J. Math. Kyoto Univ. (JMKYAZ)
22-1 (1982) 115-130

The Onsager-Machlup function for diffusion processes

By

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(Received Nov. 28, 1980)

0. Introduction

Let M be a smooth d-dimensional connected Riemannian manifold and Δ_M be the Laplace-Beltrami perator. Let $(X_t, P_p)_{p\in M}$ be the minimal diffusion processes on M generated by the operator $\frac{1}{2}\Delta_M+b$ where b is a smooth vector field on M. For a given smooth curve $\phi=(\phi_t)_{0\leq t\leq T}$ on M starting at $\phi(0)=q$, the sojourn probability around ϕ up to time T is defined by

$$\mu_\varepsilon^q(\phi)=P_q\{\rho(X_t, \phi_t)<\varepsilon, \quad \text{for all} \quad t\in[0, T]\} \tag{0.1}$$

where ρ is the Riemannian distance.

Main aim of the present paper is to obtain an asymptotic formula for the sojourn probability as $\varepsilon\downarrow 0$ and the result we obtain is the following.

Theorem.

$$\mu_\varepsilon^q(\phi)=f_1(0)\int_D f_1(x)dx \exp\left\{-\frac{\lambda_1 T}{\varepsilon^2}+\int_0^T L(\phi_t, \dot{\phi}_t)dt+o(1)\right\} \tag{0.2}$$

as $\varepsilon\downarrow 0$, *where* L *is a function on the tangent bundle* TM *defined by*

$$L(p, v)=-\frac{1}{2}|v-b(p)|_p^2-\frac{1}{2}\operatorname{div} b(p)+\frac{1}{12}R(p). \tag{0.3}$$

Here the notion and notations involved are as follows. Let $\{\lambda_k, f_k\}$ *be the eigensystem for the eigenvalue problem*

$$\frac{1}{2}\Delta_{R^d}f+\lambda f=0 \quad in \quad D=\{x\in R^d: |x|<1\}$$

$$f=0 \quad on \quad \partial D=\{x\in R^d; |x|=1\}$$

Thus λ_1 *and* f_1 *in* (0.2) *is the minimal eigenvalue and the corresponding normalized eigenfunction of the above eigenvalue problem.* $|\ |_p$ *denotes the Riemannian norm in the tangent space* $T_p(M)$ *at* p, $\operatorname{div}(b)(p)$ *is the divergence of* b *at* p *and* $R(p)$ *is the scalar curvature at* p.

This problem is related to some problem in physics such as path-integral formulations and the most probable paths of diffusion processes (c.f. [1], [2], [4], [7]): the function L of (0.3), called the Onsager-Machlup function, is regarded as Lagrangian for the most probable paths. In a probabilistic motivation, S. Watanabe considered the above asymptotic formula to obtain a probabilistic characterisation of the symmetry (the reversibility) of the diffusion processes (c.f. [3]). Also, Stratonovich ([9]) considered a similar problem and introduced a notion of probability functional of diffusion processes. The result in the form of the above theorem was conjectured for a few years and verified to hold in Einstein spaces by Y. Takahashi through probabilistic techniques such as Girsanov's formula and stochastic Stokes' theorem. Recently his idea was further extended by S. Watanabe to cover the general case (c.f. Takahashi-Watanabe [10]).

In this paper, we obtain the above theorem by a purely analytical approach: we first identify the probability $\mu_\varepsilon^q(\phi)$ with the solutions of some heat equations with small parameter ε and then carry out their asymptotic expansions.

The authors express their hearty thanks to Professor S. Watanabe for his kind advices.

1. Preliminaries

In this section, we shall make some preparations from the Riemannian geometry which will be needed in later discussions.

First of all, the notion of normal coordinates will play a fundamental role. As usual, a normal coordinate with center $q \in M$ is determined by choosing an orthonormal basis $(e_1, e_2, \cdots, e_d)$ in the tangent space $T_q(M)$: for p sufficiently close to q, its normal coordinate $(x^1, x^2, \cdots, x^d)$ is defined by

$$p = \exp(q, x^k e_k)$$

Here $\exp(q, X)$, $X \in T_q(M)$, stands for the exponential map, i.e. $t \to \exp(q, tX)$ is the geodesic $c(t)$ such that $c(0)=q$ and $\dot{c}(0)=X$. The components of the metric tensor, its inverse, the Schwarz-Christoffel symbols and the Riemannian curvature tensor in the normal coordinate are denoted by $g_{ij}(p)$, $g^{ij}(p)$, $\Gamma_{ij}^k(p)$, $R_{ijkl}(p)$ respectively.

