{C}$. Hence we have $\delta X=0$ (for all $X\in\mathfrak{J}^C$) as the coefficient of η, therefore $\delta=0$. Similarly from $\alpha\Big(\dfrac{1}{\xi}(Y\times Y)+\dot{Y}+\xi+\dfrac{1}{\xi^2}(\det Y)^{\cdot}\Big)\in\mathfrak{M}^C$, we have $\varepsilon=0$. Thus

$$\alpha=\begin{pmatrix}\beta & 0 & 0 & 0\\ 0 & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

Again the condition $\alpha(X+(X\times X)^{\cdot}+\det X+\dot{1})=\beta X+\gamma(X\times X)^{\cdot}+\det X+\dot{1}\in\mathfrak{M}^C$ implies

$$\begin{cases}\beta X\times\beta X=\gamma(X\times X),\\ (\beta X,\ \gamma(X\times X))=3\det X.\end{cases}$$

Hence $\det\beta X=\dfrac{1}{3}(\beta X,\ \beta X\times\beta X)=\dfrac{1}{3}(\beta X,\ \gamma(X\times X))=\det X$. Furthermore, $\langle\alpha X,\ \alpha Y\rangle$

$=\langle X, Y\rangle$ implies $\langle\beta X, \beta Y\rangle=\langle X, Y\rangle$. Therefore we have $\beta\in E_{6(-78)}$ and $\gamma=\tau\beta\tau\in E_{6(-78)}$. Thus Proposition 2 is proved.

It is easy to see that the group E_7 contains the following subgroup

$$U(1)=\left\{\theta=\begin{pmatrix}\theta^{-1}1 & 0 & 0 & 0\\ 0 & \theta 1 & 0 & 0\\ 0 & 0 & \theta^3 & 0\\ 0 & 0 & 0 & \theta^{-3}\end{pmatrix}\,\middle|\,\theta\in \boldsymbol{C},\ |\theta|=1\right\}$$

(where 1 is the identity mapping of $\mathfrak{J}^C$) which is isomorphic to the unitary group $\mathrm{U}(1)=\{\theta\in\boldsymbol{C}\,|\,|\theta|=1\}$.

From now on, we identify these groups $E_{6(-78)}$ and E_6, U (1) and $U(1)$ under the above correspondences.

3. Lie algebra $\mathfrak{e}_7$ of E_7.

We consider the Lie algebra $\mathfrak{e}_7$ of the group E_7:

$$\mathfrak{e}_7=\left\{\Phi\in\mathrm{Hom}_C(\mathfrak{P}^C, \mathfrak{P}^C)\,\middle|\,\begin{array}{c}\Phi L\dot{\times}L=0 \text{ for } L\in\mathfrak{M}^C\\ \{\Phi\dot{1}, \dot{1}\}+\{\dot{1}, \Phi\dot{1}\}=0\\ \langle\Phi P, Q\rangle+\langle P, \Phi Q\rangle=0 \text{ for } P, Q\in\mathfrak{P}^C\end{array}\right\},$$

Theorem 3. *Any element Φ of the Lie algebra $\mathfrak{e}_7$ is represented by the form*

$$\Phi=\begin{pmatrix}\phi-\frac{1}{3}\rho 1 & 2A & 0 & -\bar{A}\\ -2\bar{A} & \tau\phi\tau+\frac{1}{3}\rho 1 & A & 0\\ 0 & -\bar{A} & \rho & 0\\ A & 0 & 0 & -\rho\end{pmatrix}$$

where $\phi\in\mathfrak{e}_6=\{\phi\in\mathrm{Hom}_C(\mathfrak{J}^C, \mathfrak{J}^C)\,|\,(\phi X, X, X)=0,\ \langle\phi X, Y\rangle+\langle X, \phi Y\rangle=0\}$ (which is the Lie algebra of the group E_6), $A\in\mathfrak{J}^C$, $\rho\in\boldsymbol{C}$ such that $\rho+\bar{\rho}=0$ and the action of Φ on $\mathfrak{P}^C$ is defined by

$$\Phi\begin{pmatrix}X\\ Y\\ \xi\\ \eta\end{pmatrix}=\begin{pmatrix}\phi X-\frac{1}{3}\rho X+2A\times Y-\eta\bar{A}\\ -2\bar{A}\times X+\tau\phi\tau Y+\frac{1}{3}\rho Y+\xi A\\ -\langle A, Y\rangle+\rho\xi\\ (A, X)-\rho\eta\end{pmatrix}$$

In particular, the type of the Lie group E_7 is E_7 [1].

