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Collapsing of the Eilenberg-Moore spectral sequence mod 5 of the compact exceptional group E_8

By

Mamoru MIMURA and Yuriko SAMBE

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§ 1. Introduction.

A principal bundle with structure group G is said to be universal if its homotopy groups are all trivial. Such spaces exist for any compact Lie group G. The base space BG of the universal bundle EG for G is called the classifying space for G. Its importance is stated in the classification theorem that the equivalence classes of G-bundles over B are in one-to-one correspondence with the homotopy classes of maps $f: B \to BG$. So each G-bundle over B has a homomorphism $f^*: H^*(BG; R) \to H^*(B; R)$ called the characteristic map of the bundle (R is a coefficient ring). The image of f^* is the characteristic ring of the bundle. For example, the Stiefel-Whitney classes are the image of f^*: $H^*(BO(n); \boldsymbol{Z}_2) \to H^*(B; \boldsymbol{Z}_2)$ of the particular elements. Thus it is quite important to determine the ring $H^*(BG; R)$ of the universal characteristic classes for G-bundles.

Now let G be a compact, 1-connected, simple Lie group. Then, as is well known, it is classified as

classical type: $\mathrm{Spin}(n)$, $SU(n)$, $Sp(n)$,

exceptional type: G_2, F_4, E_6, E_7, E_8.

Let us recall the Borel theorems:

(1) *If $H^*(G; \boldsymbol{Z}_p)$ is the exterior algebra generated by elements of odd degrees (p: a prime), then*

$$H^*(BG; \boldsymbol{Z}_p) \cong \boldsymbol{Z}_p[y_1, \cdots, y_l], \qquad l = \operatorname{rank} G.$$

(2) *If $H^*(G; \boldsymbol{Z}_2)$ has a simple system of universally transgressive generators, then*

$$H^*(BG; \boldsymbol{Z}_2) \cong \boldsymbol{Z}_2[y_1, \cdots, y_n].$$

The assumption of (1) is satisfied when $H^*(G; \boldsymbol{Z})$ has no p-torsion, e.g.,

$G = SU(n)$, $Sp(n)$ for any p;

$G = G_2$, $\mathrm{Spin}(n)$ for $p > 2$;

$G = F_4$, E_6, E_7, for $p > 3$;

$G = E_8$ for $p > 5$.

The assumption of (2) is satisfied even when $H_*(G;\boldsymbol{Z})$ has 2-torsion, e.g.,

$$G=G_2,\ F_4,\ \mathrm{Spin}(n)\ (7\leqq n\leqq 9).$$

The cases not covered by Borel's results are

$$G=\mathrm{Spin}(n),\ E_6,\ E_7,\ E_8 \text{ for } p=2;$$

$$G=F_4,\ E_6,\ E_7,\ E_8 \text{ for } p=3;$$

$$G=E_8 \text{ for } p=5.$$

Of these cases the following were already determined:

$H^*(B\,\mathrm{Spin}(n);\boldsymbol{Z}_2)$ by Quillen;

$H^*(BF_4;\boldsymbol{Z}_3)$ by Toda;

$H^*(BE_6;\boldsymbol{Z}_2)$, $H^*(BE_6;\boldsymbol{Z}_3)$ by Kono-Mimura;

$H^*(BE_7;\boldsymbol{Z}_2)$ by Kono-Mimura-Shimada.

In this paper we will study the module structure of $H^*(BE_8;\boldsymbol{Z}_5)$.

Let E_8 be the compact, 1-connected, simple exceptional Lie group of rank 8. Denote by $\{E_8:5\}$ the set $\{X:$ compact, associative H-space such that $H^*(X;\boldsymbol{Z}_5)\cong H^*(E_8;\boldsymbol{Z}_5)$ as Hopf algebras$\}$. As is well known, every $X\in\{E_8:5\}$ has the classifying space BX.

The purpose of this paper is to determine the module structure of $H^*(BX;\boldsymbol{Z}_5)$ for any $X\in\{E_8:5\}$. The method is the Eilenberg-Moore spectral sequence mod 5 $\{E_r, d_r\}$ associated with X:

$$E_2\cong\mathrm{Cotor}_A(\boldsymbol{Z}_5,\boldsymbol{Z}_5) \text{ with } A=H^*(X;\boldsymbol{Z}_5),$$

$$E_\infty\cong \mathcal{G}r\, H^*(BX;\boldsymbol{Z}_5).$$

In Section 2 we construct an injective resolution $W=A\otimes\overline{W}$ of $\boldsymbol{Z}_5$ over A (by making use of the twisted tensor product) so that

$$H(\overline{W}:d)=\mathrm{Cotor}_A(\boldsymbol{Z}_5,\boldsymbol{Z}_5).$$

In §§ 3 and 4 we determine all the indecomposable cocycles in the polynomial subalgebra V of $\overline{W}$.

In §§ 5 and 6 we determine all the indecomposable cocycles with elements of odd degree. Thus we show that $\mathrm{Cotor}_A(\boldsymbol{Z}_5,\boldsymbol{Z}_5)$ has 1040 (indecomposable) algebra generators (Theorem 6.5).

In Section 7 we check the commutativity of these generators and prove that $\mathrm{Cotor}_A(\boldsymbol{Z}_5,\boldsymbol{Z}_5)$ is commutative (Theorem 7.1).

In the last section, § 8, we show

Theorem 8.1. *The Eilenberg-Moore spectral sequence* mod 5 *associated with X collapses for all $X\in\{E_8:5\}$.*

In particular we have

Corollary 8.3. *As modules*

$$H^*(BE_8;\boldsymbol{Z}_5)\cong \mathrm{Cotor}_A(\boldsymbol{Z}_5,\boldsymbol{Z}_5) \text{ with } A=H^*(E_8;\boldsymbol{Z}_5).$$

§ 2. An injective resolution of $\boldsymbol{Z}_5$ over $H^*(X;\boldsymbol{Z}_5)$.

Let $X\in\{E_8:5\}$. First we recall from [2] the Hopf algebra structure of $H^*(X;\boldsymbol{Z}_5)$:

(2.1) $$H^*(X;\boldsymbol{Z}_5)=\boldsymbol{Z}_5[x_{12}]/(x_{12}^5)\otimes\Lambda(x_3, x_{11}, x_{15}, x_{23}, x_{27}, x_{35}, x_{39}, x_{47}),$$

where $\deg x_i=i$;

(2.2)
$$\begin{aligned}
&\bar\phi(x_i)=0 && \text{for } i=3,11,12,\\
&\bar\phi(x_j)=x_{12}\otimes x_{j-12} && \text{for } j=15,23,\\
&\bar\phi(x_k)=2x_{12}\otimes x_{k-12}+x_{12}^2\otimes x_{k-24} && \text{for } k=27,35,\\
&\bar\phi(x_l)=3x_{12}\otimes x_{l-12}+3x_{12}^2\otimes x_{l-24}+x_{12}^3\otimes x_{l-36} && \text{for } l=39,47,
\end{aligned}$$

where $\bar\phi$ is the reduced diagonal map induced from the multiplication on X.

Notation. $A=H^*(X;\boldsymbol{Z}_5)$ and $\bar A=\tilde H^*(X;\boldsymbol{Z}_5)$.

We shall construct an injective resolution of $\boldsymbol{Z}_5$ over A using the same construction as that in § 3 of [3].

Let L be a graded submodule of $\bar A$ generated by

$$\{x_3, x_{11}, x_{12}, x_{12}^2, x_{12}^3, x_{12}^4, x_{15}, x_{23}, x_{27}, x_{35}, x_{39}, x_{47}\}.$$

Let $\theta: A\to L$ be the projection and $\iota: L\to A$ the injection such that $\iota\circ\theta=1_A$. We name the set of corresponding elements under the suspension s as

(2.3) $$sL=\{a_4, a_{12}, a_{13}, c_{25}, c_{37}, c_{49}, b_{16}, b_{24}, d_{28}, d_{36}, e_{40}, e_{48}\}.$$

Define $\bar\theta: A\to sL$ by $\bar\theta=s\circ\theta$ and $\bar\iota: sL\to A$ by $\bar\iota=\iota\circ s^{-1}$. Let $T(sL)$ be the free tensor algebra over sL with the natural product ψ. Consider the two sided ideal I of $T(sL)$ generated by $\mathrm{Im}\,(\psi\circ(\bar\theta\otimes\bar\theta)\circ\phi)(\mathrm{Ker}\,\bar\theta)$, where ϕ is the diagonal map of A. Put $\overline{W}=T(sL)/I$, that is, $\overline{W}=\boldsymbol{Z}_5\{a_i, b_j, d_k, e_l, c_m\}$ ($i=4, 12, 13$; $j=16, 24$; $k=28, 36$; $l=40, 48$; $m=25, 37, 49$). It is easy to see that I is generated by

(2.4) $[\alpha,\beta]$ *for all pairs* (α,β) *of generators* α,β *of* $T(sL)$ *except*

$$\begin{aligned}
&(a_{13}, b_j) \text{ and } (c_m, b_j) && \text{for } j=16, 24,\ m=25, 37,\\
&(a_{13}, d_k) \text{ and } (c_m, d_k) && \text{for } k=28, 36,\ m=25, 37,\\
&(a_{13}, e_l) \text{ and } (c_m, e_l) && \text{for } l=40, 48,\ m=25, 37,\\
&(a_{13}, c_m) && \text{for } m=25, 37, 49,\\
&(c_m, c_{m'}) && \text{for } m, m'=25, 37, 49\ (m\neq m');
\end{aligned}$$

$[a_{13}, b_j]+c_{25}a_{j-12}$,

$[c_{25}, b_j]+c_{37}a_{j-12}$,

$[c_{37}, b_j]+c_{49}a_{j-12}$ *for* $j=16, 24$;

$[a_{13}, d_k]+2c_{25}b_{k-12}+c_{37}a_{k-24}$,

$[c_{25}, d_k]+2c_{37}b_{k-12}+c_{49}a_{k-24}$,

$[c_{37}, d_k]+2c_{49}b_{k-12}$ *for* $k=28, 36$;

$[a_{13}, e_l]+3c_{25}d_{l-12}+3c_{37}b_{l-24}+c_{49}a_{l-36}$,

$[c_{25}, e_l]+3c_{37}d_{l-12}+3c_{49}b_{l-24}$,

$[c_{37}, e_l]+3c_{49}d_{l-12}$ *for* $l=40, 48$,

where $[\alpha, \beta]=\alpha\beta-(-1)^*\beta\alpha$ *with* $*=\deg\alpha\cdot\deg\beta$.

Note that $\overline{W}$ contains the polynomial algebra

$$V=\boldsymbol{Z}_5[a_4, a_{12}, b_{16}, b_{24}, d_{28}, d_{36}, e_{40}, e_{48}].$$

Notation. $V_4=\boldsymbol{Z}_5[a_4, b_{16}, d_{28}, e_{40}]$ and $V_{12}=\boldsymbol{Z}_5[a_{12}, b_{24}, d_{36}, e_{48}]$.

