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Dynamics of Caianiello's equation

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1. Introduction

Caianiello's equation is known as mathematical neuron model. In 1961, L. D. Harmon found an unusual and unsuspected phenomenon between the amplitude of the input pulses and the firing in experimental studies with his transister neuron model. J. Nagumo and S. Sato [1] started mathematical investigation of this model. They studied the dynamics of the periodic attractor and suggested that this complicated relationship between them takes the form of Cantor function. Mainly we will give some new results concerning the dynamics which is not the periodic attractor. In this section Caianiello's equation is reduced to a discontinuous piecewise-linear equation. We distinguish two cases, which we shall treat one in section 2 and the other in section 3.

We assume that the magnitude of the input stimulus is constant and that the neuron is forgetting past firing with exponential rate. Under these assumptions Caianiello's equation takes the form:

$$x_{n+1}=\mathbf{1}\left[A-\alpha\sum_{r=0}^{n}\frac{x_{n-r}}{b^r}-\theta\right] \qquad (\alpha>0,\ b>1),$$

where $\mathbf{1}[x]$ is the Heaviside function. The value x_n represents the state of the neuron at the instant n: $x_n=0$ represents the resting state and $x_n=1$ represents the exciting state. Constant A is the magnitude of the input stimulus and θ is the threshold value.

Letting $y_n=1+\dfrac{A-\theta}{\alpha b}-\sum_{r=0}^{n}\dfrac{x_{n-r}}{b^r}$, we obtain

$$y_{n+1}=f(y_n,\beta,c)$$

where $f(x,\beta,c)=\begin{cases}\beta(x-c)+1 & \text{if } x<c\\ \beta(x-c) & \text{if } x\geqq c,\end{cases}$

$$\beta=\frac{1}{b},\ c=1-\frac{A-\theta}{\alpha}\left(1-\frac{1}{b}\right),\ \text{ and } \ x_{n+1}=\mathbf{1}[y_n-c].$$

We assume that $0<c<1$ because the neuron always excites or rests for large instant n according to $c<0$ or $c>1$ respectively.

In the following we will investigate the dynamics of a discontinuous piecewise-linear function $f(x)=f(x, \beta, c)$ on the parameter plane $(\beta, c)\in(0, 1)\times(0, 1)$. $f(x)$ maps the right open unit interval $I=[0, 1)$ into itself. It is sufficient to examine $f(x)$ only on I, because the iterated point of the initial value out of I must fall into I after some iterations by f. The next lemma obviously follows and the proofs are omitted.

Lemma 1.1. (1) *The graph of $f^n(x)$ consists of right open segments. Each segment has the same slope β^n.*

(2) *$I\supset f(I)\supset f^2(I)\supset\cdots$ and f^n is an one to one map for $n\geqq 1$.*

(3) *Assume that $c\in f^n(I)$ for some $n\geqq 0$ and that right open interval J dose not contain any of the following $n+1$ points:*

$$c,\ f^{-1}(c),\ f^{-2}(c),\ \cdots,\ f^{-n}(c)$$

as interior points of J. Then $f^k\colon J\to f^k(J)$ is a homeomorphism for $k=1, 2, \cdots n+1$.

Divide $I=[0, 1)$ into two subintervals $I^0=[0, c)$ and $I^1=[c, 1)$. First we give some definitions. For $x\in I$, let $A(x)$ be the formal symbol I^j if x belongs to I^j, where $j=0$ or 1. We call $A(x)$ the address of x. By the itinerary $I(x)$ we mean the sequence of addresses:

$$(A(x),\ A(f(x)),\ A(f^2(x)),\ \cdots)$$

of the successive image of x. For each symbol I^j we define the sign $\varepsilon(I^j)=j$ and $\varepsilon_n(x)=\varepsilon(A(f^{n-1}(x)))$ for $n\geqq 1$. By the sign itinerary $\varepsilon(I(x))$ we mean the sequence of signs:

$$(\varepsilon_1(x),\ \varepsilon_2(x),\ \varepsilon_3(x),\ \cdots)$$

To investigate the dynamics of $f(x)$ for the parameters $(\beta, c)\in(0, 1)\times(0, 1)$, we deal with Case A and Case B separately as follows.

Case A: There exists some integer $N>1$ such that $c\in\operatorname{Int} f^i(I)$ for $i=0, 1, \cdots, N-1$ and $c\notin\operatorname{Int} f^N(I)$.

Case B: For any integer $n\geqq 1$, $c\in\operatorname{Int} f^n(I)$.

2. Dynamics in case A

In this section we describe the dynamics in Case A. We will call this case a periodic case, which is justified by the following theorem.

Theorem 2.1. *For any $x\in I$ the itinerary $I(x)$ has period $N+1$, where N is given in the definition of Case A.*

Proof. Since $c\in\operatorname{Int} f^i(I)$, there exists the inverse image of c for f^i, where $i=0, 1, \cdots, N-1$. We use the abbreviation c_{-i} instead of $f^{-i}(c)$ for $i\geqq 0$. Note that N points $\{c_0, c_{-1}, \cdots, c_{-N+1}\}$ are all distinct and non-zero. Divide I into $N+1$ right open subintervals by these N points and denote $I_1, I_2, \cdots I_{N+1}$. By Lemma 1.1, f is homeomorphic on each subinterval I_j. Assume that $c_{-j}\in\operatorname{Int} f(I_i)$

for some $0 \leqq j \leqq N-1$ and $1 \leqq i \leqq N+1$. Since $c_{-j-1} \in \operatorname{Int} I_i$, we have $j+1=N$. However, $f^N(c_{-j-1}) \in \operatorname{Int} f^N(I_i)$ since f^N is homeomorphic on I_i, so $c \in \operatorname{Int} f^N(I)$. This is a contradiction. Hence each $f(I_i)$ must be contained by some subinterval I_k. On the other hand we can show that each subinterval must contain some $f(I_i)$. So we have $f(I_i) \subset I_{\pi(i)}$ for $i=1, 2, \cdots, N+1$, where π is a permutation. Also we have $\pi^{N+1}=Id$ and $\pi^j \neq Id$ for $j=1, 2, \cdots, N$. Moreover we can show that $\pi=\sigma^p$ where $\sigma=\begin{pmatrix} 1 & 2 & \cdots & N+1 \\ 2 & 3 & \cdots & 1 \end{pmatrix}$, $1 \leqq p \leqq N$, $(p, N+1)=1$, and p is the number of $1 \leqq j \leqq N+1$ such that $I_j \subset I^1$.

Finally, for any $x \in I$, the sequence $(x, f^{N+1}(x), f^{2(N+1)}(x), \cdots)$ is monotonous and the itinerary $I(x)$ has period $N+1$. □

3. Dynamics in Case B

In this section we describe the dynamics in Case B. We will call this case a singular case. It will be justified by the Theorem 3.5 below.

Lemma 3.1. *For any* $n \geqq 1$,

(1) $0 < f^n(1) < f^n(0) < 1$

(2) *letting* $J_n = [f^n(1), f^n(0))$, *then* $\operatorname{cl} J_p \cap \operatorname{cl} J_q = \phi$ *if* $p \neq q$, $c \notin \operatorname{cl} J_n$, *and* $|J_n| = \beta^{n-1}(1-\beta)$

(3) $f^n(I) = I - \bigcup_{k=1}^{n} J_k$.