Then we have the following fundamental lemma. (c.f. [8])

Lemma 1.1. (E. Cartan)

$$g_{ij}(p) = \delta_{ij} + \frac{1}{3} R_{ikjl}(q) x^k x^l + 0(|x|^3) \tag{1.1}$$

$$\Gamma_{ij}^k(p) = \frac{1}{3} \{R_{iukj}(q) x^u + R_{kiuj}(q) x^u\} + 0(|x|^2) \tag{1.2}$$

For later use, we shall introduce a system of normal coordinates along a curve ϕ. Let ϕ be a smooth curve and choose an orthonormal basis $(e_1, e_2, \cdots, e_d)$ in $T_{\phi(0)}M$. Define a diffeomorphism Φ between some neighborhood U in $[0, T] \times R^d$ of the curve $t \to (t, 0)$ and some neighborhood V in $[0, T] \times M$ of the curve $t \to (t, \phi(t))$ by

$$\Phi(t, (x^1, x^2, \cdots, x^d))=(t, \exp(\phi(t), x^k e_k(t))$$

where $e_k(t)$ $(e_k(0)=e_k)$ is obtained as the parallel translate of e_k along the curve ϕ. It is clear that $x=(x^1, x^2, \cdots, x^d)$ is a normal coordinate of $p=\Phi(t, (x^1, x^2, \cdots, x^d))$ with center $q=\phi(t)$ for each fixed $t\in[0, T]$. The components of the vector field b, the metric tensor, Christoffel symbol, etc, in this normal coordinate $\Phi(t, \cdot)$ for each fixed $t\in[0, T]$, are denoted by $b^i(t, x)$, $g_{ij}(t, x)\Gamma^k_{ij}(t, x)$, etc. For a differential operator ∂ on V, we denote by $\tilde{\partial}$ the differential operator on U transformed by the above diffeomorphism:

$$\tilde{\partial}f(t, x)=\partial(f\circ\Phi^{-1})(\Phi(t, x)).$$

We shall calculate the operotors $\tilde{b}$, $\tilde{\Delta}_M$ and $\tilde{\partial}/\partial t$ in the following lemma.

Lemma 1.2.

$$\tilde{b}=b^i(t, x)\frac{\partial}{\partial x^i},$$

$$\tilde{\Delta}_M=g^{ij}(t, x)\frac{\partial^2}{\partial x^i\partial x^j}-g^{ij}(t, x)\Gamma^k_{ij}(t, x)\frac{\partial}{\partial x^k},$$

$$\frac{\tilde{\partial}}{\partial t}=\frac{\partial}{\partial t}-\{\dot{\phi}^i(t)+\varepsilon^i(t, x)\}\frac{\partial}{\partial x^i}$$

where $\dot{\phi}^i(t)=\lim_{u\to t}\frac{\phi^i(u)-\phi^i(t)}{u-t}$ *(*$\phi^i(u)$ *is the i-th component of* $\phi(u)$ *in the local coordinate* $\Phi(t, \cdot)$*.) and* $\varepsilon^i(t, x)$ *is a smooth function satisfying*

(1.3) $$\max_{0\le t\le T}|\varepsilon^i(t, x)|=0(|x|^2) \quad \textit{and}$$

(1.4) $$\max_{0\le t\le T}\left|\frac{\partial}{\partial x^k}\varepsilon^i(t, x)\right|=0(|x|) \textit{ for any } i \textit{ and } k.$$

Proof. Define a function $x^k(t, p)$ $(t\in[0, T], p\in M)$ by an equation

(1.5) $$\exp(\phi(t), x^k(t, p)e_k(t))=p.$$

Then we have the i-th component of $\tilde{b}$ in the base $\frac{\partial}{\partial x^k}$

$$=\tilde{b}(x^i)|_{t=t_0, x=x_0} \quad (x^i\text{: } i\text{-th coordinate function in } R^d)$$

$$=b(x^i(t, p))|_{t=t_0, p=p_0} \quad (p_0=\exp(\phi(t), x_0^k e_k(t_0)))$$

$$=b^i(t_0, x_0) \quad \text{(by the definition).}$$

Here it is evident that the $\frac{\partial}{\partial t}$-component of $\tilde{b}$ is identically zero. The proof for $\tilde{\Delta}_M$ is similar to the above. Next set

$$\frac{\tilde{\partial}}{\partial t}=c(t, x)\frac{\partial}{\partial t}+d^i(t, x)\frac{\partial}{\partial x^i}.$$

Then we see easily

$$c(t, x)=\frac{\tilde{\partial}}{\partial t}(t)=\frac{\partial}{\partial t}(t)=1\,.$$