We define a map

$$d=-\psi\circ(\bar{\theta}\otimes\bar{\theta})\circ\phi\circ\bar{\iota}: sL\to T(sL)$$

and extend it naturally over $T(sL)$ as a derivation. Since $d(I)\subset I$ holds, d induces a map $\overline{W}\to\overline{W}$, which is again denoted by $d:\overline{W}\to\overline{W}$ by abuse of notation. It is easy to check that $d\circ d=0$ and so $\overline{W}$ is a differential algebra over $\boldsymbol{Z}_5$. Using the relation

$$d\circ\bar{\theta}+\psi\circ(\bar{\theta}\otimes\bar{\theta})\circ\phi=0$$

we can construct the twisted tensor product $W=A\otimes\overline{W}$ with respect to $\bar{\theta}$ [5]. Namely, W is an A-comodule with a differential operator

$$\bar{d}=1\otimes d+(1\otimes\psi)\circ(1\otimes\bar{\theta}\otimes 1)\circ(\phi\otimes 1).$$

More explicitly, the differential operators $\bar{d}$ and d are given by

(2.5) $\bar{d}(x_i\otimes 1)=1\otimes a_{i+1}$ *for* $i=3, 11, 12$,

$\bar{d}(x_{12}^2\otimes 1)=1\otimes c_{25}+2x_{12}\otimes a_{13}$,

$\bar{d}(x_{12}^3\otimes 1)=1\otimes c_{37}+3x_{12}\otimes c_{25}+3x_{12}^2\otimes a_{13}$,

$\bar{d}(x_{12}^4\otimes 1)=1\otimes c_{49}+4x_{12}\otimes c_{37}+x_{12}^2\otimes c_{25}+4x_{12}^3\otimes a_{13}$,

$\bar{d}(x_j\otimes 1)=1\otimes b_{j+1}+x_{12}\otimes a_{j-11}$ *for* $j=15, 23$,

$\bar{d}(x_k\otimes 1)=1\otimes d_{k+1}+2x_{12}\otimes b_{k-11}+x_{12}^2\otimes a_{k-23}$ *for* $k=27, 35$,

$\bar{d}(x_l\otimes 1)=1\otimes e_{l+1}+3x_{12}\otimes d_{l-11}+3x_{12}^2\otimes b_{l-23}+x_{12}^3\otimes a_{l-35}$ *for* $l=39, 47$;

(2.6)
$$\begin{aligned}
&da_i=0 && \text{for } i=4,\ 12,\ 13\,,\\
&dc_{25}=3a_{13}^2\,,\\
&dc_{37}=2[a_{13},\ c_{25}]\,,\\
&dc_{49}=[a_{13},\ c_{37}]-c_{25}^2\,,\\
&db_j=-a_{13}a_{j-12} && \text{for } j=16,\ 24\,,\\
&dd_k=-2a_{13}b_{k-12}-c_{25}a_{k-24} && \text{for } k=28,\ 36\,,\\
&de_l=-3a_{13}d_{l-12}-3c_{25}b_{l-24}-c_{37}a_{l-36} && \text{for } l=40,\ 48\,.
\end{aligned}$$

Now we define weight in $W=A\otimes\overline{W}$ as follows:

(2.7)

A:	x_3,	x_{11},	x_{12},	x_{12}^2,	x_{12}^3,	x_{12}^4,	x_{15},	x_{23},	x_{27},	x_{35},	x_{39},	x_{47}
$\overline{W}$:	a_4,	a_{12},	a_{13},	c_{25},	c_{37},	c_{49},	b_{16},	b_{24},	d_{28},	d_{36},	e_{40},	e_{48}
weight:	0	0	1	2	3	4	2	2	6	6	12	12

(The weight of a monomial is the sum of the weights of each element.)

Define a filtration

(2.8)
$$F_r=\{x\,|\,\text{weight } x\leqq r\}\,.$$

Put $E_0W=\sum F_i/F_{i-1}$. Then it is easy to see that

$$E_0W\cong\Lambda(x_3,\ x_{11},\ x_{15},\ x_{23},\ x_{27},\ x_{35},\ x_{39},\ x_{47})$$
$$\otimes\boldsymbol{Z}_5[a_4,\ a_{12},\ b_{16},\ b_{24},\ d_{28},\ d_{36},\ e_{40},\ e_{48}]\otimes C(Q(x_{12}))\,,$$

where $C(Q(x_{12}))$ is the cobar construction of $\boldsymbol{Z}_5[x_{12}]/(x_{12}^5)$. The differential formulae (2.5) and (2.6) imply that E_0W is acyclic, and hence W is acyclic.

Theorem 2.9. *W is an injective resolution of $\boldsymbol{Z}_5$ over $A=H^*(X;\boldsymbol{Z}_5)$.*

By the definition of Cotor we have

Corollary 2.10. $H(\overline{W}: d)=\operatorname{Ker} d/\operatorname{Im} d\cong\operatorname{Cotor}_A(\boldsymbol{Z}_5,\ \boldsymbol{Z}_5)$.

§ 3. Cocycles in V_4 and V_{12}.

We define an operator ∂ by

$$\begin{aligned}
&\partial a_i=0 && \text{for } i=4,\ 12\,,\\
&\partial b_j=a_{j-12} && \text{for } j=16,\ 24\,,\\
&\partial d_k=2b_{k-12} && \text{for } k=28,\ 36\,,\\
&\partial e_l=3d_{l-12} && \text{for } l=40,\ 48\,,
\end{aligned}$$

and extend it over $V=\boldsymbol{Z}_5[a_4,\ a_{12},\ b_{16},\ b_{24},\ d_{28},\ d_{36},\ e_{40},\ e_{48}]$ so that it satisfies

$$\partial(P+Q)=\partial P+\partial Q \qquad\text{and}\qquad \partial(PQ)=\partial P\cdot Q+P\partial Q$$

for any polynomials P and Q of V.

Then we have

Lemma 3.1. *For any polynomial P in V we have*

(1) $\partial^5 P=0$;

(2) $[a_{13}, P]=-c_{25}\partial P-3c_{37}\partial^2 P-c_{49}\partial^3 P$,

$[c_{25}, P]=-c_{37}\partial P-3c_{49}\partial^2 P$,

$[c_{37}, P]=-c_{49}\partial P$;

(3) $dP=-a_{13}\partial P+2c_{25}\partial^2 P-c_{37}\partial^3 P+c_{49}\partial^4 P$.

Proof. (By induction.)

(1) Suppose that $\partial^5 P=0$ holds for any polynomial P of degree up to l. Then for a monomial xP of degree $l+1$, we have

$$\partial^5(xP)=\partial^5 x\cdot P+x\partial^5 P=0.$$

Thus $\partial^5 P=0$ holds for any polynomial of degree $l+1$.

(2) Suppose that (2) holds for any polynomial P of degree up to l. Then for a monomial xP of degree $l+1$, we have

$$\begin{aligned}
[a_{13}, xP]&=[a_{13}, x]P+x[a_{13}, P]\\
&=(-c_{25}\partial x-3c_{37}\partial^2 x-c_{49}\partial^3 x)P+x(-c_{25}\partial P-3c_{37}\partial^2 P-c_{49}\partial^3 P)\\
&=(-c_{25}\partial x-3c_{37}\partial^2 x-c_{49}\partial^3 x)P-(c_{25}x+c_{37}\partial x+3c_{49}\partial^2 x)\partial P\\
&\qquad -3(c_{37}x+c_{49}\partial x)\partial^2 P-c_{49}x\partial^3 P\\
&=-c_{25}(\partial x\cdot P+x\partial P)-3c_{37}(\partial^2 x\cdot P+2\partial x\cdot\partial P+x\partial^2 P)\\
&\qquad -c_{49}(\partial^3 x\cdot P+3\partial^2 x\cdot\partial P+3\partial x\cdot\partial^2 P+x\partial^3 P)\\
&=-c_{25}\partial(xP)-3c_{37}\partial^2(xP)-c_{49}\partial^3(xP).
\end{aligned}$$

Thus the first relation holds for any polynomial of degree $l+1$. The other two relations are proved similarly.

(3) Suppose that the differential formula holds for any polynomial P of degree up to l. Then for a monomial xP of degree $l+1$, we have

$$\begin{aligned}
d(xP)&=dx\cdot P+xdP\\
&=(-a_{13}\partial x+2c_{25}\partial^2 x-c_{37}\partial^3 x+c_{49}\partial^4 x)P\\
&\qquad +x(-a_{13}\partial P+2c_{25}\partial^2 P-c_{37}\partial^3 P+c_{49}\partial^4 P)\\
&=(-a_{13}\partial x+2c_{25}\partial^2 x-c_{37}\partial^3 x+c_{49}\partial^4 x)P\\
&\qquad -(a_{13}x+c_{25}\partial x+3c_{37}\partial^2 x+c_{49}\partial^3 x)\partial P\\
&\qquad +2(c_{25}x+c_{37}\partial x+3c_{49}\partial^2 x)\partial^2 P
\end{aligned}$$

$$-(c_{37}x+c_{49}\partial x)\partial^3 P+c_{49}x\partial^4 P$$

$$=-a_{13}(\partial x\cdot P+x\partial P)$$
$$+2c_{25}(\partial^2 x\cdot P+2\partial x\cdot\partial P+x\partial^2 P)$$
$$-c_{37}(\partial^3 x\cdot P+3\partial^2 x\cdot\partial P+3\partial x\cdot\partial^2 P+x\partial^3 P)$$
$$+c_{49}(\partial^4 x\cdot P-\partial^3 x\cdot\partial P+\partial^2 x\cdot\partial^2 P-\partial x\cdot\partial^3 P+x\partial^4 P)$$
$$=-a_{13}\partial(xP)+2c_{25}\partial^2(xP)-c_{37}\partial^3(xP)+c_{49}\partial^4(xP)\,.$$

Thus the differential formula holds for any polynomial of degree $l+1$. q. e. d.

Lemma 3.2. *Let P be non-trivial in V. Then P is a non-trivial cocycle if and only if $\partial P=0$.*

Proof. If P is a cocycle, $dP=0$. Then by the differential formula, we have $\partial P=0$.

Conversely, if $\partial P=0$, then $\partial^2 P$, $\partial^3 P$ and $\partial^4 P$ are also 0 and we have $dP=0$. Since P does not contain a_{13}, it is not in the d-image and hence P is a non-trivial cocycle. q. e. d.

We shall find cocycles without elements of odd degree, namely those in V, in the following manner. First, we shall find cocycles in $V_4=\boldsymbol{Z}_5[a_4, b_{16}, d_{28}, e_{40}]$. Cocycles in $V_{12}=\boldsymbol{Z}_5[a_{12}, b_{24}, d_{36}, e_{48}]$ will be obtained quite similarly and of the same form, since by the differential formula (2.6) we see that both V_4 and V_{12} are closed under the operator ∂ and that they are of the same ∂-structure. Then in Section 4 we shall find cocycles of "*mixed type*", the ones with both a_4 and a_{12}. There are 1002 cocycles of mixed type, of which 1001 are in the ∂^4-image. We have little interest in enumerating them, as we can easily write them down in polynomial form by mere calculations of ∂-image if necessary.

We shall find cocycles with elements of odd degree in Sections 5 and 6.

Now we shall find cocycles in $V_4=\boldsymbol{Z}_5[a_4, b_{16}, d_{28}, e_{40}]$.

Apparently a_4 and b_{16}^5 are the only indecomposable cocycles in $\boldsymbol{Z}_5[a_4, b_{16}]$.

A cocycle in $\boldsymbol{Z}_5[a_4, b_{16}, d_{28}]$ of degree 1 with respect to d_{28} is of the form

$$P=Ad_{28}-2B \qquad \text{with } A,\ B\in\boldsymbol{Z}_5[a_4, b_{16}]\,.$$

The formula $\partial P=\partial A\cdot d_{28}+2Ab_{16}-2\partial B=0$ gives rise to

$$\partial A=0 \qquad \text{and} \quad Ab_{16}=\partial B\,.$$

We obtain, for $A=a_4$, a cocycle $a_4d_{28}-b_{16}^2$. It is not hard to see that there are no more indecomposable cocycles in $\boldsymbol{Z}_5[a_4, b_{16}, d_{28}]$ except d_{28}^5. Thus we have 4 indecomposable cocycles in $\boldsymbol{Z}_5[a_4, b_{16}, d_{28}]$:

$$x_{140}=d_{28}^5=\partial^4(d_{28}e_{40}^4)\,, \tag{3.3}$$
$$x_4=a_4=\partial^3 e_{40}\,,$$

$$y_{80}=b_{16}^5=\partial^4(-b_{16}d_{28}^4),$$

$$z_{32}=a_4d_{28}-b_{16}^2=\partial^4(-e_{40}^2).$$

A cocycle P_i in $\boldsymbol{Z}_5[a_4, b_{16}, d_{28}, e_{40}]$ of degree i with respect to e_{40} $(i=1, 2, 3, 4)$ is of the form

$$P_1=Ae_{40}+2B,$$

$$P_2=Ae_{40}^2-Be_{40}-2C,$$

$$P_3=Ae_{40}^3+Be_{40}^2-Ce_{40}-2D,$$

$$P_4=Ae_{40}^4-2Be_{40}^3-2Ce_{40}^2+2De_{40}-E,$$

where $A, B, C, D, E\in\boldsymbol{Z}_5[a_4, b_{16}, d_{28}]$. Now $dP_i=0$ gives rise to

(3.4) $\quad \partial A=0 \quad$ (for P_1, P_2, P_3, P_4),

$Ad_{28}=\partial B \quad$ (for P_1, P_2, P_3, P_4),

$Bd_{28}=\partial C \quad$ (for P_2, P_3, P_4),

$Cd_{28}=\partial D \quad$ (for P_3, P_4),

$Dd_{28}=\partial E \quad$ (for P_4).