Proof. Theorem 3 is the direct consequence of the following Lemmas 4 and

5.

Lemma 4 [1]. *Any element* Φ *of the Lie algebra*

$$\mathfrak{e}_7{}^C=\left\{\Phi\in\mathrm{Hom}_C(\mathfrak{P}^C,\mathfrak{P}^C)\;\middle|\;\begin{array}{l}\Phi L\dot{\times}L=0 \text{ for } L\in\mathfrak{M}^C\\ \{\Phi 1, \dot{1}\}+\{1,\Phi\dot{1}\}=0\end{array}\right\}$$

(*which is the Lie algebra of type* E_7) *is represented by the form*

$$\Phi=\begin{pmatrix} \phi-\frac{1}{3}\rho 1 & 2A & 0 & B \\ 2B & \phi'+\frac{1}{3}\rho 1 & A & 0 \\ 0 & B & \rho & 0 \\ A & 0 & 0 & -\rho \end{pmatrix}$$

where $\phi\in\mathfrak{e}_6{}^C=\{\phi\in\mathrm{Hom}_C(\mathfrak{J}^C,\mathfrak{J}^C)\,|\,(\phi X, X, X)=0\}$ (*which is the Lie algebra of the group* $E_6{}^C=\{\alpha\in\mathrm{Iso}_C(\mathfrak{J}^C,\mathfrak{J}^C)\,|\,\det\alpha X=\det X\}$), ϕ' *is the skew-transpose of* ϕ *with respect to* (X, Y): $(\phi X, Y)+(X, \phi' Y)=0$, A, $B\in\mathfrak{J}^C$ *and* $\rho\in\boldsymbol{C}$.

Proof. Since $\Phi\in\mathfrak{e}_7{}^C$ is a linear transformation of $\mathfrak{P}^C$, we denote it by

$$\Phi=\begin{pmatrix} g & l & C & B \\ k & h & A & D \\ c & b & \rho & \lambda \\ a & d & \kappa & \sigma \end{pmatrix}$$

where g, h, k, l are linear transformations of $\mathfrak{J}^C$, a, b, c, d linear functionals of $\mathfrak{J}^C$, A, B, C, $D\in\mathfrak{J}^C$ and σ, ρ, κ, $\lambda\in\boldsymbol{C}$. Now for any $0\neq r\in\boldsymbol{C}$, the linear transformation $f_r=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & r1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^{-1} \end{pmatrix}$ of $\mathfrak{P}^C$ induces a mapping of $\mathfrak{M}^C$ and it holds that

$$f_r^{-1}\Phi f_r=\begin{pmatrix} g & rl & r^2C & r^{-1}B \\ r^{-1}k & h & rA & r^{-2}D \\ r^{-2}c & r^{-1}b & \rho & r^{-3}\lambda \\ ra & r^2d & r^3\kappa & \sigma \end{pmatrix}\in\mathfrak{e}_7{}^C.$$

Therefore Φ is decomposable in

$$\Phi=\Phi_3+\Phi_{-3}+\Phi_2+\Phi_{-2}+\Phi_1+\Phi_{-1}+\Phi_0, \qquad \Phi_i\in\mathfrak{e}_7{}^C$$

where

$$\Phi_3=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&\kappa&0\end{pmatrix},\quad \Phi_2=\begin{pmatrix}0&0&C&0\\0&0&0&0\\0&0&0&0\\0&d&0&0\end{pmatrix},\quad \Phi_1=\begin{pmatrix}0&l&0&0\\0&0&A&0\\0&0&0&0\\a&0&0&0\end{pmatrix},$$

$$\Phi_{-3}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&\lambda\\0&0&0&0\end{pmatrix}\quad \Phi_{-2}=\begin{pmatrix}0&0&0&0\\0&0&0&D\\c&0&0&0\\0&0&0&0\end{pmatrix}\quad \Phi_{-1}=\begin{pmatrix}0&0&0&B\\k&0&0&0\\0&b&0&0\\0&0&0&0\end{pmatrix}$$

$$\Phi_0=\begin{pmatrix}g&0&0&0\\0&h&0&0\\0&0&\rho&0\\0&0&0&\sigma\end{pmatrix}.$$