Moreover note

$$(1.6)\qquad d^{i}(t, x)=\frac{\partial}{\partial t}(x^{i})|_{t=t_0}$$

$$=\frac{\partial}{\partial t}(x^{i}(t, p))|_{t=t_0}\ (p=\exp(\phi(t_0),\ x^{k}e_{k}(t_0)))\,.$$

In the normal coordinate $\Phi(t_0, \cdot)$, a geodesic $(x^{i}(t))$ from (α^{i}) with a velocity (β^{i}) satisfies the following ordinary differential equations

$$(1.7)\qquad \frac{dx^{i}(u)}{du}=y^{i}(u)$$

$$\frac{dy^{i}(u)}{du}=-\Gamma^{i}_{kl}(x(u))y^{k}(u)y^{l}(u)$$

$$x^{i}(0)=\alpha^{i},\ y^{i}(0)=\beta^{i}\,.$$

We denote by $x^{i}(u, \alpha, \beta)$ and $y^{i}(u, \alpha, \beta)$ the above solutions. Then the equation (1.5) is equivalent to the following

$$(1.8)\qquad p^{i}=x^{i}(1,\ \phi(t),\ x^{k}(t, p)e_{k}(t))\,.$$

On the other hand, $e_{k}(t)$ satisfies in the normal coordinate $\Phi(t_0, \cdot)$

$$\frac{de^{i}_{k}(t)}{dt}=-\Gamma^{i}_{lm}(\phi(t))e^{l}_{k}(t)\frac{d\phi^{m}(t)}{dt}$$

$$e^{i}_{k}(t_0)=\delta^{i}_{k}$$

Since $\Gamma^{i}_{lm}(\phi(t_0))=0$, we have $\frac{de^{i}_{k}(t)}{dt}|_{t=t_0}=0$. Hence noting also $e^{i}_{k}(t_0)=\delta^{i}_{k}$, $\phi^{i}(t_0)=0$ and $x^{k}(t_0, p)=x^{k}$, we get

$$(1.9)\qquad 0=\frac{\partial x^{i}}{\partial\alpha^{j}}(1, 0, (x^{k}))\dot{\phi}^{j}(t_0)+\frac{\partial x^{i}}{\partial\beta^{j}}(1, 0, (x^{k}))\dot{x}^{j}(t_0, p)$$

by differentiating both sides of (1.8) by t and setting $t=t_0$. Since $\Gamma^{i}_{kl}(\phi(t_0))=0$, it is easy to deduce from (1.7) that

$$(1.10)\qquad \frac{\partial x^{i}}{\partial\alpha_{j}}(1, 0, (x^{k}))=\delta^{i}_{j}+0(|x|^{2})$$

$$\frac{\partial x^{i}}{\partial\beta^{j}}(1, 0, (x^{k}))=\delta^{i}_{j}+0(|x|^{2})$$

Consequently

$$d^{i}(t_0, x)=-\dot{\phi}^{i}(t_0)+0(|x|^{2})\,,$$

which implies (1.1). we can see by tracing the above calculation carefully that (1.2) is also valid.

2. A reduction of the theorem

We now consider a space-time process (t, X_t). Its generator is $\frac{\partial}{\partial t}+\frac{1}{2}\Delta_M+b$. Let $\sigma=\inf\{t\geqq 0: (t, X_t)\notin V\}$ and $(t\wedge\sigma, \tilde{X}_{t\wedge\sigma})=\Phi^{-1}(t\wedge\sigma, X_{t\wedge\sigma})$. Then by Lemma 1.2 the local generator of $\tilde{X}_t$ becomes

$$\frac{\partial}{\partial t}+\frac{1}{2}g^{ij}(t, x)\frac{\partial^2}{\partial x^i\partial x^j}+\hat{b}^i(t, x)\frac{\partial}{\partial x^i} \qquad \text{where}$$

$$\hat{b}^k(t, x)=b^k(t, x)-\frac{1}{2}g^{ij}(t, x)\Gamma^k_{ij}(t, x)-\dot{\phi}^k(t)+\varepsilon^i(t, x) \tag{2.1}$$

Let $\Psi(x)=\exp(\phi(0), x^k e_k(0))$ and $u_\varepsilon(x)=P_{\psi(x)}\{\rho(X_t, \phi_t)\leqq\varepsilon$, for any $t\in[0, T]\}$. Then we have by a property of normal coordinate

$$u_\varepsilon(x)=\tilde{P}_{0,x}\{\sup_{0\leq t\leq T}|X_t|\leqq\varepsilon\}.$$

where $\tilde{P}_{0,x}$ is the distribution of $\tilde{X}_t$ in $C([0, T]\times U)$ starting from x at time 0.