Since $\partial A=0$, A is a cocycle in $\boldsymbol{Z}_5[a_4, b_{16}, d_{28}]$, that is, in $\boldsymbol{Z}_5[x_{140}, x_4, y_{80}, z_{32}]$, for which we see if there exist B, C, D and E satisfying (3.4). If for some cocycle A there exist B, C and D but no E, then there exist cocycles P_1, P_2 and P_3 but that there exists no such P_4 as beginning with Ae_{40}^4. Similarly, if there exist B and C but no D, there exist P_1 and P_2, but no P_3 or P_4; and so on.

Although choice of B, C, D and E for a cocycle A is not unique, it is sufficient to take one choice, if any, since the difference between two cocycles that begin with the same term Ae_{40}^i is a cocycle of lower degree with respect to e_{40}.

If there exist cocycles $P_i'=A'e_{40}^i+\cdots$ and $P_i''=A''e_{40}^i+\cdots$ for A' and A'' respectively, we have

$$P_i'+P_i''=(A'+A'')e_{40}^i+\cdots$$

and

$$A'P_i''=(A'A'')e_{40}^i+\cdots.$$

Thus cocycle P_i for $A=A'+A''$ and $A'A''$ exist but are decomposable (and so are any cocycles for such A).

For $A=x_{140}=d_{28}^5$ or $A=x_4=a_4$, there is no B satisfying (3.4). Thus there is no cocycle beginning with $x_{140}e_{40}^i$ or with $x_4e_{40}^i$ $(i=1, 2, 3, 4)$. And we also see that there is neither cocycle P_i for $A=x_{140}^k+A'$ $(k=1, 2, \cdots)$ nor for $A=x_4x_{140}^k+A''$ $(k=0, 1, 2, \cdots)$ whatever cocycles A' and A'' may be.

For $A=x_4^2=\partial^4(-d_{28}^2)$ we have cocycles $P_i=\partial^4(-d_{28}^2e_{40}^i)$ $(i=1, 2, 3, 4)$ and for $A=y_{80}=b_{16}^5=\partial^4(-b_{16}d_{28}^4)$ we have $P_i=\partial^4(-b_{16}d_{28}^4e_{40}^i)$ $(i=1, 2, 3, 4)$, all of which are indecomposable.

For $A=z_{32}=a_4d_{28}-b_{16}^2=\partial^4(-e_{40}^2)$, we have $B=b_{16}d_{28}^2+(\partial\text{-kernel})$ and $C=2d_{28}^4$

+(other terms), but no D. Thus we have

$$P_1 = z_{32}e_{40} + 2b_{16}d_{28}^2 = \partial^4(-2e_{40}^3), \qquad \text{say } z_{72},$$

$$P_2 = z_{32}e_{40}^2 - b_{16}d_{28}^2 e_{40} + d_{28}^4 = \partial^4(-e_{40}^4), \qquad \text{say } z_{112},$$

but no P_3 or P_4. And we see also that there exists neither P_3 nor P_4 for $A = z_{32}x_{140}^k + A'$ ($k=0, 1, 2, \cdots$) whatever a cocycle A' may be.

For $A = x_4 z_{32} = \partial^4(-2d_{28}^3)$, we have $P_i = \partial^4(-2d_{28}^3 e_{40}^i) = x_4 z_{32} e_{40}^i + \cdots$ ($i=1, 2, 3, 4$). However P_1 and P_2 are decomposable as their beginning terms are the same as $x_4 z_{72}$ and $x_4 z_{112}$ respectively. The cocycles P_3 and P_4 are indecomposable, since there is no cocycle that begins with $x_4 e_{40}$ or with $x_4 e_{40}^2$.

For $A = z_{32}^2$, we have $P_1 = z_{32}z_{72}$, $P_2 = z_{72}^2$, $P_3 = z_{72}z_{112}$ and $P_4 = z_{112}^2$, which are all decomposable.

Now let A be a cocycle that has no term of the form x_{140}^k, $x_4 x_{140}^k$ or $z_{32}x_{140}^k$. Then, each term of A having at least one of x_4^2, y_{80}, $x_4 z_{32}$ or z_{32}^2, cocycles P_i ($i=1, 2, 3, 4$) for such A exist and are decomposable except the ones that have been found and shown to be indecomposable.

And for such a cocycle as $A = z_{32}x_{140}^k + A'$ ($k=0, 1, 2, \cdots$) with A' a cocycle that has no term of the form x_{140}^k or $x_4 x_{140}^k$, cocycles P_1 and P_2 exist and are decomposable. (There exist no P_3 or P_4 as we have mentioned.) Thus we have found all the indecomposable cocycles of degree 1, 2, 3 and 4 with respect to e_{40}.

Finally, e_{40}^5 is the only indecomposable cocycle of degree 5 with respect to e_{40}. It is easy to see that there are no more indecomposable cocycles of degree greater than 5.

Thus the following are all the indecomposable cocycles in $V_4 = \boldsymbol{Z}_5[a_4, b_{16}, d_{28}, e_{40}]$:

(3.5)
$$\begin{aligned}
x_{140} &= d_{28}^5 = \partial^4(d_{28}e_{40}^4), \\
x_{200} &= e_{40}^5, \\
x_4 &= a_4, \\
x_{48} &= a_4^2 e_{40} + \cdots = \partial^4(-d_{28}^2 e_{40}), \\
x_{88} &= a_4^2 e_{40}^2 + \cdots = \partial^4(-d_{28}^2 e_{40}^2), \\
x_{128} &= a_4^2 e_{40}^3 + \cdots = \partial^4(-d_{28}^2 e_{40}^3), \\
x_{168} &= a_4^2 e_{40}^4 + \cdots = \partial^4(-d_{28}^2 e_{40}^4), \\
y_{80} &= b_{16}^5 = \partial^4(-b_{16}d_{28}^4), \\
y_{120} &= b_{16}^5 e_{40} + \cdots = \partial^4(-b_{16}d_{28}^4 e_{40}), \\
y_{160} &= b_{16}^5 e_{40}^2 + \cdots = \partial^4(-b_{16}d_{28}^4 e_{40}^2), \\
y_{200} &= b_{16}^5 e_{40}^3 + \cdots = \partial^4(-b_{16}d_{28}^4 e_{40}^3), \\
y_{240} &= b_{16}^5 e_{40}^4 + \cdots = \partial^4(-b_{16}d_{28}^4 e_{40}^4), \\
z_{32} &= a_4 d_{28} - b_{16}^2 = \partial^4(-e_{40}^2), \\
z_{72} &= (a_4 d_{28} - b_{16}^2)e_{40} + \cdots = \partial^4(-2e_{40}^3),
\end{aligned}$$

$$z_{112}=(a_4d_{28}-b_{16}^2)e_{40}^2+\cdots=\partial^4(-2e_{40}^4),$$

$$z_{156}=a_4(a_4d_{28}-b_{16}^2)e_{40}^3+\cdots=\partial^4(-2d_{28}^3e_{40}^3),$$

$$z_{196}=a_4(a_4d_{28}-b_{16}^2)e_{40}^4+\cdots=\partial^4(-2d_{28}^3e_{40}^4).$$

Remark 3.6. (1) The generators in $V_4=\boldsymbol{Z}_5[a_4, b_{16}, d_{28}, e_{40}]$ are in the ∂^4-image except x_4 and x_{200};
(2) x_4 is in the ∂^3-image but not in the ∂^4-image;
(3) x_{200} is not in the ∂-image;
(4) We have $x_4^2=\partial^4(-d_{28}^2)$ and we see that a cocycle in V_4 is in the ∂^4-image if and only if it has no term of the form $x_4x_{200}^k$ or x_{200}^k;
(5) A cocycle is in the ∂^3-image if it has no term of the form x_{200}^k;
(6) A cocycle is not in the ∂-image if it has a term x_{200}^k.
In other words, we see that a sum $A+A'$ is in the ∂^3-image or ∂^4-image if and only if both A and A' are in the image.

Quite similarly one can see that the following are all the indecomposable cocycles in $V_{12}=\boldsymbol{Z}_5[a_{12}, b_{24}, d_{36}, e_{48}]$:

(3.5)′
$$u_{180}=d_{36}^5=\partial^5(d_{36}e_{48}^4),$$

$$u_{240}=e_{48}^5,$$

$$u_{12}=a_{12},$$

$$u_{72}=a_{12}^2e_{48}+\cdots=\partial^4(-d_{36}^2e_{48}),$$

$$u_{120}=a_{12}^2e_{48}^2+\cdots=\partial^4(-d_{36}^2e_{48}^2),$$

$$u_{168}=a_{12}^2e_{48}^3+\cdots=\partial^4(-d_{36}^2e_{48}^3),$$

$$u_{216}=a_{12}^2e_{48}^4+\cdots=\partial^4(-d_{36}^2e_{48}^4),$$

$$v_{120}=b_{24}^5=\partial^4(-b_{24}d_{36}^4),$$

$$v_{168}=b_{24}^5e_{48}+\cdots=\partial^4(-b_{24}d_{36}^4e_{48}),$$

$$v_{216}=b_{24}^5e_{48}^2+\cdots=\partial^4(-b_{24}d_{36}^4e_{48}^2),$$

$$v_{264}=b_{24}^5e_{48}^3+\cdots=\partial^4(-b_{24}d_{36}^4e_{48}^3),$$

$$v_{312}=b_{24}^5e_{48}^4+\cdots=\partial^4(-b_{24}d_{36}^4e_{48}^4),$$

$$w_{48}=a_{12}d_{36}-b_{24}^2=\partial^4(-e_{48}^2),$$

$$w_{96}=(a_{12}d_{36}-b_{24}^2)e_{48}+\cdots=\partial^4(-2e_{48}^3),$$

$$w_{144}=(a_{12}d_{36}-b_{24}^2)e_{48}^2+\cdots=\partial^4(-2e_{48}^4),$$

$$w_{204}=a_{12}(a_{12}d_{36}-b_{24}^2)e_{48}^3+\cdots=\partial^4(-2d_{36}^3e_{48}^3),$$

$$w_{252}=a_{12}(a_{12}d_{36}-b_{24}^2)e_{48}^4+\cdots=\partial^4(-2d_{36}^3e_{48}^4).$$

Remark 3.6′. (1) The generators in $V_{12}=\boldsymbol{Z}_5[a_{12}, b_{24}, d_{36}, e_{48}]$ are in the ∂^4-image except u_{12} and u_{240};

(2) u_{12} is in the ∂^3-image but not in the ∂^4-image;
(3) u_{240} is not in the ∂-image;
(4) We have $u_{12}^2=\partial^4(-d_{36}^2)$ and we see that a cocycle in V_{12} is in the ∂^4-image if and only if it has no term of the form $u_{12}u_{240}^l$ or u_{240}^l;
(5) A cocycle is in the ∂^3-image if it has no term of the form u_{240}^l;
(6) A cocycle is not in the ∂-image if it has a term u_{240}^l.

In other words, we see that a sum $P+P'$ is in the ∂^3-image or ∂^4-image if and only if both P and P' are in the image.

§4. Cocycles of mixed type.

Throughout this section the letters A, B, C, D and E will be used for elements in $V_4=\boldsymbol{Z}_5[a_4, b_{16}, d_{28}, e_{40}]$ and the letters P, Q, R, S and T will be used for those in $V_{12}=\boldsymbol{Z}_5[a_{12}, b_{24}, d_{36}, e_{48}]$.

Now we shall find cocycles of mixed type, that is, those in $V=\boldsymbol{Z}_5[a_4, b_{16}, d_{28}, e_{40}, a_{12}, b_{24}, d_{36}, e_{48}]$ with both a_4 and a_{12}. A polynomial f in V is of the form $f=\sum_i A_iP_i$ where A_i's are polynomials in V_4 and P_i's in V_{12}. We may suppose that the A_i's and P_i's are all distinct and $\deg A_1=\min_i \{\deg A_i\}$. The condition for f to be a cocycle is $\partial f=0$.