Then $\Phi_3 1\dot{\times}1=0$ implies $\kappa=0$, hence $\Phi_3=0$ and similarly $\Phi_{-3}=0$. And then $\Phi_2 1\dot{\times}1=0$ and $\Phi_2 L\dot{\times}L=0$ where $L=(Y\times Y)+\dot{Y}+1+(\det Y)^{\cdot}$ imply $C=0$ and $d=0$ respectively, hence $\Phi_2=0$, similarly $\Phi_{-2}=0$. Next $\Phi_1 L\dot{\times}L=0$ where $L=(Y\times Y)+\dot{Y}+1+(\det Y)^{\cdot}$ implies

$$l(Y)=2A\times Y\,.$$

And $\Phi_1 L\dot{\times}L=0$ where $L=X+(X\times X)^{\cdot}+\det X+\dot{1}$ implies

$$2(A\times(X\times X),\ X\times X)+(\det X)(X,\ A)=3(\det X)a(X)\,.$$

hence

$$3(\det X)(X,\ A)=3(\det X)a(X)\,.$$

Therefore we have (since a is a linear funtional of $\mathfrak{J}^C$, even if for X such that $\det X=0$)

$$a(X)=(A,\ X)\,.$$

Similarly $k(X)=2B\times X$ and $b(X)=(B,\ X)$. Finally $\Phi_0 L\dot{\times}L=0$ where $L=X+(X\times X)^{\cdot}+\det X+\dot{1}$ implies

$$\begin{cases}2gX\times X=\sigma(X\times X)+h(X\times X)\,, & \text{(i)}\\ 2h(X\times X)\times(X\times X)=(\det X)(\rho X+gX)\,, & \text{(ii)}\\ (gX,\ X\times X)+(h(X\times X),\ X)=3(\rho+\sigma)\det X\,. & \text{(iii)}\end{cases}$$

Hence if we put $\phi=g-\frac{1}{3}(\rho+2\sigma)1$, then

$$\begin{aligned}3(\phi X,\ X,\ X)&=3(gX,\ X,\ X)-(\rho+2\sigma)(X,\ X,\ X)\\ &\underset{\text{(i)}}{=}(\sigma(X\times X)+h(X\times X),\ X)+(gX,\ X,\ X)-3(\rho+2\sigma)\det X\\ &\underset{\text{(iii)}}{=}3\sigma\det X+3(\rho+\sigma)\det X-3(\rho+2\sigma)\det X=0\,.\end{aligned}$$

Therefore we have

$$\phi\in\mathfrak{e}_6{}^C.$$

Similarly $\psi=h-\frac{1}{3}(2\rho+\sigma)1\in\mathfrak{e}_6{}^C$. And from the above formula (ii),

$$2\Big(\psi(X\times X)+\frac{1}{3}(2\rho+\sigma)X\times X\Big)\times(X\times X)=(\det X)\Big(\rho X+\phi X+\frac{1}{3}(\rho+2\sigma)X\Big).$$

$$2\psi(X\times X)\times(X\times X)=(\det X)\phi X.$$

$$\psi'((X\times X)\times(X\times X))=(\det X)\phi X.$$

$$(\det X)\psi' X=(\det X)\phi X.$$

Therefore we have (even if for X such that $\det X=0$) $\psi' X=\phi X$, i.e.

$$\psi'=\phi.$$

Finally from $\{\Phi\dot{1},\ \dot{1}\}+\{1,\ \Phi\dot{1}\}=0$, we have $\rho+\sigma=0$. Thus Lemma 4 is just proved.