Let $u_0^\varepsilon(t, x)$ be the solution of the following initial boundary problem:

$$\frac{\partial u_0^\varepsilon}{\partial t}=\left\{\frac{1}{2}g^{ij}(T-t, x)\frac{\partial^2}{\partial x^i\partial x^j}+\hat{b}^i(T-t, x)\frac{\partial}{\partial x^i}\right\}u_0^\varepsilon \quad \text{on} \quad [0, T]\times\{|x|\leqq\varepsilon\}$$

$$u_0^\varepsilon(t, \cdot)|_{\partial D}=0,\ u_0^\varepsilon(0, x)=1 \quad \text{for} \quad x\in\{|x|<\varepsilon\}..$$

Then, as is well known, the hitting probability $u_\varepsilon(x)$ coincides with $u_0^\varepsilon(T, x)$.

To prove the theorem, it is therefore sufficient to see the convergence of $\exp\left(\frac{\lambda_1 T}{\varepsilon^2}\right)u_0^\varepsilon(T, 0)$ to $f_1(0)\int_D f_1(x)dx\ \exp\left(\int_0^T L(\phi_t, \dot{\phi}_t)dt\right)$ as $\varepsilon\downarrow 0$ where L is given by (0.3). The following transformation of u_0^ε makes it possible to remove the singularity of the drift coefficients. Namely set,

$$u_1^\varepsilon(t, x)=u_0^\varepsilon(t, \varepsilon x)\exp\left\{\frac{\lambda_1 t}{\varepsilon^2}+\varepsilon\sum_{k=1}^d\hat{b}^k(T-t, 0)x^k\right\}$$

Then u_1^ε is a unique solution of an equation

$$\begin{aligned}\frac{\partial u_0^\varepsilon}{\partial t}=&\frac{1}{2\varepsilon^2}g^{ij}(T-t, \varepsilon x)\frac{\partial^2 u_1^\varepsilon}{\partial x^i\partial x^j}\\&+\frac{1}{\varepsilon}\{\hat{b}^i(T-t, \varepsilon x)-g^{ij}(T-t, \varepsilon x)\hat{b}^j(T-t, 0)\}\frac{\partial u_1^\varepsilon}{\partial x^i}\\&+\left\{\frac{1}{2}g^{ij}(T-t, \varepsilon x)\hat{b}^i(T-t, 0)\hat{b}^j(T-t, 0)-\delta_{ij}\hat{b}^i(T-t, 0)\hat{b}^j(T-t, \varepsilon x)\right.\\&\left.-\varepsilon\sum_{k=1}^d\frac{\partial b^k}{\partial t}(T-t, 0)x^k+\frac{\lambda_1}{\varepsilon^2}\right\}u_1^\varepsilon \qquad \text{on} \quad [0, T]\times D\end{aligned} \tag{2.2}$$

$$u_1^\varepsilon|_{\partial D}=0, \qquad u_1^\varepsilon(0, x)=\exp\{\varepsilon\sum_{k=1}^d\hat{b}^k(T, 0)x^k\} \qquad \text{on} \quad D.$$

We denote by $L^{t,\varepsilon}$ the differential operator defined by the right-hand side.

Summing up the above, we have

Lemma 2.1. *Let* $\Psi(x)=\exp(\phi(0), x^k e_k)$ *and* $u_\varepsilon(x)=P_{\phi(x)}\{\rho(X_t, \phi_t)\leqq\varepsilon,$ *for any* $t\in[0, T]\}$. *Define* $u_1^\varepsilon(t, x)$ *as a unique solution of*

$$\frac{\partial u_1^\varepsilon}{\partial t}=L^{t,\varepsilon}u_1^\varepsilon \quad on \quad [0, T]\times D \tag{2.3}$$

$$u_1^\varepsilon(0, x)=\exp\left\{\varepsilon\sum_{k=1}^{d} b^k(T, 0)x^k\right\} \quad on \quad D$$

$$u_1^\varepsilon(t, \cdot)|_{\partial D}=0\,.$$

Then we have an identity,

$$u_\varepsilon(0)\,(=\mu_\varepsilon^q(\phi))=u_1^\varepsilon(T, 0)\exp\left(-\frac{\lambda_1}{\varepsilon^2}T\right).$$

Therefore we can reduce our problem to prove that $u_1^\varepsilon(T, 0)$ converges to $f_1(0)\int_D f_1(x)dx\exp\left(\int_0^T L(\phi_t, \dot{\phi}_t)dt\right)$ as $\varepsilon\downarrow 0$. The main difficulty here is that the diffusion coefficients change as ε tends to 0, and we tackle this problem in the next section.