Now we have $\partial f=\partial A_1\cdot P_1+A_1\partial P_1+\sum_{i\geqq 2}(\partial A_i\cdot P_i+A_i\partial P_i)$. Note that

$$\deg \partial A_1=\deg A_1-12<\deg A_i \ (i\geqq 1),$$

$$P_1\neq P_i \ (i\geqq 2).$$

Thus, in order that ∂f be 0, ∂A_1 must be 0, that is, A_1 is a cocycle.

If $\partial P_1=0$, then $f=A_1P_1$ is a (decomposable) cocycle.

If $\partial P_1\neq 0$, in order that ∂f be 0, $-\partial P_1$ must be one of P_i's ($i\geqq 2$), as $A_1\neq A_i$ ($i\geqq 2$). Let $P_2=-\partial P_1$, then A_1 must be ∂A_2. The element f is now of the form

$$f=\partial A_2\cdot P_1-A_2\partial P_1+\sum_{i\geqq 3}A_iP_i \qquad \text{with} \quad \partial^2A_2=0$$

and

$$\partial f=-A_2\partial^2P_1+\sum_{i\geqq 3}(\partial A_i\cdot P_i+A_i\partial P_i).$$

If $\partial^2P_1=0$, then $f=\partial A_2\cdot P_1-A_2\partial P_1$ is a cocycle.

If $\partial^2P_1\neq 0$, then, in order that ∂f be 0, ∂^2P_1 must be, say, P_3 and A_2 must be ∂A_3 so that

$$f=\partial^2A_3\cdot P_1-\partial A_3\cdot\partial P_1+A_3\partial^2P_1+\sum_{i\geqq 4}A_iP_i$$

with $\partial^3A_3=0$ and

$$\partial f=A_3\partial^3P_1+\sum_{i\geqq 4}(\partial A_i\cdot P_i+A_i\partial P_i).$$

If $\partial^3P_1=0$, $f=\partial^2A_3\cdot P_1-\partial A_3\cdot\partial P_1+A_3\partial^2P_1$ is a cocycle.

If $\partial^3P_1\neq 0$, then it must be, say, $-P_4$ and A_3 must be ∂A_4, so that

$$f=\partial^3A_4\cdot P_1-\partial^2A_4\cdot\partial P_1+\partial A_4\cdot\partial^2P_1-A_4\partial^3P_1+\sum_{i\geqq 5}A_iP_i$$

with $\partial^4 A_4=0$ and

$$\partial f=-A_4\partial^4 P_1+\sum_{i\geqq 5}(\partial A_i\cdot P_i+A_i\partial P_i).$$

If $\partial^4 P_1=0$, then $f=\partial^3 A_4\cdot P_1-\cdots-A_4\partial^3 P_1$ is a cocycle.

If $\partial^4 P_1\neq 0$, then it must be, say, P_5 and A_4 must be ∂A_5. This time, $\partial^5 P_1$ being 0,

$$f=\partial^4 A_5\cdot P_1-\partial^3 A_5\cdot\partial P_1+\partial^2 A_5\cdot\partial^2 P_1-\partial A_5\cdot\partial^3 P_1+A_5\partial^4 P_1$$

is a cocycle.

Thus we have had 5 possible cases, which can be written as follows. The first one AP is decomposable and is omitted.

(4.1)
$$\begin{aligned}
&(A, P)_{\mathrm{I}}=AQ-BP,\\
&(A, P)_{\mathrm{II}}=AR-BQ+CP,\\
&(A, P)_{\mathrm{III}}=AS-BR+CQ-DP,\\
&(A, P)_{\mathrm{IV}}=AT-BS+CR-DQ+EP,
\end{aligned}$$

where A is a cocycle in $V_4=\boldsymbol{Z}_5[a_4, b_{16}, d_{28}, e_{40}]$ with $A=\partial B$, $B=\partial C$, $C=\partial D$, $D=\partial E$ and P is a cocycle in $V_{12}=\boldsymbol{Z}_5[a_{12}, b_{24}, d_{36}, e_{48}]$ with $P=\partial Q$, $Q=\partial R$, $R=\partial S$, $S=\partial T$.

The notation $(A, P)_J$ for $J=\mathrm{I}$, II, III, IV will be used to denote some cocycle of mixed type of the above form, as we shall study such cocycles for pairs of cocycles A and P.

We have

(4.2)
$$\begin{aligned}
&(A, P)_J+(A', P)_J=(A+A', P)_J,\\
&(A, P)_J+(A, P')_J=(A, P+P')_J,\\
&A'(A, P)_J=(A'A, P)_J,\\
&(A, P)_J P'=(A, PP')_J,\\
&(A, P)_J(A', P')_K=(AA', PP')_L,
\end{aligned}$$

where J and K are I, II, III *or* IV, *and L is* II *if $J=K=\mathrm{I}$, is* III *if $J=\mathrm{I}$ and $K=\mathrm{II}$ or if $J=\mathrm{II}$ and $K=\mathrm{I}$, and is* IV *otherwise.*

We see that there is a cocycle of the form $(A, P)_J$ if and only if both A and P are in the ∂^j-image, where $j=1, 2, 3, 4$ according as $J=\mathrm{I}$, II, III, IV. Thus by Remarks 3.6 and 3.6′ we have

(4.3.1) *There is no cocycle of the form $(A, P)_J$ if A has a term of the form x_{200}^k or if P has a term of the form u_{240}^k;*

(4.3.2) *There exists a cocycle of the form $(A, P)_J$ for $J=\mathrm{I}$,* II *and* III *if A has no term of the form x_{200}^k and if P has no term of the form u_{240}^l;*

(4.3.3) *There exists a cocycle of the form $(A, P)_{\mathrm{IV}}$ only if A has no term of the* *form* x_{200}^k *or* $x_4x_{200}^k$ *and only if* P *has no term of the form* u_{240}^l *or* $u_{12}u_{240}^l$.

We see that, if $A=\partial^4 E$ (or $P=\partial^4 T$), cocycles in (4.1) can be written as follows:

(4.4)
$$\begin{aligned} &(A, P)_{\mathrm{I}}=\partial^4(EQ) && (\text{resp. } \partial^4(-BT)),\\ &(A, P)_{\mathrm{II}}=\partial^4(ER) && (\text{resp. } \partial^4(CT)),\\ &(A, P)_{\mathrm{III}}=\partial^4(ES) && (\text{resp. } \partial^4(-DT)),\\ &(A, P)_{\mathrm{IV}}=\partial^4(ET) && \text{if } A=\partial^4 E \text{ and } P=\partial^4 T. \end{aligned}$$

Although the choices of B, C, D, E for a cocycle A and Q, R, S, T for P, if any, are not unique, we see

(4.5) *The difference between two cocycles of the form* $(A, P)_J$ *is a cocycle with subscript less than* J *or a decomposable cocycle of the form* $A'P'$.

Notation. $[A, P]_J$ will be used to denote a cocycle of the form $(A, P)_J$ chosen explicitly.

If follows from the relations (4.2) that

(4.6) *If both* A *and* P *are indecomposable, then the chosen cocycle* $[A, P]_J$ *is also indecomposable.*

(The converse is not true.)

In the following, $\bar{A}$ will be an indecomposable cocycle in V_4 other than x_4 and x_{200}, for which $\bar{E}$ will be the element in (3.5) such that $\partial^4\bar{E}=\bar{A}$. Similarly, $\bar{P}$ will be an indecomposable cocycle in V_{12} other than u_{12} and u_{240}, for which $\bar{T}$ will be the element in (3.5)′ such that $\partial^4\bar{T}=\bar{P}$.

Now we shall determine indecomposable cocycles of mixed type.

Remark that

(4.7)
$$\deg\,[A, P]_J=\deg A+\deg P+12j,$$

where $j=1, 2, 3$ *or* 4 *according as* $J=$I, II, III *or* IV.

We choose $[A, P]_{\mathrm{I}}$ for each pair of indecomposable cocycles A other than x_{200} and P other than u_{240} as follows:

(4.8)
$$\begin{aligned} &[x_4, u_{12}]_{\mathrm{I}}=a_4b_{24}-b_{16}a_{12}=\partial^4(-2e_{40}d_{36}-d_{28}e_{48}),\\ &[x_4, \bar{P}]_{\mathrm{I}}=\partial^4(-b_{16}\bar{T}) && \textit{for each } \bar{P},\\ &[\bar{A}, u_{12}]_{\mathrm{I}}=\partial^4(\bar{E}b_{24}) && \textit{for each } \bar{A},\\ &[\bar{A}, \bar{P}]_{\mathrm{I}}=\partial^4(\bar{E}\partial^3\bar{T}) && \textit{for each pair of } \bar{A} \textit{ and } \bar{P}, \end{aligned}$$

all of which are in the ∂^4*-image and are indecomposable by* (4.6).

Any other $(A, P)_{\mathrm{I}}$ is decomposable by (4.2) and (4.5). We have thus $16\cdot 16$

$=256$ indecomposable cocycles of the form $(A, P)_{\mathrm{I}}$.

Quite similarly, we obtain cocycles of the form $(A, P)_J$ ($J=\mathrm{II}$, III) for each pair of indecomposable cocycles A other than x_{200} and P other than u_{240}, which are all indecomposable:

(4.9) $$[x_4, u_{12}]_{\mathrm{II}}=a_4d_{36}-2b_{16}b_{24}+d_{28}a_{12}=\partial^4(-2e_{40}e_{48}),$$

$$[x_4, \bar{P}]_{\mathrm{II}}=\partial^4(-2d_{28}\bar{T}) \quad \textit{for each } \bar{P},$$

$$[\bar{A}, u_{12}]_{\mathrm{II}}=\partial^4(-2\bar{E}d_{36}) \quad \textit{for each } \bar{A},$$

$$[\bar{A}, \bar{P}]_{\mathrm{II}}=\partial^4(\bar{E}\partial^2\bar{T}) \quad \textit{for each pair of } \bar{A} \textit{ and } \bar{P};$$

(4.10) $$[x_4, u_{12}]_{\mathrm{III}}=a_4e_{48}+2b_{16}d_{36}-2d_{28}b_{24}-e_{40}a_{12},$$

$$[x_4, \bar{P}]_{\mathrm{III}}=\partial^4(-e_{40}\bar{T}) \quad \textit{for each } \bar{P},$$

$$[\bar{A}, u_{12}]_{\mathrm{III}}=\partial^4(\bar{E}e_{48}) \quad \textit{for each } \bar{A},$$

$$[\bar{A}, \bar{P}]_{\mathrm{III}}=\partial^4(\bar{E}\partial\bar{T}) \quad \textit{for each pair of } \bar{A} \textit{ and } \bar{P},$$

which are all in the ∂^4-image except for $[x_4, u_{12}]_{\mathrm{III}}$.

The cocycle $[x_4, u_{12}]_{\mathrm{III}}$ is not in the ∂-image. So we call it m_{52}:

$$m_{52}=a_4e_{48}+2b_{16}d_{36}-2d_{28}b_{24}-e_{40}a_{12}.$$

There are 256 cocycles in (4.9) and 256 in (4.10), which are all the indecomposable cocycles of the form $(A, P)_{\mathrm{II}}$ and $(A, P)_{\mathrm{III}}$, respectively.

Finally, we shall determine indecomposable cocycles of the form $(A, P)_{\mathrm{IV}}$. Recall that there is no cocycle of the form $(x_4, P)_{\mathrm{IV}}$ or $(A, u_{12})_{\mathrm{IV}}$.

We have

$$[\bar{A}, \bar{P}]_{\mathrm{IV}}=\partial^4(\bar{E}\bar{T}) \quad \text{for each pair of } \bar{A} \text{ and } \bar{P},$$

$$[x_4^2, u_{12}^2]_{\mathrm{IV}}=\partial^4(d_{28}^2d_{36}^2),$$

$$[x_4^2, \bar{P}]_{\mathrm{IV}}=\partial^4(-d_{28}^2\bar{T}) \quad \text{for each } \bar{P},$$

$$[\bar{A}, u_{12}^2]_{\mathrm{IV}}=\partial^4(-\bar{E}d_{36}^2) \quad \text{for each } \bar{A},$$

and any other $(A, P)_{\mathrm{IV}}$ is decomposable, since each term in A has x_4^2 or $\bar{A}$ and each term in P has u_{12}^2 or $\bar{P}$ by (4.3.3).

The cocycles $[\bar{A}, \bar{P}]_{\mathrm{IV}}$ are indecomposable by (4.6).