Proof. We shall prove by an induction on n. It is trivial for $n=1$. Assume that (1), (2) and (3) hold for $n=k$. By the property (3) $f^k(I)$ consists of $k+1$ components. Since f is homeomorphic on J_i for $i=1, 2, \cdots, k$, (1), (2) and (3) hold for $n=k+1$. □

Theorem 3.2. *f has no periodic points and the itinerary $I(x)$ is not periodic for any $x \in I$.*

Proof. Assume that f has a periodic point P with period N. By Lemma 3.1, $f^n(0)$ and $f^n(1)$ are not periodic for any $n \geqq 1$. So P is neither $f^n(0)$ nor $f^n(1)$. Assume that $P \in \operatorname{Int} J_n$ for some $n \geqq 1$, then $f^N(P) \in \operatorname{Int} J_{n+N}$. This is a contradiction. Hence $P \notin \operatorname{cl} J_n$ and $P \in \operatorname{Int} f^n(I)$ for any $n \geqq 1$. By the property $\sum_{n=1}^{\infty} |J_n| = 1$, for any $\varepsilon > 0$, there exists J_N such that $|J_N - P| < \varepsilon$. Now we can take the neighborhood U of P such that $\{U, f(U), \cdots, f^{N-1}(U)\}$ are disjoint one another since $P \in \operatorname{Int} f^N(I)$. Let U_l and U_r be the left and right half of the neighborhood U of P respectively. And take small J_p and J_q such that $J_p \subset U_l$, $J_q \subset U_r$, and $p < q$. Since f^N is a linear function on U with positive slope, we have $f^n(x) \notin U_r$ for any $n \geqq 1$ and $x \in U_l$. However, $f^{q-p}(J_p) = J_q$. This is a contradiction. Hence f has no periodic points.

Next assume that the itinerary $I(x)$ has period N for some $x \in I$. Let $L_j = [\inf_{m \geq 0} f^{Nm+j}(x), \sup_{m \geq 0} f^{Nm+j}(x))$ for $j=0, 1, \cdots, N-1$. Each L_j has positive measure since f has no periodic points. By assumption, we have $L_j \subset I^0$ or $L_j \subset I^1$ for each j. Since f is homeomorphic on each L_j, we have $f(L_j) \subset L_{j+1}$ for $j=0, 1, \cdots, N-2$ and $f(L_{N-1}) \subset L_0$. Hence $f^N(L_j) \subset L_j$ and f^N is homeomorphic on each L_j. Note that $L_0, L_1, \cdots, L_{N-1}$ are disjoint since f has no periodic points. Let $n_0 = \min_{n \geq 1} \{n \mid f^n(0) \in L_0\}$. Then we have $J_{n_0+kN} \subset L_0$ for any $k \geqq 1$. By Lemma 3.1, there exists an interval $J^* = [f^{n_0+N}(0), f^{n_0+2N}(1))$ which lies between J_{n_0+N} and J_{n_0+2N}. However $J_n \cap J^* = \phi$ for any $n \geqq 1$, and this contradicts to $\sum_{n=1}^{\infty} |J_n| = 1$. This completes the proof. □

Now we define $L(n) = \inf_{x \in I} |x - f^n(x)|$ for $n \geqq 1$. Obviously we have $L(j) > 0$ for $j=1, 2, \cdots, N$ and $L(N+1)=0$ in the periodic case. And we have the following lemma in the singular case.

Lemma 3.3. $L(n) > 0$ *for any* $n \geqq 1$.

Proof. Assume that $L(M)=0$ for some $M \geqq 1$. Then there exists an interval $\tilde{I} = [a, b)$ such that f^M is homeomorphic on $\tilde{I}$ and $f^M(x) \to b$ as $x \to b-$. Thus the sequence $\{f^{kM}(x)\}_{k \geq 1}$ is monotone increasing for any $x \in \tilde{I}$. This shows the existence of a point $x_0 \in \tilde{I}$ such that the itinerary $I(x_0)$ has period M. This contradicts to Theorem 3.2. □

Lemma 3.4. *For any* $n \geqq 0$ *there exists a set of non-negative integer* $\{j_1, j_2, \cdots, j_{n+1}\}$ *such that* $c_{-j_i} \in \operatorname{Int} K_i$ *for* $i=1, 2, \cdots, n+1$, *where* $K_1, K_2, \cdots, K_{n+1}$ *are* $n+1$ *components of* $f^n(I)$.

Proof. Assume that there exists a positive integer M and a right open interval K, which is one of the components of $f^M(I)$, such that $c_{-j} \notin \operatorname{Int} K$ for any $j \geqq 0$. By Lemma 3.1, we have $K = [f^p(0), f^q(1))$ where $0 \leqq p, q \leqq M$ and $p \neq q$. Also f^j is homeomorphic on K and $f^j(K)$ lies between J_{p+j} and J_{q+j} for any $j \geqq 1$. Then, for any $x \in J_p$ and $y \in J_q$, we have

$$\begin{aligned} |f^j(x) - f^j(y)| &< |J_{p+j}| + |J_{q+j}| + |f^j(K)| \\ &= \beta^j(|J_p| + |J_q| + |K|). \end{aligned}$$

In the case of $p > q$, by Lemma 3.3, we have $L(p-q) > 0$. So there exists a positive integer j_0 such that

$$\min\{|J_{p+j}|, |f^j(x) - f^j(y)|\} < \frac{L(p-q)}{4} \qquad \text{for any} \quad j \geqq j_0.$$

Then we have

$$\begin{aligned} L(p-q) = \inf_{z \in I} |z - f^{p-q}(z)| &\leqq |f^j(y) - f^{p-q+j}(y)| \\ &\leqq |f^j(y) - f^j(x)| + |f^j(x) - f^{p-q+j}(y)|. \end{aligned}$$

However $f^j(x) \in J_{p+j}$ and $f^{p-q+j}(y) \in J_{p+j}$, so the last expression is less than

$\frac{L(p-q)}{2}$. This is a contradiction. Similarly we get the same result in the case of $p<q$. □

Let $\Lambda=\bigcap_{n=1}^{\infty}\mathrm{cl}\, f^n(I)$. This is an f-invarient Cantor set by Lemma 3.1 and has null Lebesque measure. Note that Λ is the closure of the orbit of c for f^{-1}. Moreover the next theorem follows.

Theorem 3.5. *For any $x\in\Lambda$, Λ is the ω-limit set of x. Remark that the ω-limit set of x is the closure of the orbit of x for f.*

Proof. Fix $n\geqq 1$ and let $m_0=|J_n|$. By Lemma 3.4, there exists a positive integer M such that each component of $f^n(I)$ contains some point of $\{c, c_{-1}, \cdots, c_{-M}\}$. Also, for any $x\in\Lambda$, there exists a positive integer $N>M$ satisfying the following properties:

(1) $|x-c_{-N}|<\frac{m_0}{2}$

(2) there exists an open interval U_N such that $x\in U_N$, $c_{-N}\in U_N$, $c_{-j}\notin U_N$ for $j=0, 1, \cdots, N-1$.

Then f^k is homeomorphic on U_N for $k=1, 2, \cdots, N$. Therefore we have

$$|f^k(x)-c_{-N+k}|<\beta^k|x-c_{-N}|<\frac{m_0}{2}.$$

This inequality shows that two points $f^k(x)$ and c_{-N+k} must be contained by same component of $f^n(I)$. Hence each component of $f^n(I)$ contains some point of $\{x, f(x), \cdots, f^N(x)\}$. This completes the proof. □

We call Λ a Cantor attractor.

4. Farey fractions

This section describes Farey fractions and some of their properties without proofs. Farey fractions are closely related to the distribution of the periodic cases on the parameter plane $(\beta, c)\in(0, 1)\times(0, 1)$.

Definition 4.1. The set of Farey fractions of order $n\geqq 1$, denoted F_n, is the set of reduced fractions in the interval $[0, 1]$ with denominater$\leqq n$.