3. Convergence of u_1^ε

In this section, we first consider the convergence of u_1^ε in $L^2([0, T]\times D)$ and then using this L^2-convergence, we show the pointwise convergence of $u_1^\varepsilon(T, 0)$ by averaging $u_1^\varepsilon(T, x)$ over the unit sphere. Following is a well-known result in the theory of partial differential equations (c.f. [6] p. 238).

Lemma 3.1. *Let* $\{a^{ij}(x), b^j(x), c(x)\}$ *be bounded functions on* D *with a bound* K *and define an operator*

$$L=a^{ij}(x)\frac{\partial^2}{\partial x^i\partial x^j}+b^i(x)\frac{\partial}{\partial x^i}+c(x)$$

Then $\|L\Delta^{-1}u\|_{L^2(D)}\leqq$*constant* $\|u\|_{L^2(D)}$ *for* $u\in C_0^\infty(D)$ *where this constant depends only on* K *and* Δ^{-1} *is the inverse of Laplacian with the Dirichlet condition on* ∂D *(i.e,* $u|_{\partial D}=0$*).*

Now we introduce $u_2^\varepsilon(t, x)$ as the solution of the following equation

$$\frac{\partial u_2^\varepsilon}{\partial t}=\frac{1}{\varepsilon^2}\left(\frac{1}{2}\Delta+\lambda_1\right)u_2^\varepsilon \quad \text{on} \quad [0, T]\times D$$

$$u_2^\varepsilon|_{\partial D}=0\,,\; u_2^\varepsilon(0, x)=\exp\left\{\varepsilon\sum_{k=1}^{d}\hat{b}^k(T, 0)x^k\right\}$$

and consider (2.3) as a perturbation of the above equation. Since u_1^ε satisfies (2.3), we see from the trivial identity

$$\frac{\partial u_1^\varepsilon}{\partial t}=\frac{1}{\varepsilon^2}\left(\frac{1}{2}\Delta+\lambda_1\right)u_1^\varepsilon+\left\{L^{t,\varepsilon}-\frac{1}{\varepsilon^2}\left(\frac{1}{2}\Delta+\lambda_1\right)\right\}u_1^\varepsilon,$$

that it can be obtained as the unique solution of

(3.1) $$u_1^\varepsilon(t, x)-u_2^\varepsilon(t, x)=\int_0^t\int_D \hat{p}\Big(\frac{t-s}{\varepsilon^2}, x, y\Big)\Big\{L^{s,\varepsilon}-\frac{1}{\varepsilon^2}\Big(\frac{1}{2}\Delta+\lambda_1\Big)\Big\}u_1^\varepsilon(s, y)dyds,$$

where $\hat{p}(t, x, y)=\exp(\lambda_1 t)p(t, x, y)$ and $p(t, x, y)$ is the fundamental solution of $\dfrac{\partial u}{\partial t}=\dfrac{1}{2}\Delta u$ on D, $u(t, \cdot)|_{\partial D}=0$.

we prove a lemma which gives an asymptotic of solutions depending on a small parameter ε under some conditions on the coefficients, which are satisfied in the present case. Let $Z^{t,\varepsilon}=L^{t,\varepsilon}-\frac{1}{\varepsilon^2}\Big(\frac{1}{2}\Delta+\lambda_1\Big)$ be represented as

$$Z^{t,\varepsilon}=a^{ij}(t, x, \varepsilon)\frac{\partial^2}{\partial x^i\partial x^j}+b^i(t, x, \varepsilon)\frac{\partial}{\partial x^i}+c(t, x, \varepsilon).$$

Then it follows from Lemma 1.1. and 1.2. that

(3.2) $$D^\alpha(a^{ij}(t, x, \varepsilon))=D^\alpha\Big(-\frac{1}{6}R_{ikjl}(T-t, 0)x^k x^l\Big)+0(\varepsilon)$$

for each $\alpha(|\alpha|\leqq 2)$

$$D^\alpha(b^i(t, x, \varepsilon))=D^\alpha\Big(-\frac{\partial\hat{b}^i}{\partial x^j}(T-t, 0)x^j\Big)+0(\varepsilon)$$

for each $\alpha(|\alpha|\leqq 1)$

$$c(t, x, \varepsilon)=-\frac{1}{2}\sum_{k=1}^d b^k(T-t, 0)^2+0(\varepsilon),$$

holds uniformly with respect to $(t, x)\in[0, T]\times D$ as $\varepsilon\downarrow 0$.