The cocycle $[x_4^2, u_{12}^2]_{\mathrm{IV}}$ can be shown by direct calculation to be decomposed as $-\{[x_4, u_{12}]_{\mathrm{II}}\}^2-2z_{32}w_{48}$.

We have now to check decomposability of the cocycles $[x_4^2, \bar{P}]_{\mathrm{IV}}$ and $[\bar{A}, u_{12}^2]_{\mathrm{IV}}$.

If $[x_4^2, P]_{\mathrm{IV}}=x_4^2T+\cdots$ is ever decomposable, it is decomposed as

$$(x_4T_1+\cdots)(x_4T_2+\cdots)+(\text{other terms})$$

with $x_4T_1+\cdots$ and $x_4T_2+\cdots$ cocycles. Since there is no cocycle of the form $(x_4, \partial^4T_i)_{\mathrm{IV}}$ ($i=1, 2$), it is necessary that $\partial^4T_1=\partial^4T_2=0$. And if $T=T_1T_2$ with

$\partial^4 T_1=\partial^4 T_2=0$, then $[x_4^2, P]_{\mathrm{IV}}=\partial^4(-d_{28}^2 T)$ is rewritten as

$$
\begin{aligned}
(4.11)\quad & (a_4 T_1-b_{16}\partial T_1-2d_{28}\partial^2 T_1-e_{40}\partial^3 T_1)\cdot(a_4 T_2-b_{16}\partial T_2-2d_{28}\partial^2 T_2-e_{40}\partial^3 T_2) \\
&\quad -\{2(a_4 d_{28}-b_{16}^2)(\partial^2 T_1\cdot T_2-\partial T_1\cdot\partial T_2+T_1\partial^2 T_2) \\
&\quad -(a_4 e_{40}-b_{16}d_{28})(\partial^3 T_1\cdot T_2+T_1\partial^3 T_2) \\
&\quad +(b_{16}e_{40}-d_{28}^2)(\partial^3 T_1\cdot\partial T_2+\partial T_1\cdot\partial^3 T_2) \\
&\quad +2d_{28}e_{40}(\partial^3 T_1\cdot\partial^2 T_2+\partial^2 T_1\cdot\partial^3 T_2)+e_{40}^2\partial^3 T_1\cdot\partial^3 T_2\} \\
&=(x_4, \partial^k T_1)_K(x_4, \partial^l T_2)_L \\
&\quad -2(z_{32}, \partial^m(\partial^2 T_1\cdot T_2-\partial T_1\cdot\partial T_2+T_1\partial^2 T_2))_M,
\end{aligned}
$$

where k is such that $\partial^k T_1\neq 0$ and $\partial^{k+1}T_1=0$, $K=\mathrm{I}$, II or III according as $k=1$, 2 or 3; l and L, m and M are similar except that m can be 4 and then M is IV.

Actually, we have the following decomposition of $\bar{T}$ for $\bar{P}$:

(4.12)

$\bar{P}$	$\bar{T}$	T_1	T_2
u_{180}	$d_{36}e_{48}^4$	$d_{36}e_{48}^3$	e_{48}
u_{72}	$-d_{36}^2 e_{48}$	$-d_{36}$	$d_{36}e_{48}$
u_{120}	$-d_{36}^2 e_{48}^2$	$-d_{36}e_{48}$	$d_{36}e_{48}$
u_{168}	$-d_{36}^2 e_{48}^3$	$-d_{36}$	$d_{36}e_{48}^3$
u_{216}	$-d_{36}^2 e_{48}^4$	$-d_{36}e_{48}^2$	$d_{36}e_{48}^2$
v_{120}	$-b_{24}d_{36}^4$	$-d_{36}$	$b_{24}d_{36}^3$
v_{168}	$-b_{24}d_{36}^4 e_{48}$	$-b_{24}d_{36}^3$	$d_{36}e_{48}$
v_{216}	$-b_{24}d_{36}^4 e_{48}^2$	$-b_{24}d_{36}^3$	$d_{36}e_{48}^2$
v_{264}	$-b_{24}d_{36}^4 e_{48}^3$	$-b_{24}d_{36}^3$	$d_{36}e_{48}^3$
v_{312}	$-b_{24}d_{36}^4 e_{48}^4$	$-b_{24}d_{36}^3 e_{48}$	$d_{36}e_{48}^3$
w_{48}	$-e_{48}^2$	$-e_{48}$	e_{48}
w_{96}	$-2e_{48}^3$	/	/
w_{144}	$-2e_{48}^4$	/	/
w_{204}	$-2d_{36}^3 e_{48}^3$	/	/
w_{252}	$-2d_{36}^3 e_{48}^4$	/	/

There is no decomposition such as $\bar{T}=T_1T_2$ for $\bar{P}=w_{96}$, w_{144}, w_{204} or w_{252}, for which $[x_4^2, \bar{P}]_{\mathrm{IV}}$ is indecomposable.

For $\bar{P}=w_{48}$, $T_1=-e_{48}$ and $T_2=e_{48}$ is the only decomposition of $\bar{T}=-e_{48}^2$ except for $T_1=e_{48}$ and $T_2=-e_{48}$, and the decomposition (4.11) for $[x_4^2, w_{48}]_{\mathrm{IV}}$ turns out to be

$$(4.13)\qquad [x_4^2, w_{48}]_{\mathrm{IV}}=-\{[x_4, u_{12}]_{\mathrm{III}}\}^2-(z_{32}, u_{12}^2)_{\mathrm{IV}}.$$

We shall choose the cocycle $[x_4^2, w_{48}]_{\mathrm{IV}}=\partial^4(d_{28}^2 e_{48}^2)$ to be indecomposable. Then the cocycle $(z_{32}, u_{12}^2)_{\mathrm{IV}}$ in (4.13) becomes decomposable and so does the cocycle $[z_{32}, u_{12}^2]_{\mathrm{IV}}=\partial^4(e_{40}^2 d_{36}^2)$.

For $\bar{P}$ other than w_i ($i=48$, 96, 144, 204 and 252) the cocycle $[x_4^2, \bar{P}]_{\mathrm{IV}}$ is decomposable, since the cocycle of the form $(z_{32}, P)_M$ in (4.11) is with $P\neq u_{12}^2$

and thus is a cocycle that has been already studied.

Decomposability of the cocycles $[\bar{A}, u_{12}^2]_{\mathrm{IV}}=\partial^4(-\bar{E}d_{36}^2)$ is checked similarly: In order that they are decomposable, it is necessary and, this time, sufficient that $\bar{E}$ be decomposed in a product E_1E_2 with $\partial^4E_1=\partial^4E_2=0$. We have a list similar to (4.12) and we have four indecomposable cocycles, namely, for $\bar{A}=z_{72}, z_{112}, z_{156}$ and z_{196}.

Thus we have shown that the following are all the indecomposable cocycles of the form $(A, P)_{\mathrm{IV}}$:

$$
\begin{aligned}
(4.14)\qquad & [\bar{A}, \bar{P}]_{\mathrm{IV}}=\partial^4(\bar{E}\bar{T}) && \textit{for each pair of } \bar{A} \textit{ and } \bar{P},\\
& [x_4^2, \bar{P}]_{\mathrm{IV}}=\partial^4(-d_{28}^2\bar{T}) && \textit{for } \bar{P}=w_{48}, w_{96}, w_{144}, w_{204}, w_{252},\\
& [\bar{A}, u_{12}^2]_{\mathrm{IV}}=\partial^4(-\bar{E}d_{36}^2) && \textit{for } \bar{A}=z_{72}, z_{112}, z_{156}, z_{196}.
\end{aligned}
$$

We have $15\cdot 15+9=234$ *cocycles here, all of which are in the* ∂^4*-image.*

Proposition 4.15. *We have* 1002 *indecomposable cocycles of mixed type, namely,* 256 *in each of* (4.8), (4.9) *and* (4.10) *and* 234 *in* (4.14), *all of which are in the* ∂^4*-image except* $m_{52}=[x_4, u_{12}]_{\mathrm{III}}$ *in* (4.10).

We have little interest in listing them up, but we can easily write them down in a polynomial form, if necessary, by mere calculations of ∂-image. (Recall that $\bar{A}$ is a cocycle in V_4 other than x_4 and x_{200}, $\bar{E}$ is the element in (3.5) such that $\partial^4\bar{E}=\bar{A}$, $\bar{P}$ is a cocycle in V_{12} other than u_{12} and u_{240} and $\bar{T}$ is the element in (3.5)$'$ such that $\partial^4\bar{T}=\bar{E}$. Recall also that $\deg[A, P]_J=\deg A+\deg P+12j$ with $j=1, 2, 3, 4$ according as $J=\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}$.)

Thus we have

Proposition 4.16. *In* V *we have* 1036 *indecomposable cocycles, namely,* 34 *ones listed in* (3.5) *and* (3.5)$'$ *and* 1002 *ones of mixed type in* Proposition 4.15.

Remark 4.17. (1) The indecomposable cocycles in V are in the ∂^4-image except x_4, x_{200}, u_{12}, u_{240} and m_{52};
(2) x_4 and u_{12} are in the ∂^3-image, but not in the ∂^4-image;
(3) x_{200}, u_{240} and m_{52} are not in the ∂-image.

We have the following products in the ∂^4-image:

$$
\begin{aligned}
(4.18)\qquad & x_4^2=\partial^4(-d_{28}^2), \quad x_{12}^2=\partial^4(-d_{36}^2), \quad x_4u_{12}=\partial^4(-d_{28}d_{36}),\\
& x_4m_{52}=\partial^4(-d_{28}^2e_{48}-2e_{40}^2b_{24}), \quad u_{12}m_{52}=\partial^4(e_{40}d_{36}^2+2b_{15}e_{48}^2),\\
& m_{52}^2=\partial^4(-d_{28}^2e_{48}^2-2e_{40}^2d_{36}^2+e_{40}^2b_{24}e_{48}).
\end{aligned}
$$

Therefore, monomials in cocycles are divided into three groups:
(4.19.1) A monomial in cocycles is in the ∂^4-image except the following;
(4.19.2) $x_4x_{200}^ku_{240}^l$ and $u_{12}x_{200}^ku_{240}^l$ which are in the ∂^3-image but not in the ∂^4-image;

(4.19.3) $x_{200}^k u_{240}^l$ and $m_{52}x_{200}^k u_{240}^l$ which are not in the ∂-image.

Finally we resume that

(4.20.1) *A cocycle in V is in the ∂^4-image if and only if it has no term of the form $x_4x_{200}^k u_{240}^l$, $u_{12}x_{200}^k u_{240}^l$, $x_{200}^k u_{240}^l$ or $m_{52}x_{200}^k u_{240}^l$;*
(4.20.2) *A cocycle is in the ∂^3-image if and only if it has no term of the form $x_{200}^k u_{240}^l$ or $m_{52}x_{200}^k u_{240}^l$;*
(4.20.3) *A cocycle is not in the ∂-image if it has a term $x_{200}^k u_{240}^l$ or $m_{52}x_{200}^k u_{240}^l$.*

In other words, we have

Remark 4.21. (1) A cocycle in the ∂-image is in the ∂^3-image;
(2) A cocycle is in the ∂^3-image or in the ∂^4-image if and only if each of its terms is in the image.

§5. Cocycles with elements of odd degree-I.

Next we shall determine cocycles with elements of odd degree, that is, with a_{13}, c_{25}, c_{37} and c_{49}.

First we study elements in the free tensor algebra $\boldsymbol{Z}_5\{a_{13}, c_{25}, c_{37}, c_{49}\}$, where d is closed. Clearly the elements a_{13} and $y_{62}=[a_{13}, c_{49}]+2[c_{25}, c_{37}]$ are cocycles.

Let ξ be an element of the form $c_{25}\lambda+c_{37}\mu+c_{49}\nu$, where λ, μ, ν are elements in $\boldsymbol{Z}_5\{a_{13}, c_{25}, c_{37}, c_{49}\}$. By abuse of notation, we write, for example,

$$\xi+\xi=\xi\,,$$

$$\xi\cdot a_{13}=\xi\,,$$

$$\xi\cdot c_j=\xi \qquad \text{for} \quad j=25,\ 37,\ 49\,.$$

Then an element f_l in $\boldsymbol{Z}_5\{a_{13}, c_{25}, c_{37}, c_{49}\}$ of degree l can be written as follows:

Lemma 5.1.

$$f_{2n}=d\Big(\sum_{i=0}^{n-1} y_{62}^i\xi\Big)+\sum_{i=0}^{n-1} y_{62}^i\xi+\alpha y_{62}^n\,,$$

$$f_{2n+1}=d\Big(\sum_{i=0}^{n-1} y_{62}^i\xi\Big)+\sum_{i=0}^{n} y_{62}^i\xi+\alpha y_{62}^n a_{13}\,,$$

where α is a number in $\boldsymbol{Z}_5$.