Examples. $F_1: \frac{0}{1}, \frac{1}{1}$

$F_2: \frac{0}{1}, \frac{1}{2}, \frac{1}{1}$

$F_3: \frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}$

$F_4: \frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}$

If $\frac{a}{b}<\frac{c}{d}$ are consecutive in some F_n, then they satisfy the unimodular relation $bc-ad=1$. Conversely, if four non-negative integers a, b, c, d satisfy $bc-ad=1$, then $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive terms in F_n only for

$$\max(b, d)\leqq n\leqq b+d-1 .$$

By these properties, the next special induction we call Farey induction. A proposition which is defined for all reduced fractions in the interval [0, 1] is true if we check the following two properties. We denote this proposition by P.

(1) Both $P(0)$ and $P(1)$ are true.

(2) $P\left(\frac{a+c}{b+d}\right)$ is true under assuming that both $P\left(\frac{a}{b}\right)$ and $P\left(\frac{c}{d}\right)$ are true, where $bc-ad=1$ and $\frac{a}{b}, \frac{c}{d}\in[0, 1]$.

5. Distribution of periodic cases and singular cases on the parameter plane

First, fix the value of $\beta\in(0, 1)$. We will investigate the two orbits which start from 0 and 1, and these orbits may be regarded as the sequences of functions of variable c. This idea is analogous to the investigation of the orbit of a critical point of one dimensional endomorphism.

Let

$$f(x, c)=\begin{cases}\beta(x-c)+1 & \text{if } x<c\\ \beta(x-c) & \text{if } x\geqq c .\end{cases}$$

And define $F_n(c)$ and $G_n(c)$ inductively as follows.

$$F_1(c)=1-\beta c , \qquad F_{n+1}(c)=f(F_n(c), c) ,$$

$$G_1(c)=\beta(1-c) , \qquad G_{n+1}(c)=f(G_n(c), c) .$$

Lemma 5.1. (1) *$F_n(c)$ (resp. $G_n(c)$) is a piecewise-linear function and each segment of $F_n(c)$ (resp. $G_n(c)$) has the same slope $\frac{\beta^{n+1}-\beta}{1-\beta}$ for any $n\geqq 1$.*

(2) *P is a discontinuity point of $F_n(c)$ (resp. $G_n(c)$) if and only if there exists $1\leqq m\leqq n-1$ such that P is a fixed point of $F_m(c)$ (resp. $G_m(c)$) for any $n\geqq 2$.*

Proof. We shall prove by an induction on n. It is trivial for $n=2$. Assume that (1) and (2) hold for $n=k$. By definition,

$$F_{k+1}(c)=\begin{cases}\beta(F_k(c)-c)+1 & \text{if } F_k(c)<c\\ \beta(F_k(c)-c) & \text{if } F_k(c)\geqq c .\end{cases}$$

So $F_{k+1}(c)$ is obviously a piecewise-linear function and the slope equals to

$$\beta\left(\frac{\beta^{k+1}-\beta}{1-\beta}-1\right)=\frac{\beta^{k+2}-\beta}{1-\beta}.$$

If $F_k(c)$ is continuous at c_0 and c_0 is not a fixed point of $F_k(c)$, then c_0 is a continuity point of $F_{k+1}(c)$. Therefore, if P is a discontinuity point of $F_{k+1}(c)$, then P is a discontinuity point or fixed point of $F_k(c)$. Hence, there exists $1\leqq m\leqq k$ such that P is a fixed point of $F_m(c)$. Conversely, let P be a primitive fixed point of $F_m(c)$ for some $1\leqq m\leqq k$. Then P is a discontinuity point of $F_{m+1}(c)$. By the following Lemma 5.2, we conclude that P is also a discontinuity point of $F_{k+1}(c)$. Similarly we get the same results for $G_n(c)$. □

Lemma 5.2. *Let P be a discontinuity point of $F_n(c)$. Then P is also a discontinuity point of $F_{n+1}(c)$.*

Proof. By definition, we have $F_{n+1}(P\pm)=\beta(F_n(P\pm)-P)+\delta_\pm$ where $\delta_\pm=0$ or 1. Assume that $F_n(P+)\neq F_n(P-)$ and $F_{n+1}(P+)=F_{n+1}(P-)$. Then we have

$$\beta(F_n(P+)-F_n(P-))=\delta_- - \delta_+ .$$

However $F_n(P+)\neq F_n(P-)$ implies $\delta_+\neq\delta_-$. Therefore we have

$$\beta|F_n(P+)-F_n(P-)|=1. \qquad \text{Hence} \quad |F_n(P+)-F_n(P-)|>1 .$$

This is a contradiction since $F_n(P\pm)\in[0, 1]$. □

Lemma 5.3. *For any $n\geqq 1$,*

$$F_n(0+)=\beta^{n-1},\ G_n(0+)=\beta^n,\ F_n(1-)=1-\beta^n, \text{ and } G_n(1-)=1-\beta^{n-1} .$$

Proof. We shall prove by an induction on n. Since $F_1(c)=1-\beta c$, it is trivial for $n=1$. Assume that $F_k(0+)=\beta^{k-1}$. By definition, $F_{k+1}(c)=\beta(F_k(c)-c)$ for small $c>0$. Then $F_{k+1}(0+)=\beta F_k(0+)=\beta^k$. Similarly we can get other formulas. □

The itinerary of x was defined in section 1 as follows.

$$I(x)=(A(x),\ A(f(x)),\ A(f^2(x)),\ \cdots) .$$

This may be regarded as the sequence of functions of c. So we write

$$I(x,\ c)=(A(x),\ A(f(x,\ c)),\ A(f^2(x,\ c)),\ \cdots) .$$

Definition 5.4.

$$I(x,\ c\pm)=(A(x),\ \lim_{\xi\to c\pm} A(f(x,\ \xi)),\ \lim_{\xi\to c\pm} A(f^2(x,\ \xi)),\ \cdots) .$$

Each limit certainly exists because of the piecewise-linearity. In particular,

$$I(0,\ c\pm)=(I^0,\ \lim_{\xi\to c\pm} A(F_1(\xi)),\ \lim_{\xi\to c\pm} A(F_2(\xi)),\ \cdots)$$

and

$$I(1,\ c\pm)=(I^0,\ \lim_{\xi\to c\pm} A(G_1(\xi)),\ \lim_{\xi\to c\pm} A(G_2(\xi)),\ \cdots) ,$$

where we define exceptionally that $A(1)=I^0$.

Lemma 5.5. *A discontinuity point of $G_n(c)$ (resp. $F_n(c)$) is a continuity point of $F_m(c)$ (resp. $G_m(c)$) for any $m\geqq 1$.*

Proof. Assume that ξ is a fixed point of both $G_n(c)$ and $F_m(c)$. Then we have $f^n(1)=\xi$ and $f^m(0)=\xi$. Therefore 0 is a periodic point and $f^n(f^{m-n}(0))=\xi$. Since f^n is an one to one map on $[0, 1]$, we obtain $f^{m-n}(0)=1$. This is a contradiction. □

The next lemma easily follows from the results in sections 2 and 3, and the proofs are omitted.