Lemma 3.2. *The solution u_1^ε of* (2.3) *converges to u_1 in $L^2([0, T]\times D)$, where u_1 is defined by the equation*

(3.3) $$u_1(t, x)=u_2(t, x)+\int_0^t\int_D f_1(x)f_1(y)Z^{s,0}u_1(s, y)dyds$$

Here

$$u_2(t, x)=f_1(x)\int_D f_1(x)dx,$$

and

$$Z^{t,0}=-\frac{1}{6}R_{ikjl}(T-t, 0)x^k x^l\frac{\partial^2}{\partial x^i\partial x^j}+\frac{\partial\hat{b}^i}{\partial x^j}(T-t, 0)x^j\frac{\partial}{\partial x^i}$$
$$-\frac{1}{2}\sum_{k=1}^d\hat{b}^k(T-t, 0)^2$$

Proof. For any smooth function f, let

$$G^\varepsilon f(t, x)=\int_0^t\int_D\hat{p}\Big(\frac{t-s}{\varepsilon^2}, x, z\Big)Z^{s,\varepsilon}f(s, y)dyds$$

and

$$Gf(t, x)=\int_0^t\int_D f_1(x)f_1(y)Z^{s,0}f(s, y)dyds.$$

Since the kernel $\hat{p}$ can be expanded by the normalized eigenfunctions $\{f_k\}$ of $-\frac{1}{2}\Delta$ (see § 0.). we have

$$\hat{p}(t, x, y)=f_1(x)f_1(y)+\sum_{k=2}^{\infty}\exp(\lambda_1-\lambda_k)t\, f_k(x)f_k(y).$$

Therefore

$$\begin{aligned}(G^\varepsilon-G)f(t, x)&=\int_0^t\int_D(Z^{s,\varepsilon}-Z^{s,0})^{*1}(f_1(y))f(s, y)dyds\, f_1(x)\\&\quad+\sum_{k=2}^{\infty}f_k(x)\int_0^t\exp\Big(\frac{\lambda_1-\lambda_k}{\varepsilon^2}(t-s)\Big)ds\int_D(Z^{s,\varepsilon})^*f_k(y)f(s, y)dy\\&=I_1^\varepsilon(t, x)+I_2^\varepsilon(t, x).\end{aligned}$$

First we estimate I_1^ε. Since f_1 is smooth, noting the estimate (3.2), we have as $\varepsilon\downarrow 0$,

$$(Z^{s,\varepsilon}-Z^{s,0})^*f_1(y)=0(\varepsilon)\qquad\text{uniformly in}\quad (s, y)\in[0, T]\times D.$$

So it follows obviously that as $\varepsilon\downarrow 0$,

$$\|I_1^\varepsilon\|_{L^2([0,T]\times D)}=0(\varepsilon)\|f\|_{L^2([0,T]\times D)}.$$

On the other hand, f_k is the eigenfunction of $-\frac{1}{2}\Delta$ corresponding to λ_k, we see

$$\begin{aligned}\int_D(Z^{s,\varepsilon})^*\{f_k(y)\}f(s, y)dy&=\int_D(Z^{s,\varepsilon})^*\{\Delta^{-1}\Delta f_k(y)\}f(s, y)dy\\&=-2\lambda_k\int_D\{(Z^{s,\varepsilon})^*\Delta^{-1}f_k(y)\}f(s, y)dy\end{aligned}$$

Setting

$$e_k(s)=2\lambda_k I_{[0,T]}(s)\exp\Big(\frac{\lambda_1-\lambda_k}{\varepsilon^2}s\Big),$$

$$g_k(s)=I_{[0,T]}(s)\int_D\{(Z^{s,\varepsilon})^*\Delta^{-1}f_k(y)\}f(s, y)dy,$$

we have

$$I_2^\varepsilon(t, x)=\sum_{k=2}^{\infty}f_k(x)(e_k*g_k)(t)$$

where e_k*g_k is the usual convolution of e_k and g_k on R. The orthonormality of $\{f_k\}$ in $L^2(D)$ implies

$$\|I_2^\varepsilon\|_{L^2([0,T]\times D)}^2=\sum_{k=2}^{\infty}\|e_k*g_k\|_{L^2(R)}^2.$$

Note an obvious estimate

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1) Generally for a differential operator L, L^* denotes the formal adjoint operator of L with respect to Lebesgue measure.