Proof. We prove the lemma by induction on degree. Obviously f_1 is of the above form. Suppose that the lemma is true up to degree $2n-1$. Then an element f_{2n} of degree $2n$ is expressible as a sum

$$f_{2n}=f_{2n-1}a_{13}+f'_{2n-1}c_{25}+f''_{2n-1}c_{37}+f'''_{2n-1}c_{49}\,,$$

where f_{2n-1}, f'_{2n-1}, f''_{2n-1}, f'''_{2n-1} satisfy the lemma. So by the assumption we have

$$f_{2n-1}a_{13}=d\Big(\sum_{i=0}^{n-2} y_{62}^i\xi a_{13}\Big)+\sum_{i=0}^{n-1} y_{62}^i\xi a_{13}+\alpha y_{62}^{n-1}a_{13}^2$$

$$=d\Big(\sum_{i=1}^{n-2} y_{62}^i\xi\Big)+\sum_{i=0}^{n-1} y_{62}^i\xi+d(\alpha y_{62}^{n-1}2c_{25})$$

$$=d\Big(\sum_{i=0}^{n-1} y_{62}^i\xi\Big)+\sum_{i=0}^{n-1} y_{62}^i\xi\,.$$

Also we have

$$f'_{2n-1}c_{25}=d\Big(\sum_{i=0}^{n-2} y_{62}^i\xi\Big)c_{25}+\sum_{i=0}^{n-1} y_{62}^i\xi c_{25}+\alpha y_{62}^{n-1}a_{13}c_{25}$$

$$=d\Big(\sum_{i=0}^{n-2} y_{62}^i\xi c_{25}\Big)-\sum_{i=0}^{n-2} y_{62}^i\xi dc_{25}+\sum_{i=0}^{n-1} y_{62}^i\xi c_{25}$$

$$+d(3\alpha y_{62}^{n-1}c_{37})-\alpha y_{62}^{n-1}c_{25}a_{13}$$

$$=d\Big(\sum_{i=0}^{n-1} y_{62}^i\xi\Big)+\sum_{i=0}^{n-1} y_{62}^i\xi\,.$$

By a similar calculation, we have

$$f''_{2n-1}c_{37}=d\Big(\sum_{i=0}^{n-1} y_{62}^i\xi\Big)+\sum_{i=0}^{n-1} y_{62}^i\xi\,,$$

and

$$f'''_{2n-1}c_{49}=d\Big(\sum_{i=0}^{n-2} y_{62}^i\xi\Big)c_{49}+\sum_{i=0}^{n-1} y_{62}^i\xi c_{49}+\alpha y_{62}^{n-1}a_{13}c_{49}$$

$$=d\Big(\sum_{i=0}^{n-2} y_{62}^i\xi c_{49}\Big)-\sum_{i=0}^{n-2} y_{62}^i\xi dc_{49}+\sum_{i=0}^{n-1} y_{62}^i\xi c_{49}$$

$$+\alpha y_{62}^n-\alpha y_{62}^{n-1}c_{49}a_{13}+\alpha y_{62}^{n-1}2[c_{25},\ c_{37}]$$

$$=d\Big(\sum_{i=0}^{n-1} y_{62}^i\xi\Big)+\sum_{i=0}^{n-1} y_{62}^i\xi+\alpha y_{62}^n\,.$$

Thus, the sum f_{2n} is of the required form:

$$d\Big(\sum_{i=0}^{n-1} y_{62}^i\xi\Big)+\sum_{i=0}^{n-1} y_{62}^i\xi+\alpha y_{62}^n \qquad \text{with} \quad \alpha\in \boldsymbol{Z}_5\,.$$

Quite similarly, an element f_{2n+1} of degree $2n+1$ is expressible as a sum

$$f_{2n+1}=f_{2n}a_{13}+f'_{2n}c_{25}+f''_{2n}c_{37}+f'''_{2n}c_{49}\,,$$

where

$$f_{2n}a_{13}=d\Big(\sum_{i=0}^{n-1} y_{62}^i\xi\Big)a_{13}+\sum_{i=0}^{n-1} y_{62}^i\xi a_{13}+\alpha y_{62}^n a_{13}$$

$$=d\Big(\sum_{i=0}^{n-1} y_{62}^i\xi\Big)+\sum_{i=0}^{n-1} y_{62}^i\xi+\alpha y_{62}^n a_{13}\,,$$

$$f'_{2n}c_{25}=d\Big(\sum_{i=0}^{n-1} y_{62}^i\xi\Big)c_{25}+\sum_{i=0}^{n-1} y_{62}^i\xi c_{25}+\alpha y_{62}^n c_{25}$$

$$=d\Big(\sum_{i=0}^{n-1} y_{62}^i\xi c_{25}\Big)+\sum_{i=0}^{n-1} y_{62}^i\xi d c_{25}+\sum_{i=0}^{n-1} y_{62}^i\xi+\alpha y_{62}^n c_{25}$$

$$=d\Big(\sum_{i=0}^{n-1} y_{62}^i\xi\Big)+\sum_{i=0}^{n-1} y_{62}^i\xi+\alpha y_{62}^n c_{25}\,,$$

$$f''_{2n}c_{37}=d\Big(\sum_{i=0}^{n-1} y_{62}^i\xi\Big)+\sum_{i=0}^{n-1} y_{62}^i\xi c_{37}+\alpha y_{62}^n c_{37}\,,$$

$$f'''_{2n}c_{49}=d\Big(\sum_{i=0}^{n-1} y_{62}^i\xi\Big)+\sum_{i=0}^{n-1} y_{62}^i\xi+\alpha y_{62}^n c_{49}\,.$$

Therefore, as required, we have

$$f_{2n+1}=f_{2n}a_{13}+f'_{2n}c_{25}+f''_{2n}c_{37}+f'''_{2n}c_{49}$$

$$=d\Big(\sum_{i=0}^{n-1} y_{62}^i\xi\Big)+\sum_{i=0}^{n-1} y_{62}^i\xi+y_{62}^n(\alpha a_{13}+\alpha' c_{25}+\alpha'' c_{37}+\alpha''' c_{49})$$

$$=d\Big(\sum_{i=0}^{n-1} y_{62}^i\xi\Big)+\sum_{i=0}^{n} y_{62}^i\xi+\alpha y_{62}^n a_{13} \qquad \text{with} \quad \alpha\in \boldsymbol{Z}_5\,. \qquad \text{q. e. d.}$$

Suppose ξ is as above. Then

$$d\xi=d(c_{25}\lambda+c_{37}\mu+c_{49}\nu)$$

$$=3a_{13}^2\lambda+c_{25}d\lambda+2[a_{13},\ c_{25}]\mu+c_{37}d\mu$$

$$+([a_{13},\ c_{37}]-c_{25}^2)\nu+c_{49}d\nu$$

$$=3a_{13}^2\lambda+2a_{13}c_{25}\mu+a_{13}c_{37}\nu+c_{25}d\lambda-2c_{25}a_{13}\mu$$

$$+c_{37}d\mu-c_{37}a_{13}\nu-c_{25}^2\nu+c_{49}d\nu\,.$$

We see that the first 3 terms are the only terms that begin with a_{13}^2, $a_{13}c_{25}$ and $a_{13}c_{37}$, respectively. Thus, $d\xi$ is 0 only if λ, μ and ν are 0. The converse is obvious and so we have

(5.2) $\qquad d\xi=0 \quad$ *if and only if* $\xi=0$.

Note that $d\xi$ has no term ($y_{62}\times$ some other term).

Writing an element f_{2n} as in Lemma 5.1, we have

$$df_{2n}=\sum_{i=0}^{n-1} y_{62}^i d\xi_i\,,$$

where ξ_i are elements of the same form as ξ. (The suffix i of ξ_i does not indicate its degree.) Therefore, df_{2n} is 0 only if each $d\xi_i$ is 0, that is, each ξ_i is 0. Thus, the only cocycle of degree $2n$ is αy_{62}^n.

Similarly, writing a polynomial f_{2n+1} as in Lemma 5.1, we have

$$df_{2n+1}=\sum_{i=0}^{n} y_{62}^i d\xi_i .$$

Therefore, df_{2n+1} is 0 only if each $d\xi_i$ is 0, that is, each ξ_i is 0. Thus, $\alpha y_{62}^n a_{13}$ is the only cocycle of degree $2n+1$.

Thus we have shown

Proposition 5.3. *The elements a_{13} and y_{62} are all the indecomposable cocycles in $\boldsymbol{Z}_5\{a_{13}, c_{25}, c_{37}, c_{49}\}$.*

§ 6. Cocycles with elements of odd degree-II.

Denote by F_k an element in $\overline{W}$ of degree k with respect to a_{13} and c_j $(j=25, 37, 49)$. The argument here is quite similar to that for f_k, though $\overline{W}$ is not a free tensor algebra as $\boldsymbol{Z}_5\{a_{13}, c_{25}, c_{37}, c_{49}\}$ is.

Lemma 6.1. *F_k can be written as follows:*

$$F_{2n}=\sum_{i=0}^{n-1} y_{62}^i(c_{25}F_{2n-2i-1}+c_{37}F'_{2n-2i-1}+c_{49}F''_{2n-2i-1})+y_{62}^n P+(d\text{-}image),$$

$$F_{2n+1}=\sum_{i=0}^{n-1} y_{62}^i(c_{25}F_{2n-2i}+c_{37}F'_{2n-2i}+c_{49}F''_{2n-2i})$$
$$+y_{62}^n(a_{13}S+c_{25}R+c_{37}Q+c_{49}P)+(d\text{-}image),$$

where $P, Q, R, S\in V$.

Proof. Using (2) of Lemma 3.1, we can put the elements of odd degree before elements in V. Thus each term of F_k is of the form f_kP, where f_k is as before and P is an element in V.

Writing f_{2n} as in Lemma 5.1, $f_{2n}P$ can be written as

$$f_{2n}P=d\Big(\sum_{i=0}^{n-1} y_{62}^i\xi\Big)P+\sum_{i=0}^{n-1} y_{62}^i\xi' P+\alpha y_{62}^n P$$
$$=d\Big(\sum_{i=0}^{n-1} y_{62}^i\xi P\Big)+\sum_{i=0}^{n-1} y_{62}^i(\xi dP+\xi' P)+\alpha y_{62}^n P.$$

Thus any element F_{2n} of degree $2n$ is of the form

$$F_{2n}=\sum_{i=0}^{n-1} y_{62}^i(c_{25}F_{2n-2i-1}+c_{37}F'_{2n-2i-1}+c_{49}F''_{2n-2i-1})+y_{62}^n P+(d\text{-image}).$$

Similarly,

$$f_{2n+1}P=\sum_{i=0}^{n-1} y_{62}^{i}(-\xi dP+\xi' P)$$

$$+y_{62}^{n}(\alpha a_{13}+\alpha' c_{25}+\alpha'' c_{37}+\alpha''' c_{49})P+(d\text{-image}),$$

and any element F_{2n+1} of degree $2n+1$ is of the form

$$F_{2n+1}=\sum_{i=0}^{n-1} y_{62}^{i}(c_{25}F_{2n-2i}+c_{37}F'_{2n-2i}+c_{49}F''_{2n-2i})$$

$$+y_{62}^{n}(a_{13}S+c_{25}R+c_{37}Q+c_{49}P)+(d\text{-image}),$$

where $P, Q, R, S\in V$. q.e.d.

By an argument similar to (5.2) we see that there is no term ($y_{62}\times$ some other term) in $d(c_{25}F_k+c_{37}F'_k+c_{49}F''_k)$ and that $d(c_{25}F_k+c_{37}F'_k+c_{49}F''_k)=0$ if and only if $c_{25}F_k+c_{37}F'_k+c_{49}F''_k$ itself is 0.