Lemma 5.6. (1) *In the singular case, $I(0)=I(1)$.*

(2) *In the periodic case $A(f^j(0))=A(f^j(1))$ for $j=1, 2, \cdots, N-1$ and $c\in [f^N(1), f^N(0)]$. Also $I(0)$ has period $N+1$ and $I(1)$ has same period without the first term.*

Lemma 5.7. (1) *If ξ is a primitive fixed point of $G_n(c)$, then $I(0, \xi-)=I(1, \xi-)$. And these have period $n+1$.*

(2) *If η is a primitive fixed point of $F_n(c)$, then $I(0, \eta+)=I(1, \eta+)$. And these have period $n+1$ without the first term.*

(3) *$F_m(\xi-)-G_m(\xi-)=F_m(\eta+)-G_m(\eta+)=\beta^{m-1}-\beta^m$ for any $m\geqq 1$.*

Proof. Let ξ be a primitive fixed point of $G_n(c)$. Then ξ is a continuity point of $G_j(c)$ for $j=1, 2, \cdots, n$ and of $F_j(c)$ for any $j\geqq 1$. Since $f^n(1)=\xi$, we can show that $\xi\in \mathrm{Int}\, f^j(I)$ for $j=1, 2, \cdots, n-1$ and $\xi\notin \mathrm{Int}\, f^n(I)$. Therefore this is the periodic case with period $n+1$. Obviously we have $\lim_{c\to\xi-} A(G_n(c))=I^1$ and $\lim_{c\to\xi-} A(G_{n+1}(c))=I^0$. Also we have $\lim_{c\to\xi-} G_{n+1+j}(c)=F_j(\xi)$ for any $j\geqq 1$. Thus $\lim_{c\to\xi-} A(G_{n+1+j}(c))=A(F_j(\xi))$. Similarly we obtain $\lim_{c\to\xi-} A(G_j(c))=A(G_j(\xi))$ for $j=1, 2, \cdots, n-1$ and $\lim_{c\to\xi-} A(F_j(c))=A(F_j(\xi))$ for any $j\geqq 1$. This means that $I(0, \xi-)=I(0, \xi)$. Hence we obtain that $I(0, \xi-)=I(1, \xi-)$ and these have period $n+1$ by Lemma 5.6. Since $F_m(\xi-)$ and $G_m(\xi-)$ have same address for any $m\geqq 1$, we can show inductively that $F_m(\xi-)-G_m(\xi-)=\beta^{m-1}-\beta^m$. Similarly we get the same results for a primitive fixed point of $F_n(c)$. □

The following theorem gives the distribution of the periodic cases.

Theorem 5.8. *For any reduced fraction $\frac{m}{n}\in(0, 1)$, there exists a closed interval $\Delta\left(\frac{m}{n}\right)=[\alpha, \xi]$ satisfying the following properties.*

(1) *α (resp. ξ) is a primitive fixed point of $G_{n-1}(c)$ (resp. $F_{n-1}(c)$).*

(2) *For any $c\in\Delta\left(\frac{m}{n}\right)$, the periodic case is valid with period n.*

(3) *For any $c\in \mathrm{Int}\,\Delta\left(\frac{m}{n}\right)$, $G_{n-1}(c)<c<F_{n-1}(c)$*

and

$$\sum_{j=1}^{n-1} \varepsilon(A(F_j(c)))=n-m\,.$$

(4) *Let* $\frac{q}{p}<\frac{s}{r}$ *be consecutive in the set of Farey fractions of some order. Then* $\lim_{c\to\xi+} A(F_j(c))=\lim_{c\to\gamma-} A(F_j(c))$ *for* $j=1, 2, \cdots, p+r-2$, *where*

$$\Delta\Big(\frac{q}{p}\Big)=[\alpha,\xi] \quad \textit{and} \quad \Delta\Big(\frac{s}{r}\Big)=[\gamma,\delta]\,.$$

Proof. We shall prove by Farey induction defined in section 4. Let $\Delta\Big(\frac{0}{1}\Big)=(-\infty, 0]$ and $\Delta\Big(\frac{1}{1}\Big)=[1, \infty)$. And define $\Delta\Big(\frac{1}{2}\Big)=\Big[\frac{\beta}{1+\beta}, \frac{1}{1+\beta}\Big]$ where $\frac{\beta}{1+\beta}\Big(\text{resp. } \frac{1}{1+\beta}\Big)$ is a unique fixed point of $G_1(c)$(resp. $F_1(c)$). Obviously (1), (2), (3) and (4) hold. Assume that (1), (2), (3) and (4) hold for both $\Delta\Big(\frac{q}{p}\Big)$ and $\Delta\Big(\frac{s}{r}\Big)$, where $\frac{q}{p}<\frac{s}{r}$ are consecutive in th the set of Farey fractions of some order. Now we must define a closed interval $\Delta\Big(\frac{q+s}{p+r}\Big)$ and show that (1), (2), (3) and (4) hold for $\Delta\Big(\frac{q+s}{p+r}\Big)$. Let $\Delta\Big(\frac{q}{p}\Big)=[\alpha, \xi]$ and $\Delta\Big(\frac{s}{r}\Big)=[\gamma, \delta]$ where $\alpha<\xi<\gamma<\delta$. By Lemma 5.7, $I(0, \xi+)$ has period p without the first term and $I(0, \gamma-)$ has period r. By the property (4) and Lemma 5.7, both $F_j(c)$ and $G_j(c)$ have no fixed points in (ξ, γ) for $j=1, 2, \cdots, p+r-2$. Hence both $F_{p+r-1}(c)$ and $G_{p+r-1}(c)$ are continuous in (ξ, γ). Using the periodic property,

$$\lim_{c\to\xi+} A(F_{p+r-1}(c))=\lim_{c\to\xi+} A(F_{r-1}(c))=\lim_{c\to\gamma-} A(F_{r-1}(c))=I^1$$

and

$$\lim_{c\to\gamma-} A(F_{p+r-1}(c))=\lim_{c\to\gamma-} A(F_{p-1}(c))=\lim_{c\to\xi+} A(F_{p-1}(c))=I^0.$$

Also $G_{p+r-1}(c)$ has same properties by Lemma 5.7. Thus we conclude that

$$\xi<G_{p+r-1}(\xi+)<F_{p+r-1}(\xi+) \quad \text{and} \quad G_{p+r-1}(\gamma-)<F_{p+r-1}(\gamma-)<\gamma\,.$$

Hence $F_{p+r-1}(c)$ (resp. $G_{p+r-1}(c)$) has a unique fixed point ζ (resp. λ) in (ξ, γ) where $\xi<\lambda<\zeta<\gamma$. Now we define $\Delta\Big(\frac{q+s}{p+r}\Big)=[\lambda, \zeta]$ to satisfy (1). For any $c\in\Delta\Big(\frac{q+s}{p+r}\Big)$, we can show that $c\in\operatorname{Int} f^j(I)$ for $j=1, 2, \cdots, p+r-2$ and $c\notin \operatorname{Int} f^{p+r-1}(I)$. So this is the periodic case with period $p+r$. Since ξ is a continuity point of $F_j(c)$ for $j=1, 2, \cdots, p-1$, we have

$$\sum_{j=1}^{p-2} \varepsilon(A(F_j(\xi+)))=p-q-1\,.$$

Similarly we have $\sum_{j=1}^{r-1} \varepsilon(A(F_j(\gamma-)))=r-s$. Hence, for $c\in\operatorname{Int}\Delta\Big(\frac{q+s}{p+r}\Big)$,

$$\sum_{j=1}^{p+r-1} \varepsilon(A(F_j(c)))=\lim_{c\to\xi+} \sum_{j=1}^{p+r-1} \varepsilon(A(F_j(c)))$$

$$=\sum_{j=1}^{p-2}\varepsilon(A(F_j(\xi+)))+\lim_{c\to\xi+}\varepsilon(A(F_{p-1}(c)))+\lim_{c\to\xi+}\varepsilon(A(F_p(c)))+\sum_{j=1}^{r-1}\varepsilon(A(F_j(\gamma-)))$$