Lemma 5. *Any linear transformation Φ of $\mathfrak{P}^C$ satisfying*

$$\langle\Phi P,\ Q\rangle+\langle P,\ \Phi Q\rangle=0$$

is represented by the form

$$\phi=\begin{pmatrix} g & l & C & B\\ k & h & A & D\\ -\bar{C} & -\bar{A} & \rho & -\bar{\kappa}\\ -\bar{B} & -\bar{D} & \kappa & \sigma\end{pmatrix}$$

where $g,\ h,\ k,\ l$ are linear transformations of $\mathfrak{J}^C$ satisfying

$$\langle gX,\ Y\rangle+\langle X,\ gY\rangle=0,$$

$$\langle hX,\ Y\rangle+\langle X,\ hY\rangle=0,$$

$$\langle kX,\ Y\rangle+\langle X,\ lY\rangle=0,$$

$A,\ B,\ C,\ D\in\mathfrak{J}^C$ and $\rho,\ \kappa,\ \sigma\in\boldsymbol{C}$ such that $\rho+\bar{\rho}=0$, $\sigma+\bar{\sigma}=0$.

Proof. Analogously in Lemma 4, we denote Φ by

$$\Phi=\begin{pmatrix} g & l & C & B\\ k & h & A & D\\ c & b & \rho & \lambda\\ a & d & \kappa & \sigma\end{pmatrix}$$

Then $\langle\Phi X,\ Y\rangle+\langle X,\ \Phi Y\rangle=0$, $\langle\Phi\dot{X},\ \dot{Y}\rangle+\langle\dot{X},\ \Phi\dot{Y}\rangle=0$ and $\langle\Phi X,\ \dot{Y}\rangle+\langle X,\ \Phi\dot{Y}\rangle=0$ imply $\langle gX,\ Y\rangle+\langle X,\ gY\rangle=0$, $\langle hX,\ Y\rangle+\langle X,\ hY\rangle=0$ and $\langle kX,\ Y\rangle+\langle X,\ lY\rangle=0$ respectively. And $\langle\Phi X,\ 1\rangle+\langle X,\ \Phi 1\rangle=0$, $\langle\Phi X,\ \dot{1}\rangle+\langle X,\ \Phi\dot{1}\rangle=0$, $\langle\Phi\dot{X},\ 1\rangle+\langle\dot{X},\ \Phi 1\rangle$

$=0$, $\langle \Phi\dot{X}, \dot{1}\rangle+\langle\dot{X}, \Phi\dot{1}\rangle=0$ imply $c(X)=-\langle C, X\rangle$, $a(X)=-\langle B, X\rangle$, $b(X)=-\langle A, X\rangle$ and $d(X)=-\langle D, X\rangle$ respectively. Finally $\langle\Phi 1, \dot{1}\rangle+\langle 1, \Phi\dot{1}\rangle=0$ implies $\bar{\kappa}+\lambda=0$. Thus Lemma 5 is proved.

4. Center $z(E_7)$ of E_7.

Theorem 6. *The center $z(E_7)$ of the group E_7 is isomorphic to the cyclic group $\boldsymbol{Z}_2$ of order 2:*

$$z(E_7)=\{1, -1\}\cong\boldsymbol{Z}_2.$$

Proof. Let $\alpha\in z(E_7)$. From the commutativity with $\beta\in E_6\subset E_7$, we have $\beta\alpha 1=\alpha\beta 1=\alpha 1$. If we denote $\alpha 1=M+\dot{N}+\mu+\dot{\nu}$, then $\beta M+(\tau\beta\tau N)^{\cdot}+\mu+\dot{\nu}=M+\dot{N}+\mu+\dot{\nu}$, hence

$$\beta M=M, \quad \tau\beta\tau N=N, \qquad \text{for all } \beta\in E_6.$$

Therefore $M=N=0$, so $\alpha 1=\mu+\dot{\nu}$, where $\mu\nu=0$ (since $\alpha 1\in\mathfrak{M}^C$). Suppose that $\mu=0$, i.e. $\alpha 1=\dot{\nu}\neq 0$, then from the commutativity with

$$\theta=\begin{pmatrix} \theta^{-1}1 & 0 & 0 & 0 \\ 0 & \theta 1 & 0 & 0 \\ 0 & 0 & \theta^3 & 0 \\ 0 & 0 & 0 & \theta^{-3} \end{pmatrix}\in U(1)\subset E_7,$$

we have

$$(\theta^{-3}\nu)^{\cdot}=\theta\dot{\nu}=\theta\alpha 1=\alpha\theta 1=\alpha\theta^3=(\theta^3\nu)^{\cdot}, \qquad \text{for all } \theta\in U(1).$$