Lemma 6.2. (1) *$dF_{2n}=0$ if and only if F_{2n} is of the form*

$$F_{2n}=y_{62}^{n}A \text{ with } A \text{ a cocycle in } V;$$

(2) *$dF_{2n+1}=0$ if and only if F_{2n+1} is of the form*

$$F_{2n+1}=y_{62}^{n}(a_{13}S-2c_{25}\partial S+c_{37}\partial^2 S-c_{49}\partial^3 S)$$

with $\partial^4 S=0$.

Proof. (1) Writing F_{2n} as in Lemma 6.1, we have

$$dF_{2n}=\sum_{i=0}^{n-1} y_{62}^{i}d(c_{25}F_{2n-2i-1}+c_{37}F'_{2n-2i-1}+c_{49}F''_{2n-2i-1})+y_{62}^{n}dP.$$

Since the term $y_{62}^{i}d(c_{25}F_{2n-2i-1}+\cdots)$ is the only term that begins with y_{62}^{i} but not with y_{62}^{i+1}, the relation $dF_{2n}=0$ gives rise to $d(c_{25}F_{2n-2i-1}+\cdots)=0$ (for each i) and $dP=0$, and thus to $c_{25}F_{2n-2i-1}+\cdots=0$ (for each i) and $dP=0$. Therefore $dF_{2n}=0$ only if F_{2n} is of the form $y_{62}^{n}A$ with A a cocycle.
(2) Writing F_{2n+1} also as in Lemma 6.1, we have

$$dF_{2n+1}=\sum_{i=0}^{n-1} y_{62}^{i}d(c_{25}F_{2n-2i}+c_{37}F'_{2n-2i}+c_{49}F''_{2n-2i})$$

$$+y_{62}^{n}d(a_{13}S+c_{25}R+c_{37}Q+c_{49}P),$$

and $dF_{2n+1}=0$ if and only if $c_{25}F_{2n-2i}+\cdots=0$ (for each i) and $d(a_{13}S+c_{25}R+c_{37}Q+c_{49}P)=0$.

Now we have

$$\begin{aligned} &d(a_{13}S+c_{25}R+c_{37}Q+c_{49}P)\\ &\quad=y_{62}(-\partial^4 S)-2a_{13}^2(R+2\partial S)+2[a_{13}, c_{25}](Q-\partial^2 S)\\ &\qquad+[a_{13}, c_{25}](P+\partial^3 S)+c_{25}a_{13}(\partial R+2\partial^2 S)\end{aligned}$$

$$-c_{25}^2(P+2\partial^2 R)+c_{25}c_{37}(\partial^3 R+2\partial^4 S)-c_{25}c_{49}\partial^4 R$$
$$+c_{37}a_{13}(\partial Q-\partial^3 S)+c_{37}c_{25}(-2\partial^2 Q+2\partial^4 S)$$
$$+c_{37}^2\partial^3 Q-c_{37}c_{49}\partial^4 Q+c_{49}a_{13}(\partial P+\partial^4 S)$$
$$-2c_{49}c_{25}\partial^2 P+c_{49}c_{37}\partial^3 P-c_{49}^2\partial^4 P.$$

Thus $d(a_{13}S+c_{25}R+c_{37}Q+c_{49}P)=0$ if and only if

$$\partial^4 S=0, \qquad P=-\partial^3 S, \qquad Q=\partial^2 S, \qquad R=-2\partial S.$$

Therefore, $dF_{2n+1}=0$ if and only if F_{2n+1} is of the form

$$F_{2n+1}=y_{62}^n(a_{13}S-2c_{25}\partial S+c_{37}\partial^2 S-c_{49}\partial^3 S)$$

with $\partial^4 S=0$. q. e. d.

We have now only to find cocycles of the form

$$F_1=a_{13}S-2c_{25}\partial S+c_{37}\partial^2 S-c_{49}\partial^3 S \qquad \text{with} \quad \partial^4 S=0.$$

We divide into the following four cases:

(i) If $\partial S=0$, then $F_1=a_{13}S$ with S a cocycle.

(ii) If $\partial^2 S=0$, then $F_1=a_{13}S-2c_{25}\partial S$ with ∂S a cocycle. If $\partial S'=\partial S$, then the difference of two cocycles $F_1'=a_{13}S'-2c_{25}\partial S$ and $F_1=a_{13}S-2c_{25}\partial S$ is a cocycle $a_{13}(S'-S)$, a cocycle in (i). Thus we may choose one S for a cocycle of the form ∂S.

Now, by Remark 4.21, a cocycle of the form ∂S is in the ∂^3-image, say $\partial^3 T$. Choosing S to be $\partial^2 T$, we have

$$F_1=a_{13}\partial^2 T-2c_{25}\partial^3 T=d(-\partial T).$$

(iii) If $\partial^3 S=0$, then $F_1=a_{13}S-2c_{25}\partial S+c_{37}\partial^2 S$ with $\partial^2 S$ a cocycle. Again, it is sufficient to choose one S for a cocycle of the form $\partial^2 S$. Again by Remark 4.21, a cocycle in the ∂^2-image is of the form $\partial^3 T$. Thus choosing $S=\partial T$, we have

$$F_1=a_{13}\partial T-2c_{25}\partial^2 T+c_{37}\partial^2 T=d(-T).$$

(iv) Finally, as $\partial^4 S=0$, $\partial^3 S$ is a cocycle. If $\partial^3 S$ is $\partial^4 U$ for some U, then choosing $S=\partial U$, we have $F_1=d(-U)$.

A cocycle of the form $\partial^3 S$ but not in the ∂^4-image is, by (4.19.2), expressible as

$$\sum \alpha(k, l)x_4x_{200}^k u_{240}^l+\sum \beta(k, l)u_{12}x_{200}^k u_{240}^l+\partial^4 U,$$

where $\alpha(k, l)$ and $\beta(k, l)$ are numbers in $\boldsymbol{Z}_5$.

In particular, for $x_4=\partial^3 e_{40}$ and $u_{12}=\partial^3 e_{48}$, we have

$$y_{53}=a_{13}e_{40}+c_{25}d_{28}+c_{37}b_{16}+c_{49}a_4$$

and

$$y_{61}=a_{13}e_{48}+c_{25}d_{36}+c_{37}b_{24}+c_{49}a_{12}.$$

For $\partial^3 S=\sum \alpha(k, l)x_4x_{200}^k u_{240}^l+\sum \beta(k, l)u_{12}x_{200}^k u_{240}^l+\partial^4 U$, we have a decomposable cocycle

$$F_1=\sum \alpha(k,\ l)y_{53}x_{200}^k u_{240}^l+\sum \beta(k,\ l)y_{61}x_{200}^k u_{240}^l+d(-U)\,.$$

Thus we have

Proposition 6.3. *The elements a_{13}, y_{62}, y_{53} and y_{61} are all the indecomposable cocycles with elements of odd degree.*

For later use, we shall state a more concrete form of Lemma 6.2.

Lemma 6.4. (1) *A cocycle of degree $2n$ with respect to elements of odd degree is as in* Lemma 6.2:

$$y_{62}^n A \text{ with } A \text{ a cocycle in } V\,;$$

(2) *A cocycle of degree $2n+1$ is*

$$y_{62}^n a_{13}A+\sum_{k,l}\alpha(k,\ l)y_{62}^n y_{53}x_{200}^k u_{240}^l+\sum_{k,l}\beta(k,\ l)y_{62}^n y_{61}x_{200}^k u_{240}^l$$

with A a cocycle in V and $\alpha(k,\ l)$, $\beta(k,\ l)\in \boldsymbol{Z}_5$.

Thus we have found all the indecomposable cocycles:

Theorem 6.5. *All the indecomposable cocycles in $\overline{W}$ are the* 34 *listed in* (3.5) *and* (3.5)$'$, *the* 1002 *of mixed type and a_{13}, y_{62}, y_{53} and y_{61}.*

§7. Commutativity of generators.

Now that we have found all the indecomposable cocycles, we shall check for commutativity and find relations among them.

Theorem 7.1. *$H(\overline{W}:d)$ is commutative.*

Proof. Since the cocycles in V satisfy $\partial P=0$, they commute with a_{13}, c_{25}, c_{37} and c_{49}, and hence with a_{13}, y_{62}, y_{53} and y_{61}.

We have

$$[a_{13},\ y_{62}]=d(2[c_{25},\ c_{49}]+c_{37}^2)\,,$$

$$[a_{13},\ y_{53}]=d(-c_{25}e_{40}+c_{37}d_{28}-c_{49}b_{16})\,,$$

$$[a_{13},\ y_{61}]=d(-c_{25}e_{48}+c_{37}d_{36}-c_{49}b_{24})\,,$$

$$[y_{53},\ y_{61}]=d(c_{25}e_{40}e_{48}+c_{37}(d_{28}e_{48}-e_{40}d_{36})$$
$$-c_{49}(b_{16}e_{48}+d_{28}d_{36}-e_{40}b_{24}))\,,$$

$$[y_{53},\ y_{62}]=d(2[c_{25},\ c_{49}]e_{40}+c_{37}^2e_{40}-2[c_{37},\ c_{49}]d_{28}+c_{49}^2b_{16})\,,$$

$$[y_{61},\ y_{62}]=d(2[c_{25},\ c_{49}]e_{48}+c_{37}^2e_{48}-2[c_{37},\ c_{49}]d_{36}+c_{49}^2b_{24})\,.$$

Thus commutativity holds in $H(\overline{W}: d)$. q. e. d.

Lemma 7.2. *The following elements are non-trivial and they are linearly independent:*

$$y_{62}^i x_{200}^j u_{240}^k, \quad y_{62}^i m_{52} x_{200}^j u_{240}^k, \quad y_{62}^i x_4 x_{200}^j u_{240}^k,$$

$$y_{62}^i u_{12} x_{200}^j u_{240}^k, \quad y_{62}^i a_{13} x_{200}^j u_{240}^k, \quad y_{62}^i a_{13} m_{52} x_{200}^j u_{240}^k,$$

$$y_{62}^i y_{53} x_{200}^j u_{240}^k, \quad y_{62}^i y_{61} x_{200}^j u_{240}^k.$$

where i, j, k are non-negative integers.

Proof. Note first that the differential operator d augments the degree by 1 with respect to elements of odd degree. Thus $\sum F_k \in d$-image for different degrees k occurs only when each F_k is in the d-image.

Now, a cocycle F_{2n+1} of degree $2n+1$ is, by Lemma 6.4, of the form:

$$\begin{aligned} F_{2n+1} &= y_{62}^n(a_{13}A + \sum \alpha(i, j) y_{53} x_{200}^i u_{240}^j + \sum \beta(i, j) y_{61} x_{200}^i u_{240}^j) \\ &= y_{62}^n\{a_{13}(A + \sum \alpha(i, j) e_{40} x_{200}^i u_{240}^j + \sum \beta(i, j) e_{48} x_{200}^i u_{240}^j) \\ &\quad + c_{25}(\sum \alpha(i, j) d_{28} x_{200}^i u_{240}^j + \sum \beta(i, j) d_{36} x_{200}^i u_{240}^j) \\ &\quad + c_{37}(\sum \alpha(i, j) b_{16} x_{200}^i u_{240}^j + \sum \beta(i, j) b_{24} x_{200}^i u_{240}^j) \\ &\quad + c_{49}(\sum \alpha(i, j) a_4 x_{200}^i u_{240}^j + \sum \beta(i, j) a_{12} x_{200}^i u_{240}^j)\}, \end{aligned}$$

where A is a cocycle in V and

$$\sum \alpha(i, j) a_4 x_{200}^i u_{240}^j + \sum \beta(i, j) a_{12} x_{200}^i u_{240}^j$$

is not in the ∂^4-image by (4.20.1).

On the other hand any element F_{2n} of degree $2n$ can be written as in Lemma 6.1 and its d-image is calculated as

$$\begin{aligned} dF_{2n} &= \sum_{i=0}^{n-1} y_{62}^i d(c_{25} F_{2n-2i-1} + c_{37} F'_{2n-2i-1} + c_{49} F''_{2n-2i-1}) \\ &\quad + y_{62}^n(-a_{13}\partial P + 2c_{25}\partial^2 P - c_{37}\partial^3 P + c_{49}\partial^4 P). \end{aligned}$$

Comparing our cocycle F_{2n+1} with dF_{2n}, we see that F_{2n+1} is not in the d-image so long as it has a term $y_{62}^n y_{53} x_{200}^i u_{240}^j$ or $y_{62}^n y_{61} x_{200}^i u_{240}^j$. That is to say, $y_{62}^n y_{53} x_{200}^i u_{240}^j$ and $y_{62}^n y_{61} x_{200}^i u_{240}^j$ are non-trivial, and they and $y_{62}^n a_{13} A$ (if it is not trivial) are linearly independent.