$$=p+r-q-s\,.$$

Finally we consider $\Delta\left(\frac{q+s}{p+r}\right)$ and $\Delta\left(\frac{s}{r}\right)$, where $\frac{q+s}{p+r}<\frac{s}{r}$ are consecutive in the set of Farey fractions of some order. By Lemma 5.7, $I(0, \zeta+)$ has period $p+r$ without the first term. Therefore we obtain

$$\lim_{c\to\zeta+}A(F_j(c))=\lim_{c\to\gamma-}A(F_j(c))=\lim_{c\to\xi+}A(F_j(c)) \qquad \text{for } j=1, 2, \cdots, p+r-2$$

$$\lim_{c\to\zeta+}A(F_{p+r-1}(c))=I^0$$

$$\lim_{c\to\gamma-}A(F_{p+r-1}(c))=\lim_{c\to\gamma-}A(F_{p-1}(c))=\lim_{c\to\xi+}A(F_{p-1}(c))=I^0$$

$$\lim_{c\to\zeta+}A(F_{p+r}(c))=I^1$$

$$\lim_{c\to\gamma-}A(F_{p+r}(c))=\lim_{c\to\gamma-}A(F_p(c))=\lim_{c\to\xi+}A(F_p(c))=I^1\,.$$

Moreover we have

$$\lim_{c\to\zeta+}A(F_{p+r+j}(c))=\lim_{c\to\zeta+}A(F_j(c))$$

$$\lim_{c\to\gamma-}A(F_{p+r+j}(c))=\lim_{c\to\gamma-}A(F_{p+j}(c))=\lim_{c\to\xi+}A(F_{p+j}(c))=\lim_{c\to\xi+}A(F_j(c))$$

for $j=1, 2, \cdots, r-2$. Similarly we get the same results for $\Delta\left(\frac{q}{p}\right)$ and $\Delta\left(\frac{q+s}{p+r}\right)$. □

This proof shows that $\Delta\left(\frac{q}{p}\right)\cap\Delta\left(\frac{s}{r}\right)=\phi$ if $\frac{q}{p}\neq\frac{s}{r}$. Moreover we can compute the sum of $\Delta\left(\frac{q}{p}\right)$ over all reduced fractions in (0, 1).

Theorem 5.9.

$$\sum_{\substack{0<(q/p)<1\\(p,q)=1}}\left|\Delta\left(\frac{q}{p}\right)\right|=1$$

Proof. By Lemma 5.7, we have $F_{p-1}(c)-G_{p-1}(c)=\beta^{p-2}(1-\beta)$ for $c\in\Delta\left(\frac{q}{p}\right)$. Hence $\left|\Delta\left(\frac{q}{p}\right)\right|=\left(\frac{1-\beta}{\beta}\right)^2\frac{\beta^p}{1-\beta^p}$. For fixed $p\geqq 2$, there are $\phi(p)$ reduced fractions with denominator p in (0, 1), where $\phi(p)$ is the Euler's function. Therefore we have

$$\sum_{\substack{0<(q/p)<1\\(p,q)=1}}\left|\Delta\left(\frac{q}{p}\right)\right|=\left(\frac{1-\beta}{\beta}\right)^2\sum_{p=2}^{\infty}\phi(p)\frac{\beta^p}{1-\beta^p}\,.$$

This Lambert series is equal to 1. □

Let Σ be the remainder set; $(0, 1)-\bigcup_{\substack{0<(q/p)<1 \\ (p,q)=1}} \Delta\left(\frac{q}{p}\right)$. Σ is the set removed countable set from Cantor set and has null Lebesque measure. The following theorem gives the distribution of the singular cases.

Theorem 5.10. *The singular case is valid for any $c \in \Sigma$.*

Proof. We use the same notations as in the proof of Theorem 5.8. Then we have

$$G_j(c)<F_j(c)<c \quad \text{or} \quad c<G_j(c)<F_j(c)$$

for $c \in (\xi, \gamma)$ and $j=1, 2, \cdots, p+r-2$. Hence, for any $j \geqq 1$, we have

$$G_j(c)<F_j(c)<c \quad \text{or} \quad c<G_j(c)<F_j(c)$$

for any $c \in \Sigma$. This is the singular case. □

The coordinates of each $\Delta\left(\frac{q}{p}\right)$ will be computed in section 7.

6. Average firing rate

In this section we will define a number called the average firing rate, which is analogous to the rotation number of a homeomorphism of the circle.

First of all, we define this number in the periodic case. Throughout this section, fix the value of $\beta \in (0, 1)$.

Definition 6.1. The following value $\rho(x, c)$ we call the average firing rate.

$$\rho(x, c)=\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \varepsilon_j(x) \quad \text{where} \quad \varepsilon_j(x)=\varepsilon(A(f^{j-1}(x, c))) .$$

In the periodic case, by the periodicity, this limit certainly exists. And the next lemma easily follows from Theorem 2.1.

Lemma 6.2. *If $c \in \Delta\left(\frac{m}{n}\right)$, then we have $\rho(x, c)=\rho(0, c)$ for any $x \in [0, 1)$.*

Now we investigate the sign itinerary of 0. So we write

$$\rho(c)=\rho(0, c)=\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \varepsilon(A(F_j(c))) \quad \text{and} \quad \{\varepsilon_j\}=\{\varepsilon_j(0)\} .$$

Theorem 6.3. $\rho(c)=1-\frac{m}{n}$ *for any* $c \in \Delta\left(\frac{m}{n}\right)$.

Proof. By the property (3) of Theorem 5.8, we have

$$\sum_{j=1}^{n-1} \varepsilon(A(F_j(c)))=n-m .$$

Hence, by the periodicity,

$$\rho(c)=\frac{1}{n} \sum_{j=1}^{n} \varepsilon(A(F_j(c)))=\frac{n-m}{n} . \quad \square$$

Now we will show the existence of the limit $\rho(c)$ in the singular case. We need following lemmas.

Lemma 6.4. *Let* $\{\varepsilon_j\}$, $\{\eta_j\}$ *and* $\{\zeta_j\}$ *be the sign itineraries for* $c\in\Delta\left(\frac{q}{p}\right)$, $\Delta\left(\frac{s}{r}\right)$ *and* $\Delta\left(\frac{q+s}{p+r}\right)$ *respectively, where* $\frac{q}{p}<\frac{s}{r}$ *are consecutive in the set of Farey fractions of some order. Then* $\eta_j=\zeta_j$ *for* $j=1, 2, \cdots r$ *and* $\varepsilon_j=\zeta_{r+j}$ *for* $j=1, 2, \cdots, p$.