This is a contradiction. Hence $\alpha 1=\mu$. Similarly $\alpha\dot{1}=\dot{\lambda}$. The condition $\{\alpha 1, \alpha\dot{1}\}=1$ implies $\mu\lambda=1$, hence

$$\alpha 1=\mu, \quad \alpha\dot{1}=(\mu^{-1})^{\cdot}.$$

Next, note that

$$\iota=\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

belongs to E_7, then the commutativity condition $\iota\alpha=\alpha\iota$ implies

$$\dot{\mu}=\iota\mu=\iota\alpha 1=\alpha\iota 1=\alpha\dot{1}=(\mu^{-1})^{\cdot},$$

hence $\mu=\mu^{-1}$, i.e. $\mu=\pm 1$. In the case of $\mu=1$, $\alpha\in E_6$, so $\alpha\in z(E_6)$ which is $\{1, \omega 1, \omega^2 1\}$, $\omega\in\boldsymbol{C}$, $\omega^3=1$, $\omega\neq 1$ [3]. Hence

$$\alpha=\begin{pmatrix} \omega 1 & 0 & 0 & 0 \\ 0 & \omega^{-1}1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \omega\in\boldsymbol{C}, \ \omega^3=1.$$

Again from the commutativity condition $\iota\alpha=\alpha\iota$, we have

$$(\omega X)^{\cdot}=\iota(\omega X)=\iota\alpha X=\alpha\iota X=\alpha\dot{X}=(\omega^{-1}X)^{\cdot}, \qquad \text{for all } X\in\mathfrak{J}^C,$$

hence $\omega=\omega^{-1}$, i.e. $\omega=1$, therefore $\alpha=1$. In the case of $\mu=-1$, $-\alpha\in z(E_6)$, so $-\alpha=1$. Thus we see that $z(E_7)=\{1, -1\}$.

5. Connectedness of E_7.

We shall prove that the group E_7 is connected. We denote, for a while, the connected component of E_7 containing the identity 1 by $(E_7)_0$.

Lemma 7. *For $a\in C$, the linear transformation of $\mathfrak{P}^C$ defined by*

$$\alpha_1(a)=\begin{pmatrix} 1+(\cos|a|-1)p_1 & 2a\dfrac{\sin|a|}{|a|}E_1 & 0 & -\bar{a}\dfrac{\sin|a|}{|a|}E_1 \\ -2\bar{a}\dfrac{\sin|a|}{|a|}E_1 & 1+(\cos|a|-1)p_1 & a\dfrac{\sin|a|}{|a|}E_1 & 0 \\ 0 & -\bar{a}\dfrac{\sin|a|}{|a|}E_1 & \cos|a| & 0 \\ a\dfrac{\sin|a|}{|a|}E_1 & 0 & 0 & \cos|a| \end{pmatrix}$$

$\left(\text{if } a=0, \text{ then } a\dfrac{\sin|a|}{|a|} \text{ means } 0\right)$ belongs to $(E_7)_0$, where $E_1=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\in\mathfrak{J}^C$, *the mapping p_1: $\mathfrak{J}^C\to\mathfrak{J}^C$ is defined by*

$$p_1\begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix}=\begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & x_1 \\ 0 & \bar{x}_1 & \xi_3 \end{pmatrix}$$

and the action $\alpha_1(a)$ on $\mathfrak{P}^C$ is defined as similar to that of Theorem 3. Similarly we can define mappings $\alpha_2(a)$, $\alpha_3(a)\in(E_7)_0$.

Proof. For $\Phi_1(a)=\begin{pmatrix} 0 & 2aE_1 & 0 & -\bar{a}E_1 \\ -2\bar{a}E_1 & 0 & aE_1 & 0 \\ 0 & -\bar{a}E_1 & 0 & 0 \\ aE_1 & 0 & 0 & 0 \end{pmatrix}\in\mathfrak{e}_7$, we have $\alpha_1(a)=\exp\Phi_1(a)$, hence $\alpha_1(a)\in(E_7)_0$.