$$\|e_k\|_{L^1(R)}=\frac{\lambda_k\varepsilon^2}{\lambda_k-\lambda_1}\left(1-\exp\left(\frac{\lambda_1-\lambda_k}{\varepsilon^2}T\right)\right)$$
$$\leqq C_1\varepsilon^2$$

where C_1 is a constant independent of k and ε. Introduce the operator $H^{s,\varepsilon}=(Z^{s,\varepsilon})^*\Delta^{-1}$. Then from Lemma 3.1. and the estimate (3.2), it follows that there exists a constant C_2 independent of s and ε such that

$$\|H^{s,\varepsilon}\|_{L^2(D)\to L^2(D)}\leqq C_2$$

Therefore we have

$$\sum_{k=2}^{\infty}\|g_k\|^2_{L^2(R)}=\sum_{k=2}^{\infty}\int_0^T ds\left|\int_D H^{s,\varepsilon}f_k(y)f(s,y)dy\right|^2$$
$$=\int_0^T ds\sum_{k=2}^{\infty}\left|\int_D f_k(y)(H^{s,\varepsilon})^*f(s,y)dy\right|^2$$
$$\leqq\int_0^T ds\|(H^{s,\varepsilon})^*f(s,\cdot)\|_{L^2(D)}$$
$$=(C_2)^2\|f\|^2_{L^2([0,T]\times D)}$$

These relations show

$$\|I_2^\varepsilon\|^2_{L^2([0,T]\times D)}\leqq\sum_{k=2}^{\infty}\|e_k\|^2_{L^1(R)}\|g_k\|^2_{L^2(R)}$$
$$\leqq(C_1)^2\varepsilon^4\sum_{k=2}^{\infty}\|g_k\|^2_{L^2(R)}$$
$$\leqq(C_1C_2)^2\varepsilon^4\|f\|^2_{L^2([0,T]\times D)}.$$

Consequently we have

$$\|(G^\varepsilon-G)f\|_{L^2([0,T]\times D)}\leqq o(1)\|f\|_{L^2([0,T]\times D)},$$

and this implies the convergence of G^ε to G in $L^2([0,T]\times D)$-operator norm. Obviously $(I-G)^{-1}$ exists as a bounded operator on $L^2([0,T]\times D)$, and so does $(I-G^\varepsilon)^{-1}$ for sufficiently small ε. Consequently $(I-G^\varepsilon)^{-1}$ converges to $(I-G)^{-1}$ in the operator norm. Clearly we have

(3.4) $$u_1^\varepsilon=(I-G^\varepsilon)^{-1}(u_2^\varepsilon)\quad\text{and}\quad u_1=(I-G)^{-1}(u_2).$$

It is not difficult to see that u_2^ε converges to u_2 as $\varepsilon\downarrow 0$ in $L^2([0,T]\times D)$. Now we can conclude that $u_1^\varepsilon\to u_1$ in $L^2([0,T]\times D)$.

In order to prove the convergence of $u_1^\varepsilon(0)$, we discuss the equation (3.1) using polar coordinates (r,θ). Let

(3.5) $$\bar{f}(x)=\int_{\partial D}f(|x|\theta)d\theta\qquad\text{for}\quad f\in C(\bar{D}),$$

where $d\theta$ is the normalized uniform measure on ∂D. Since $\bar{f}$ is rotation invariant, we can regard $\bar{f}$ as a function of radius $r=|x|$, which we denote by the same notation $\bar{f}$. As is well-known (c.f. Spivak [8]), if Δ_M is represented in geodesic polar coordinates, the coefficient of $\frac{\partial^2}{\partial r^2}$ is equal to that of $\Delta=\Delta_{R^d}$. Therefore $Z^{t,\varepsilon}$ has the following form:

$$Z^{t,\varepsilon}=A(t, r, \theta, \varepsilon)\frac{\partial}{\partial r}+L_\theta^{t,r,\varepsilon}+C(t, r, \theta, \varepsilon),$$

where $L_\theta^{t,r,\varepsilon}$ is a second order differential operator on ∂D such that $L_\theta^{t,r,\varepsilon}(1)=0$.

Lemma 3.3. *$u_1^\varepsilon(t, 0) \longrightarrow u_1(t, 0)$ as $\varepsilon \downarrow 0$ for any $0<t\leqq T$.*

Proof. Decompose $Z^{t,\varepsilon}$ into two parts;

$$D_r^{t,\varepsilon}=A(t, r, \theta, \varepsilon)\frac{\partial}{\partial r}+C(t, r, \theta, \varepsilon)+(L_\theta^{t,r,\varepsilon})^{*}(1)^{2)}$$

$$D_\theta^{t,\varepsilon}=L_\theta^{t,r,\varepsilon}-(L_\theta^{t,r,\varepsilon})^*(1).$$