Proof. Let $\Delta\left(\frac{q}{p}\right)=[\alpha, \xi]$ and $\Delta\left(\frac{s}{r}\right)=[\gamma, \delta]$. Then $\{\varepsilon_j\}$ has period p and $I(0, \xi+)$ has same period without the first term. And they satisfy the following relations.

$$\varepsilon_j=\lim_{c\to\xi+}\varepsilon(A(F_{j-1}(c))) \qquad \text{for} \quad j=1, 2, \cdots, p-1,$$

$$\varepsilon_p=1, \quad \text{and} \quad \lim_{c\to\xi+}\varepsilon(A(F_{p-1}(c)))=0.$$

Similarly $\{\eta_j\}$ and $I(0, \gamma-)$ have period r, and

$$\eta_j=\lim_{c\to\gamma-}\varepsilon(A(F_{j-1}(c))) \qquad \text{for any} \quad j\geqq 1.$$

Also we have

$$\zeta_j=\lim_{c\to\xi+}\varepsilon(A(F_{j-1}(c))) \qquad \text{for} \quad j=1, 2, \cdots, p+r.$$

Hence

$$\zeta_j=\lim_{c\to\xi+}\varepsilon(A(F_{j-1}(c)))=\lim_{c\to r-}\varepsilon(A(F_{j-1}(c)))=\eta_j \qquad \text{for} \quad j=1, 2, \cdots, r,$$

$$\zeta_{r+j}=\lim_{c\to\xi+}\varepsilon(A(F_{r+j-1}(c)))=\eta_{r+j}=\eta_j=\varepsilon_j \qquad \text{for} \quad j=1, 2, \cdots, p-1,$$

and $\zeta_{r+p}=\varepsilon_p=1$. This completes the proof. $\square$

Lemma 6.5. *For any* $c\in\Delta\left(\frac{q}{p}\right)$, *we have*

$$\varepsilon_j=\mathbf{1}\left[p-q-1-(p-q)j+p\left[\left(1-\frac{q}{p}\right)j\right]\right] \qquad \textit{for} \quad j\geqq 1.$$

Proof. We shall prove by Farey induction. For any $j\geqq 1$, $\varepsilon_j=1$ for $c\in\Delta\left(\frac{0}{1}\right)$ and $\varepsilon_j=0$ for $c\in\Delta\left(\frac{1}{1}\right)$. Assume that it holds for $\Delta\left(\frac{q}{p}\right)$ and $\Delta\left(\frac{s}{r}\right)$. Using the same notations as in Lemma 6.4, we have

$$\zeta_j=\eta_j=\mathbf{1}[W(r, s, j)] \qquad \text{for} \quad j=1, 2, \cdots, r$$

where we define

$$W(r, s, j)=r-s-1-(r-s)j+r\left[\left(1-\frac{s}{r}\right)j\right].$$

Let $(r-s)j=ur+v$ where $0\leqq v<r$. Then $W(r, s, j)=r-s-1-v$. Also, using

the unimodular relation, we have

$$(p-q)j=pu+\frac{1}{r}(pv+j).$$

Then we have

$$W(p+r, q+s, j)=\left(1+\frac{p}{r}\right)W(r, s, j)+\frac{1}{r}(1-j+p).$$

If $W(r, s, j)\geqq 1$, then $W(p+r, q+s, j)\geqq\frac{1}{r}(1+2p)>0$, since $1\leqq j\leqq r$. If $W(r, s, j)\leqq -1$, then $W(p+r, q+s, j)\leqq -1$. Finally, if $W(r, s, j)=0$, then $W(p+r, q+s, j)=\frac{1}{r}(1-j+p)$. If $r>p+1$, then $j=p+1$, since $(r-s)(p+1)\equiv r-s-1 \bmod r$. So anyway, we have $W(p+r, q+s, j)\geqq 0$. Hence we have

$$\zeta_j=\mathbf{1}[W(p+r, q+s, j)] \qquad \text{for} \quad j=1, 2, \cdots, r.$$

Similarly we obtain the same formula for $j=r+1, \cdots, r+p$. $\square$

Lemma 6.6. *For any $c\in\Delta\left(\frac{q}{p}\right)$, we have*

$$\sum_{j=1}^{n}\varepsilon_j=\left[\left(1-\frac{q}{p}\right)n\right] \qquad \textit{for any} \quad n\geqq 1.$$

Proof. We use the notation $W(p, q, j)$ defined in the proof of Lemma 6.5. For any $n\geqq 1$, we have $0\leqq\left[\left(1-\frac{q}{p}\right)n\right]-\left[\left(1-\frac{q}{p}\right)(n-1)\right]\leqq 1$. If $\left[\left(1-\frac{q}{p}\right)n\right]=\left[\left(1-\frac{q}{p}\right)(n-1)\right]+1$, then

$$W(p, q, n)>p-q-1-(p-q)n+(p-q)(n-1)=-1.$$

Therefore $W(p, q, n)$ is not negative because it must be an integer. If $\left[\left(1-\frac{q}{p}\right)n\right]=\left[\left(1-\frac{q}{p}\right)(n-1)\right]$, then

$$W(p, q, n)<p-q-1-(p-q)n+(p-q)(n-1)=-1.$$

Hence we obtain

$$\left[\left(1-\frac{q}{p}\right)n\right]-\left[\left(1-\frac{q}{p}\right)(n-1)\right]=\mathbf{1}[W(p, q, n)]=\varepsilon_n. \quad \square$$

For any irrational number $\alpha\in(0, 1)$, we can choose two numbers $\frac{q_n}{p_n}$ and $\frac{s_n}{r_n}$ in the set of Farey fractions with order n so that $\frac{q_n}{p_n}$ and $\frac{s_n}{r_n}$ are the best lower and upper approximate values of α respectively. Thus we obtain the approximation sequences of α; $\left\{\frac{q_n}{p_n}, \frac{s_n}{r_n}\right\}$, which we call Farey approximation sequences of α.

Corresponding these Farey approximation sequences, there exists a unique parameter value $c \in \Sigma$ which lies between $\Delta\left(\frac{q_n}{p_n}\right)$ and $\Delta\left(\frac{s_n}{r_n}\right)$ for any $n \geqq 1$.

Conversely, for a given $c \in \Sigma$, a number which is approximated by corresponding reduced fractions must be irrational. So there exists an one to one correspondence between Σ and irrational numbers in $(0, 1)$.

The following theorem shows that this irrational number determined by a given $c \in \Sigma$ is equal to the average firing rate in the singular case. So this guarantees the existence of the limit $\rho(c)$ in the singular case.

Theorem 6.7. *For any $c \in \Sigma$, we have $\sum_{j=1}^{m} \varepsilon_j = [\alpha m]$ for any $m \geqq 1$, where α is an irrational number corresponding to c.*

Proof. Let $\left\{\frac{q_n}{p_n}, \frac{s_n}{r_n}\right\}$ be Farey approximation sequences. And let $\{\eta_j^n\}$ and $\{\zeta_j^n\}$ be the sign itineraries for $c \in \Delta\left(\frac{q_n}{p_n}\right)$ and $\Delta\left(\frac{s_n}{r_n}\right)$ respectively. Assume that $\frac{q_n}{p_n} < \alpha < \frac{q_n+s_n}{p_n+r_n}$. Then $\left|\alpha - \frac{q_n}{p_n}\right| < \frac{1}{p_n(p_n+r_n)}$. Therefore $\sum_{j=1}^{m} \varepsilon_j = \sum_{j=1}^{m} \eta_j^n = \left[\frac{q_n}{p_n} m\right] = [\alpha m]$ for $m = 1, 2, \cdots, p_n - 1$.

Similarly we obtain the same formulas for $m = 1, 2, \cdots, r_n - 1$ in the case where $\frac{q_n+s_n}{p_n+r_n} < \alpha < \frac{s_n}{r_n}$. Hence $\sum_{j=1}^{m} \varepsilon_j = [\alpha m]$ for $m = 1, 2, \cdots, \min(p_n, r_n) - 1$. So we obtain the desired results by passing n to infinity. □

The next theorems easily follow from Theorem 6.3 and Theorem 6.7.

Theorem 6.8. *If $\rho(c) \in \boldsymbol{Q}$, then the periodic case is valid. If $\rho(c) \notin \boldsymbol{Q}$, then the singular case is valid.*

Theorem 6.9. *$\rho(c)$ is a continuous monotone decreasing function of c.*

In the periodic case, $\rho(x, c)$ is independent of x by Lemma 6.2. And this is true for the singular case by the following theorem.