Proposition 8. *Any element $L\in\mathfrak{M}^C$ can be transformed in a diagonal form by a certain element $\alpha\in(E_7)_0$:*

$$\alpha L=M+\dot{N}+\mu+\dot{\nu}, \qquad M, N \text{ are diagonal forms.}$$

Moreover we can choose $\alpha\in(E_7)_0$ so that μ is a non-negative real number (if $L\neq 0$, then $\mu>0$).

Proof. Let $L=M+\dot{N}+\mu+\dot{\nu}\in\mathfrak{M}^C$. First assume that $\mu\neq 0$, then $M=\dfrac{1}{\mu}(N\times N)$, Choose $\beta\in E_6$ such that $\tau\beta\tau N$ is a diagonal form [3], then

$$\beta M=\frac{1}{\mu}\beta(N\times N)=\frac{1}{\mu}\tau\beta\tau N\times\tau\beta\tau N$$

is also diagonal, so βL is a diagonal form (where $\beta\in E_6\subset(E_7)_0$). In the case of $\nu\neq 0$, the statement is also valid. Next we consider the case $L=M+\dot{N}\in\mathfrak{M}^C$, $N\neq 0$. Choose $\beta\in E_6$ such that

$$\tau\beta\tau N=\begin{pmatrix}\nu_1 & 0 & 0\\ 0 & \nu_2 & 0\\ 0 & 0 & \nu_3\end{pmatrix},\quad \nu_i\in\boldsymbol{C}.$$

Since $\tau\beta\tau N\neq 0$, we may assume $\nu_1\neq 0$. Operate $\alpha_1\left(-\dfrac{\pi}{2}\right)\in(E_7)_0$ of Lemma 7 on $(\tau\beta\tau N)^{\cdot}$, then

$$\alpha_1\left(-\frac{\pi}{2}\right)(\tau\beta\tau N)^{\cdot}=*+\dot{*}+\nu_1+\dot{*}\,.$$

So, we can reduce to the first case $\mu\neq 0$. In the case of $M\neq 0$, the statement is also valid. Finally, operate some $\theta\in U(1)\subset(E_7)_0$ on αL (if necessary), then μ becomes a non-negative real number. $\Big($If $L\neq 0$, noting $\alpha_3\left(-\dfrac{\pi}{2}\right)\alpha_2\left(\dfrac{\pi}{2}\right)\alpha_1\left(\dfrac{\pi}{2}\right)\dot{1}=1$, then we can always reduce to the case $\mu\neq 0\Big)$. Thus Proposition 8 is proved.

We consider a subspace $\mathfrak{M}_1$ of $\mathfrak{M}^C$ such that

$$\mathfrak{M}_1=\{L\in\mathfrak{M}^C\,|\,\langle L,\, L\rangle=1\}.$$

Theorem 9. *The group E_7 acts transitively on $\mathfrak{M}_1$ (which is connected) and the isotropy subgroup of E_7 at $1\in\mathfrak{M}_1$ is E_6. Therefore the homogeneous space E_7/E_6 is homeomorphic to $\mathfrak{M}_1$:*

$$E_7/E_6\simeq\mathfrak{M}_1.$$

In particular, the group E_7 is connected.

Proof. Obviously the group E_7 acts on $\mathfrak{M}_1$. First we shall prove that the group $(E_7)_0$ acts transitively on $\mathfrak{M}_1$. For a given $L=M+\dot{N}+\mu+\dot{\nu}\in\mathfrak{M}_1$, from Proposition 8, there exists $\alpha\in(E_7)_0$ such that

$$\alpha L=\frac{1}{\mu}\begin{pmatrix}\nu_2\nu_3 & 0 & 0\\ 0 & \nu_3\nu_1 & 0\\ 0 & 0 & \nu_1\nu_2\end{pmatrix}+\begin{pmatrix}\nu_1 & 0 & 0\\ 0 & \nu_2 & 0\\ 0 & 0 & \nu_3\end{pmatrix}^{\cdot}+\mu+\left(\frac{1}{\mu^2}\nu_1\nu_2\nu_3\right)^{\cdot},\ \mu>0\,.$$