Comparing $y_{62}^n a_{13} A$ again with dF_{2n}, we see that $y_{62}^n a_{13} A$ is in the d-image only when $A = -\partial P$ and $y_{62}^n a_{13}\partial(-P) = d(y_{62}^n P)$. Referring to (4.20.3) we see that $y_{62}^n a_{13} x_{200}^i u_{240}^j$, $y_{62}^n a_{13} m_{52} x_{200}^i u_{240}^j$ and their sum remain non-trivial.

Similarly, a cocycle of degree $2n+2$ is, by Lemma 6.4, of the form $y_{62}^{n+1} A$ with A a cocycle in V. And any element F_{2n+1} of degree $2n+1$ can be written as in Lemma 6.1:

$$F_{2n+1}=\sum_{i=0}^{n-1} y_{62}^{i}(c_{25}F_{2n-2i}+c_{37}F'_{2n-2i}+c_{49}F''_{2n-2i})$$

$$+y_{62}^{n}(a_{13}S+c_{25}R+c_{37}Q+c_{49}P)+(d\text{-image}),$$

whence we have

$$dF_{2n+1}=\sum_{i=0}^{n-1} y_{62}^{i}d(c_{25}F_{2n-2i}+c_{37}F'_{2n-2i}+c_{49}F''_{2n-2i})$$

$$+y_{62}^{n}\times(\text{some terms})+y_{62}^{n+1}(-\partial^4 S),$$

where we observe that there is no term $y_{62}^{n+1}\times$(some other term) except the last one.

Comparing $y_{62}^{n+1}A$ with dF_{2n+1}, we see that $y_{62}^{n+1}A$ is in the d-image only when $A=\partial^4 T$ for some T. And

$$y_{62}^{n+1}\partial^4 T=d(y_{62}^{n}(-a_{13}T+2c_{25}\partial T-c_{37}\partial^2 T+c_{49}\partial^3 T)).$$

By (4.20.1), we see that

$$y_{62}^{n+1}x_{200}^{i}u_{240}^{j}, \qquad y_{62}^{n+1}m_{52}x_{200}^{i}u_{240}^{j},$$

$$y_{62}^{n+1}x_4x_{200}^{i}u_{240}^{j}, \qquad y_{62}^{n+1}u_{12}x_{200}^{i}u_{240}^{j}$$

and their sum remain non-trivial. q. e. d.

Lemma 7.3. *A cocycle with elements of odd degree is either trivial or a linear combination of the cocycles in* Lemma 7.2.

Proof. We have

$$a_{13}^2=d(2c_{25}),$$

$$y_{53}^2=d(2c_{25}e_{40}^2+c_{37}d_{28}e_{40}-c_{49}(b_{16}e_{40}-2d_{28}^2)),$$

$$y_{61}^2=d(2c_{25}e_{48}^2+c_{37}d_{36}e_{48}-c_{49}(b_{24}e_{48}-2d_{36}^2)),$$

$$a_{13}y_{53}=d(2c_{25}e_{40}+2c_{37}d_{28}+c_{49}b_{16})-y_{62}x_4,$$

$$a_{13}y_{61}=d(2c_{25}e_{48}+2c_{37}d_{36}+c_{49}b_{24})-y_{62}u_{12},$$

$$y_{53}y_{61}=d(-2c_{25}e_{40}e_{48}-c_{37}(d_{28}e_{48}-2e_{40}d_{36})$$

$$-c_{49}(2b_{16}e_{48}-2d_{28}d_{36}-e_{40}b_{24}))+y_{62}m_{52}.$$

Thus any monomial in cocycles is equivalent to a monomial in cocycles each term of which has at most one of a_{13}, y_{53} or y_{61}.

We have shown

$$y_{62}\partial^4 T=d(-a_{13}T+2c_{25}\partial T-c_{37}\partial^2 T+c_{49}\partial^3 T),$$

and $a_{13}\partial T=d(-T)$. In particular

$$a_{13}\partial^4 T=d(-\partial^3 T), \qquad a_{13}x_4=d(-b_{16}), \qquad a_{13}u_{12}=d(-b_{24}).$$

Finally we have the following d-images:

$$y_{53}\partial^4 T=d(-a_4 T-\partial(b_{16}T)+2\partial^2(d_{28}T)-\partial^3(e_{40}T)),$$

$$y_{61}\partial^4 T=d(-a_{12}T-\partial(b_{24}T)+2\partial^2(d_{36}T)-\partial^3(e_{48}T)),$$

$$y_{53}x_4=d(2d_{28}^2-b_{16}e_{40}),$$

$$y_{61}x_4+a_{13}m_{52}=d(-2b_{16}e_{48}+2d_{28}d_{36}+e_{40}b_{24}),$$

$$y_{53}u_{12}-a_{13}m_{52}=d(b_{16}e_{48}+2d_{28}d_{36}-2e_{40}b_{24}),$$

$$y_{61}u_{12}=d(2d_{36}^2-b_{24}e_{48}),$$

$$y_{53}m_{52}=d((2d_{28}^2-b_{16}e_{40})e_{48}-2d_{28}e_{40}d_{36}+e_{40}^2b_{24}),$$

$$y_{61}m_{52}=d(-b_{16}e_{48}^2+2d_{28}d_{36}e_{48}+e_{40}(-2d_{36}^2+b_{24}e_{48})),$$

Using these rèlations we see that any monomial in cocycles is either trivial or equivalent to one of cocycles in Lemma 7.2. q. e. d.

Theorem 7.4. $\mathrm{Cotor}_A(\boldsymbol{Z}_5, \boldsymbol{Z}_5)$ *with* $A=H^*(X;\boldsymbol{Z}_5)$ *is generated as a commutative algebra by the elements in* Proposition 4.16 *and* a_{13}, y_{62}, y_{53} *and* y_{61}, *where*

$$y_{62}=[a_{13}, c_{49}]+2[c_{25}, c_{37}],$$

$$y_{53}=a_{13}e_{40}+c_{25}d_{28}+c_{37}b_{16}+c_{49}a_4$$

and

$$y_{61}=a_{13}e_{48}+c_{25}d_{36}+c_{37}b_{24}+c_{49}a_{12}.$$

The elements satisfy the relations

$$a_{13}^2=0, \quad a_{13}y_{53}=-y_{62}x_4, \quad a_{13}y_{61}=-y_{62}u_{12},$$

$$y_{53}^2=0, \quad y_{61}^2=0, \quad y_{53}y_{61}=y_{62}m_{52},$$

$$a_{13}\partial^4 T=0, \quad y_{53}\partial^4 T=0, \quad y_{61}\partial^4 T=0, \quad y_{62}\partial^4 T=0,$$

$$a_{13}x_4=0, \quad a_{13}u_{12}=0,$$

$$y_{53}x_4=0, \quad y_{61}x_4=-a_{13}m_{52},$$

$$y_{53}u_{12}=a_{13}m_{52}, \quad y_{61}u_{12}=0,$$

$$y_{53}m_{52}=0, \quad y_{61}m_{52}=0.$$

§ 8. Collapsing of the Eilenberg-Moore spectral sequence.

Consider the Eilenberg-Moore spectral sequence mod 5 $\{E_r, d_r\}$ associated with $X\in\{E_8:5\}$:

$$E_2\cong\mathrm{Cotor}_A(\boldsymbol{Z}_5, \boldsymbol{Z}_5) \quad \text{with} \quad A=H^*(X;\boldsymbol{Z}_5),$$

$$E_\infty\cong\mathscr{G}r\, H^*(BX;\boldsymbol{Z}_5).$$

To begin with we recall that the differential $d_r: E_r^{s,t}\to E_r^{s+r,t-r+1}$ $(r\geqq 2)$ augments the total degree by 1 and the homological degree by r.

Obviously $d_r x_4=0$ for $r\geqq 2$, since there is no element of degree 5.

There is no element of degree 14 or 54, since any element of even degree and of degree less than 62 is of degree $4n$ for some n. Thus $d_r a_{13}=0$ and $d_r y_{53}=0$ for $r\geqq 2$.

The element y_{62} is the only element of degree 62, but y_{61} and y_{62} are of the same homological degree and hence y_{62} cannot be $d_r y_{61}$. Thus $d_r y_{61}=0$ for $r\geqq 2$.

As $d_r y_{62}$ is of odd degree, referring to Lemmas 7.2 and 7.3, we see that $d_r y_{62}=0$ for $r\geqq 2$.

Referring again to Lemmas 7.2 and 7.3, we see that there is no element of degree 201 or 241 and hence $d_r x_{200}$ and $d_r u_{240}$ are also 0 for $r\geqq 2$.

Clearly a_{13} is the only element of degree 13, but u_{12} and a_{13} are of the same homological degree, whence a_{13} cannot be $d_r u_{12}$. Thus $d_r u_{12}=0$ for $r\geqq 2$. Similarly $d_r m_{52}=0$ for $r\geqq 2$, since there is no element of degree 53 except y_{53}, but y_{53} is of the same homological degree as m_{52}.

We have shown that a_{13}, y_{53}, y_{61}, y_{62}, x_4, u_{12}, m_{52}, x_{200} and u_{240} survive to E_∞. Remark that any other generators are in the ∂^4-image and of even degree.

Suppose that they all survive to E_r. Then we have $E_r\cong E_2$. The only possibility for their d_r-image Q is, by Lemmas 3.23 and 3.24, a sum of $y_{62}^i a_{13} x_{200}^j u_{240}^k$, $y_{62}^i a_{13} m_{52} x_{200}^j u_{240}^k$, $y_{62}^i y_{53} x_{200}^j u_{240}^k$ and $y_{62}^i y_{61} x_{200}^j u_{240}^k$.

Suppose now that we have a possibility of the relation $d_r P=Q$, with $P\in\partial^4$-image and Q as above. Multiply both sides by y_{62}. Then the right hand side is not 0, since $y_{62}Q$ (with Q as above) is not 0 in E_2 by Lemma 7.2, and hence in E_r. On the other hand, $y_{62}d_r P=d_r(y_{62}P)=0$, since, P being in the ∂^4-image, $y_{62}P$ is trivial in E_2 by Theorem 7.4. Therefore there is no such possibility as $d_r P=Q$. Thus, all the generators survive to E_{r+1}.

Now by induction on r we can see that all the generators survive to E_∞.

Theorem 8.1. *The Eilenberg-Moore spectral sequence* mod 5 *associated with X collapses for all $X\in\{E_8:5\}$.*

And

Theorem 3.2. *As modules, for $X\in\{E_8:5\}$,*

$$H^*(BX;\boldsymbol{Z}_5)\cong \mathrm{Cotor}_A(\boldsymbol{Z}_5,\boldsymbol{Z}_5) \quad \textit{with} \quad A=H^*(X;\boldsymbol{Z}_5).$$

In particular we have

Corollary 8.3. *As modules*

$$H^*(BE_8;\boldsymbol{Z}_5)\cong \mathrm{Cotor}_A(\boldsymbol{Z}_5,\boldsymbol{Z}_5) \quad \textit{with} \quad A=H^*(E_8;\boldsymbol{Z}_5).$$

INSTITUTE OF MATHEMATICS
YOSHIDA COLLEGE
KYOTO UNIVERSITY;
SCHOOL OF BUSINESS ADMINISTRATION
SENSHU UNIVERSITY

References

[1] A. Iwai-N. Shimada, A remark on resolutions for Hopf algebras, Publ. RIMS of Kyoto Univ., **1** (1966), 187-198.

[2] A. Kono, Hopf algebra structure of simple Lie groups, J. Math. Kyoto Univ., **17** (1977), 259-298.

[3] A. Kono-M. Mimura-N. Shimada, Cohomology of classifying space of certain associative H-spaces, J. Math. Kyoto Univ., **15** (1975), 607-617.

[4] M. Rothenberg-N. E. Steenrod, The cohomology of classifying spaces of H-spaces, Bull. AMS, **71** (1965), 872-875.

[5] N. Shimada-A. Iwai, On the cohomology of some Hopf algebra, Nagoya Math. J.. **30** (1971), 103-111.