Theorem 6.10. *In the singular case, for any $(x, c) \in [0, 1) \times (0, 1)$, there exists the limit $\rho(x, c)$ in the definition* 6.1. *Moreover we have $\rho(x, c) = \rho(c)$.*

Proof. First of all, assume that $x \in \mathrm{cl}\, J_n = [f^n(1), f^n(0)]$ for some $n \geqq 1$. Then, by Lemma 3.1, we have $A(f^k(x)) = A(f^{k+n}(0))$ for any $k \geqq 0$. Hence $\rho(x, c) = \rho(c)$. Next assume that $x = c_{-n}$ for some $n \geqq 0$. Since $A(f^k(x)) = A(c_{k-n}) = A(f^{k-n-1}(0))$ for $k \geqq 0$, we have also $\rho(x, c) = \rho(c)$. Finally assume that $x \in \Lambda - \{c_m\}_{m=-\infty}^{\infty}$. For any $k \geqq 1$, there exists a positive integer n_k such that $A(f^j(x)) = A(f^{n_k+j-1}(0))$ for $j = 0, 1, \cdots, k-1$. Hence, by Theorem 6.7,

$$\frac{1}{k} \sum_{j=1}^{k} \varepsilon(A(f^{j-1}(x))) = \frac{1}{k} \sum_{j=1}^{k} \varepsilon_{n_k+j} = \frac{1}{k}([\alpha(n_k+k)] - [\alpha n_k]).$$

Therefore we have

$$\alpha-\frac{1}{k}\leqq\frac{1}{k}\sum_{j=1}^{k}\varepsilon(A(f^{j-1}(x)))\leqq\alpha+\frac{1}{k}.$$

This completes the proof. □

7. Conjugacy problem

We say that f is topologically conjugate to g if $g=h\circ f\circ h^{-1}$ for some homeomorphism h. If $f(x, \beta, c)$ is topologically conjugate to $f(x, \xi, \lambda)$, then obviously, $\rho(\beta, c)=\rho(\xi, \lambda)$ where $\rho(\beta, c)$ is the average firing rate of $f(0, \beta, c)$. Conversely assume that $\rho(\beta, c)=\rho(\xi, \lambda)$. Then can we conclude that $f(x, \beta, c)$ is topologically conjugate to $f(x, \xi, \lambda)$? The following theorem answers this question in the singular case.

Theorem 7.1. *If $\rho(\beta, c)=\rho(\xi, \lambda)\notin \boldsymbol{Q}$, then $f(x, \beta, c)$ is topologically conjugate to $f(x, \xi, \lambda)$.*

Proof. We shall constract a homeomorphism $h(x)$. Assume that $\beta<\xi$. First of all, we define $h(x)$ at $x=f^n(1, \beta, c)$ for any $n\geqq 1$ as follows.

$$h(f^n(1))=\frac{1-\xi}{\xi}\sum_{\substack{m\geqq 1\\ f^m(0)<f^n(0)}}\xi^m$$

Next define $h(x)$ on $J_n=[f^n(1), f^n(0))$ as follows.

$$h(x)=\frac{\xi^{n-1}(1-\xi)}{\beta^{n-1}(1-\beta)}(x-f^n(1))+h(f^n(1)) \qquad \text{for} \quad x\in J_n\,.$$

Finally extend $h(x)$ continuously for all $x\in[0, 1]$. Then $h(x)$ is a continuous strictly monotone increasing function. Also we have $h(0)=0$ and $h(1)=\frac{1-\xi}{\xi}\sum_{m\geqq 1}\xi^m=1$. Let $g(x)=h\circ f\circ h^{-1}(x)$. Obviously $g(x)$ has a unique discontinuity point $h(c)$. Assume that $c<f^p(1)<f^q(1)$. Then $h(c)<h\circ f^p(1)<h\circ f^q(1)$. Hence

$$\frac{g\circ h\circ f^q(1)-g\circ h\circ f^p(1)}{h\circ f^q(1)-h\circ f^p(1)}=\frac{h\circ f^{q+1}(1)-h\circ f^{p+1}(1)}{h\circ f^q(1)-h\circ f^p(1)}=\frac{\sum\limits_{\substack{k\geqq 1\\ f^{p+1}(0)<f^k(0)<f^{q+1}(0)}}\xi^k}{\sum\limits_{\substack{k\geqq 1\\ f^p(0)<f^k(0)<f^q(0)}}\xi^k}=\xi$$

since f is homeomorphism on $[f^p(1), f^q(1)]$. Similarly we obtain the same result in the case where $f^p(1)<f^q(1)<c$.

Next, for $x\in\operatorname{Int} J_p$,

$$\frac{g\circ h(x)-g\circ h\circ f^p(1)}{h(x)-h\circ f^p(1)}=\frac{h\circ f(x)-h\circ f^{p+1}(1)}{h(x)-h\circ f^p(1)}=\frac{\xi(f(x)-f^{p+1}(1))}{\beta(x-f^p(1))}=\xi\,.$$

By these properties, we can conclude that $g(x)=f(x, \xi, h(c))$. Therefore we have $\rho(\xi, \lambda)=\rho(\xi, h(c))\notin \boldsymbol{Q}$. Now the parameter value c at which the average firing rate is irrational is determined uniquely for a given ξ. Hence $\lambda=h(c)$. This completes the proof. □

In this theorem, we have

$$\lambda=h(c)=\frac{1-\xi}{\xi}\sum_{\substack{m\geqq 1\\ f^{m}(0)<c}}\xi^{m}.$$

Hence we have

$$\lambda=\frac{1-\xi}{\xi}\sum_{m=1}^{\infty}(1-\varepsilon_{m+1})\xi^{m}=1-\frac{1-\xi}{\xi^{2}}\sum_{m=2}^{\infty}\varepsilon_{m}\xi^{m}.$$

By Theorem 6.7 the next theorem easily follows.

Theorem 7.2. *The singular case with the average firing rate* $\alpha\notin\boldsymbol{Q}$ *is valid if and only if* $c=1-\left(\frac{1-\beta}{\beta}\right)^{2}\sum_{m=2}^{\infty}[\alpha m]\beta^{m}$.

Now we can compute the coordinates of each closed interval $\Delta\left(\frac{q}{p}\right)=[\alpha, \xi]$ defined in section 6. Fix the value of $\beta\in(0, 1)$. By Theorem 6.9, we have

$$\alpha=1-\lim_{n\to\infty}\left(\frac{1-\beta}{\beta}\right)^{2}\sum_{k=2}^{\infty}[r_{n}k]\beta^{k},$$

$$\xi=1-\lim_{n\to\infty}\left(\frac{1-\beta}{\beta}\right)^{2}\sum_{k=2}^{\infty}[s_{n}k]\beta^{k}$$

where $\{r_{n}\}$ and $\{s_{n}\}$ are irrational upper and lower approximation sequences of $1-\frac{q}{p}$ respectively. Computing these limits, we obtain the following theorem

Theorem 7.3. *Let* $\Delta\left(\frac{q}{p}\right)=[\alpha, \xi]$. *Then*

$$\alpha=1-(p-q)\frac{\beta^{p-1}(1-\beta)}{1-\beta^{p}}-\frac{1}{1-\beta^{p}}\left(\frac{1-\beta}{\beta}\right)^{2}\sum_{j=1}^{p}\left[\left(1-\frac{q}{p}\right)j\right]\beta^{j},$$

$$\xi=\alpha+\left(\frac{1-\beta}{\beta}\right)^{2}\frac{\beta^{p}}{1-\beta^{p}}.$$

We say that f is topologically semi-conjugate to g if $h\circ f=g\circ h$ for some continuous monotone onto map h. We shall prove that $f(x, \beta, c)$ in the singular case is topologically semi-conjugate to R_{α}, where R_{α} is a rigid rotation on the circle and α is not only the rotation number of R_{α} but also the average firing rate of $f(x, \beta, c)$. Moreover, for a given $f(x, \beta, c)$ in the singular case, the rigid rotation R_{α} which is topologically semi-conjugate to $f(x, \beta, c)$ is uniquely determined. The next lemma was proved in Theorem 2.1.