Then the condition $\langle L, L\rangle=1$ is

$$\frac{1}{\mu^2}(|\nu_2|^2|\nu_3|^2+|\nu_3|^2|\nu_1|^2+|\nu_1|^2|\nu_2|^2)+(|\nu_1|^2+|\nu_2|^2+|\nu_3|^2)$$
$$+\mu^2+\frac{1}{\mu^4}|\nu_1|^2|\nu_2|^2|\nu_3|^2=1\,,$$

that is

$$\left(1+\frac{|\nu_1|^2}{\mu^2}\right)\left(1+\frac{|\nu_2|^2}{\mu^2}\right)\left(1+\frac{|\nu_3|^2}{\mu^2}\right)=\frac{1}{\mu^2}\,.$$

Choose real numbers r_1, r_2, r_3 $(0\leqq r_i<\frac{\pi}{2}, i=1, 2, 3)$ such that

$$\tan r_1=\frac{|\nu_1|}{\mu}\,,\quad \tan r_2=\frac{|\nu_2|}{\mu}\,,\quad \tan r_3=\frac{|\nu_3|}{\mu}\,,$$

then we have

$$\mu=\cos r_1\cos r_2\cos r_3\,.$$

And put

$$a_1=\frac{\nu_1}{|\nu_1|}r_1\,,\quad a_2=\frac{\nu_2}{|\nu_2|}r_2\,,\quad a_3=\frac{\nu_3}{|\nu_3|}r_3$$

(if $\nu_i=0$, then $\frac{\nu_i}{|\nu_i|}r_i$ means 0), then

$$\alpha L=\begin{pmatrix} \cos|a_1|\,a_2\frac{\sin|a_2|}{|a_2|}a_3\frac{\sin|a_3|}{|a_3|} & 0 & 0 \\ 0 & a_1\frac{\sin|a_1|}{|a_1|}\cos|a_2|\,a_3\frac{\sin|a_3|}{|a_3|} & 0 \\ 0 & 0 & a_1\frac{\sin|a_1|}{|a_1|}a_2\frac{\sin|a_2|}{|a_2|}\cos|a_3| \end{pmatrix}$$
$$+\begin{pmatrix} a_1\frac{\sin|a_1|}{|a_1|}\cos|a_2|\cos|a_3| & 0 & 0 \\ 0 & \cos|a_1|\,a_2\frac{\sin|a_2|}{|a_2|}\cos|a_3| & 0 \\ 0 & 0 & \cos|a_1|\cos|a_2|\,a_3\frac{\sin|a_3|}{|a_3|} \end{pmatrix}^{\cdot}$$
$$+(\cos|a_1|\cos|a_2|\cos|a_3|)+\left(a_1\frac{\sin|a_1|}{|a_1|}a_2\frac{\sin|a_2|}{|a_2|}a_3\frac{\sin|a_3|}{|a_3|}\right)^{\cdot}$$
$$=\alpha_3(a_3)\alpha_2(a_2)\alpha_1(a_1)1 \quad (\alpha_i(a_i)\in(E_7)_0 \text{ are in Lemma 7})\,,$$

that is

$$L=\alpha^{-1}\alpha_3(a_3)\alpha_2(a_2)\alpha_1(a_1)1\,.$$

This shows the transitivity of $(E_7)_0$ on $\mathfrak{M}_1$. Thus we have $\mathfrak{M}_1=(E_7)_0 1$, hence $\mathfrak{M}_1$

is connected. Since the group E_7 acts transitively on $\mathfrak{M}_1$ and the isotropy subgroup of E_7 at 1 is E_6, we have the following homeomorphism

$$E_7/E_6 \simeq \mathfrak{M}_1 .$$

From Proposition 2, E_6 is connected, so E_7 is also connected. Thus the proof of Theorem 9 is completed.

From the general theory of the compact Lie groups, it is known that the center $z(E_{7(-133)})$ of the simply connected compact simple Lie group $E_{7(-133)}$ of type E_7 is $\boldsymbol{Z}_2$ [2]. Thus from Theorems 6 and 9, we have the following

Theorem 10. *The group* $E_7 = \{\alpha \in \mathrm{Iso}_C(\mathfrak{P}^C, \mathfrak{P}^C) \mid \alpha\mathfrak{M}^C = \mathfrak{M}^C,\ \{\alpha 1, \alpha \dot{1}\} = 1,\ \langle \alpha P, \alpha Q \rangle = \langle P, Q \rangle\}$ *is a simply connected compact simple Lie group of type* E_7.

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