Lemma 7.4. *In the periodic case with period* $N+1$, $f(x, \beta, c)$ *is regarded as a permutation* $\pi=\sigma^{N+1-Q}$ *on the subintervals*: $I_{1}, I_{2}, \cdots, I_{N+1}$ *where* $\sigma=\begin{pmatrix}1 & 2 & \cdots & N+1\\ 2 & 3 & \cdots & 1\end{pmatrix}$ *and* Q *is the number of* $1\leqq j\leqq N+1$ *such that* $I_{j}\subset I^{0}$.

Lemma 7.5. *For* $c\in\Delta\left(\frac{q}{p}\right)$, *the following correspondence*

$$\zeta: f^j(0) \longleftrightarrow \left\{\left(1-\frac{q}{p}\right)j\right\} \qquad \textit{for} \quad j=0, 1, \cdots, p-1$$

is an order isomorphism.

Proof. By Theorem 5.8, for $c \in \Delta\left(\frac{q}{p}\right)$, the periodic case with period p is valid and

$$\sum_{j=1}^{p-1} \varepsilon(A(F_j(c))) = p-q\,.$$

Therefore the number of $0 \leqq j \leqq p-1$ such that $f^j(0) \in I^0$ is q. So, by Lemma 7.4, f is regarded as a permutation $\pi = \sigma^{p-q}$. This completes the proof. □

Lemma 7.6. *For $c \in \Sigma$, the following correspondence*

$$\zeta: f^j(0) \longleftrightarrow \{\alpha j\} \qquad \textit{for} \quad j \geqq 0$$

is an order isomorphism, where α is the average firing rate of $f(x, \beta, c)$.

Proof. Let $\left\{\frac{q_n}{p_n}, \frac{s_n}{r_n}\right\}$ be Farey approximation sequences of α. Assume that $\frac{q_n}{p_n} < \alpha < \frac{q_n+s_n}{p_n+r_n}$. Then

$$\left|\alpha j - \frac{q_n}{p_n} j\right| < \frac{j}{p_n(p_n+r_n)} < \frac{1}{2p_n}$$

for $p=0, 1, \cdots, \left[\frac{p_n}{2}\right]$. So,

$$\zeta_1: \{\alpha j\} \longleftrightarrow \left\{\frac{q_n}{p_n} j\right\} \qquad \text{for} \quad j=0, 1, \cdots, \left[\frac{p_n}{2}\right]$$

is an order isomorphism. Now note that $\frac{q_n}{p_n} < \frac{s_n}{r_n}$ are consecutive in the set of Farey fractions of order n. Let $\Delta\left(\frac{q_n}{p_n}\right) = [\alpha_n, \xi_n]$ and $\Delta\left(\frac{s_n}{r_n}\right) = [\gamma_n, \delta_n]$. Then, for any $c_1, c_2 \in (\alpha_n, \delta_n)$,

$$\zeta_2: F_j(c_1) \longleftrightarrow F_j(c_2) \qquad \text{for} \quad j=0, 1, \cdots, \min(p_n, r_n)-1$$

is an order isomorphism. So, for $c \in \Sigma$, we obtain that

$$\zeta_3: f^j(0) \longleftrightarrow \{\alpha j\} \qquad \text{for} \quad j=0, 1, \cdots, \min\left(\left[\frac{p_n}{2}\right], \left[\frac{r_n}{2}\right]\right)$$

is also an order isomorphism. Similarly we obtain the same result in the case where $\frac{q_n+s_n}{p_n+r_n} < \alpha < \frac{s_n}{r_n}$. We obtain the desired result by passing n to infinity. □

Theorem 7.7. *For $c\in\Sigma$, $f(x, \beta, c)$ is topologically semi-conjugate to R_α, where α is the average firing rate of $f(x, \beta, c)$. Moreover the rigid rotation R_α is uniquely determined.*

Proof. We shall constract a continuous monotone increasing onto map $h(x)$ such that $h\circ f=R_\alpha\circ h$. Let $J_n=[f^n(1), f^n(0))$. First of all, define $h(x)=\{\alpha n\}$ for any $x\in J_n$. So $h(x)$ is constant on each J_n. Next, for $x\in\Lambda$, define

$$h(x+)=\lim_{j\to\infty} h(J_{n_j})$$

where $\{J_{n_j}\}$ is a monotone decreasing sequence of intervals which converges to x. Then $\{h(J_{n_j})\}$ is also monotone decreasing by Lemma 7.6 and the limit certainly exists. Similarly we define $h(x-)$ by a monotone increasing sequence of intervals which converges to x. Since α is an irrational number, $\{\alpha n\}$ is dense in $[0, 1]$. Hence we have $h(x+)=h(x-)$ for any $x\in\Lambda$. So, for $x\in\Lambda$, define

$$h(x)=h(x+)=h(x-).$$

Then $h(0)=h(0+)=0$, $h(1)=h(1-)=1$ and $h(x)$ is a continuous monotone increasing onto map on $[0, 1]$.

Now $h\circ f(J_n)=h(J_{n+1})=\{\alpha(n+1)\}$ and $R_\alpha\circ h(J_n)=R_\alpha(\{\alpha n\})=\{\alpha(n+1)\}$ since $R_\alpha^n(0)=\{\alpha n\}$. So we have $h\circ f=R_\alpha\circ h$ on each J_n. Also there uniquely exists $x_0\in\Lambda$ such that $h(x_0)=1-\alpha$ since $1-\alpha$ is never equal to $\{\alpha m\}$ for $m\geqq 1$. Then we have $\lim_{x\to x_0+} R_\alpha\circ h(x)=0$ and $\lim_{x\to x_0-} R_\alpha\circ h(x)=1$. Let $\{J_{n_k}\}<\{J_{m_j}\}$ be monotone increasing and decreasing sequences of intervals which converge to x_0 respectively. Then we have

$$\lim_{j\to\infty} h\circ f(J_{m_j})=0 \quad\text{and}\quad \lim_{k\to\infty} h\circ f(J_{n_k})=1.$$

Therefore $h\circ f(x)$ has a unique discontinuity point x_0. Hence $x_0=c$. Finally by the continuity property, for any $x\in[0, 1]$, we have $h\circ f(x)=R_\alpha\circ h(x)$.

Assume that $H\circ f(x)=R_\beta\circ H(x)$ for some continuous monotone increasing onto map $H(x)$. Then $H\circ f(1)=R_\beta(1)=\beta$ and $H\circ f(0)=R_\beta(0)=\beta$. Hence we have $H(x)=\beta$ for $x\in J_1$. Also we can show that $H(J_n)=\{\beta n\}$ for $n\geqq 1$. So β must be an irrational number since $H(x)$ is continuous. Therefore we obtain $H(c)=1-\beta$. And $c<J_p$ is equivalent to $\{\beta(p+1)\}<\beta$. Hence $\beta=\alpha$. This completes the proof. □

Remark that the function $h(x)$ gives an one to one correspondence between Cantor attractor Λ and $I^*=I-\{\alpha n\}_{n=1}^\infty$.

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