Further let

$$G_r^\varepsilon f(t, x)=\int_0^t\int_D \hat{p}\Big(\frac{t-s}{\varepsilon^2}, x, y\Big)(D_r^{s,\varepsilon})f(s, y)dyds,$$

$$G_\theta^\varepsilon f(t, x)=\int_0^t\int_D \hat{p}\Big(\frac{t-s}{\varepsilon^2}, x, y\Big)(D_\theta^{s,\varepsilon})f(s, y)dyds$$

for any smooth function f. Then the equation (3.1) can be written as

$$u_1^\varepsilon=u_2^\varepsilon+G_r^\varepsilon u_1+G_\theta^\varepsilon u_1^\varepsilon. \tag{3.6}$$

Step 1. *For any smooth function f, $\overline{G_\theta^\varepsilon f}(t, r)=0$ identically.*

Indeed,

$$\overline{G_\theta^\varepsilon f}(t, r)=\int_{\partial D}d\theta'\int_0^t ds\int_0^1\int_{\partial D}\hat{p}\Big(\frac{t-s}{\varepsilon^2}, r'\theta', r\theta\Big)(D_\theta^{s,\varepsilon}f)(s, r\theta)r^{n-1}drd\theta$$

$$=\int_0^t ds\int_0^1\int_{\partial D}\Big\{\int_{\partial D}\hat{p}\Big(\frac{t-s}{2}, r'\theta', r\theta\Big)d\theta'\Big\}(D_\theta^{s,\varepsilon}f)(s, r\theta)r^{n-1}drd\theta.$$

Since $\hat{p}(t, x, y)$ is rotation invariant, $\int_{\partial D}\hat{p}\Big(\frac{t-s}{\varepsilon^2}, r'\theta', r\theta\Big)d\theta'$ is independent of θ. Moreover, by the definition of the (*)-operation, we see

$$\int_{\partial D}D_\theta^{s,\varepsilon}f(s, r\theta)d\theta=\int_{\partial D}L_\theta^{s,r,\varepsilon}f(s, r\theta)d\theta-\int_{\partial D}(L_\theta^{s,r,\varepsilon})^*(1)f(s, r\theta)d\theta=0.$$

Therefore we have

$$\overline{G_\theta^\varepsilon f}(t, r)=\int_0^t ds\int_0^1\Big\{\int_{\partial D}\int_{\partial D}\hat{p}\Big(\frac{t-s}{\varepsilon^2}, r'\theta' r\theta\Big)d\theta' d\theta\Big\}\Big\{\int_{\partial D}D_\theta^{s,\varepsilon}f(s, r\theta)d\theta\Big\}r^{n-1}dr$$

$$=0.$$

Let

2) *denotes the adjoint operator with respect to $d\theta$.

$$I_1(t)=1 \quad \text{if} \quad t\geqq 1$$
$$=0 \quad \text{if} \quad 0\leqq t<1,$$
$$I_2(t)=0 \quad \text{if} \quad t\geqq 1$$
$$=1 \quad \text{if} \quad 0\leqq t<1$$

and decompose G_r^ε into

$$G_r^{i,\varepsilon}f(t, x)=\int_0^t\int_D I_i\Big(\frac{t-s}{\varepsilon^2}\Big)\hat{p}\Big(\frac{t-s}{\varepsilon^2}, x, y\Big)(D_r^{s,\varepsilon}f)(s, y)dyds$$

for $i=1, 2$.

Then we want to show

Step 2. *There exists a constant* C_3 (*independent of* ε) *such that*

$$\|G_r^{1,\varepsilon}\|_{L^2([0,T]\times D\to C([0,T]\times\bar{D})}\leqq C_3 \tag{3.7}$$

holds. Also we have

$$\|G_r^{1,\varepsilon}-G_r^{1,\varepsilon'}\|_{L^2([0,T]\times D)\to C([0,T]\times\bar{D})}\longrightarrow 0 \tag{3.8}$$

as ε *and* ε' *tends to* 0.

First note that for any multi-index α, there exists a constant C_α such that

$$|D^\alpha\hat{p}(t, x, y)|\leqq C_\alpha \qquad \text{for any} \quad (t, x, y)\in[1, \infty)\times\bar{D}\times\bar{D}. \tag{3.9}$$

(Recall $\hat{p}(t, x, y)=\exp\lambda_1 t\, p(t, x, y)$.) Indeed $\hat{p}$ can be expanded as

$$\hat{p}(t, x, y)=\sum_{k=1}^{\infty}\exp-(\lambda_k-\lambda_1)t\, f_k(x)f_k(y).$$

Sobolev's lemma implies that $|D^\alpha f_k(x)|\leqq$constant $\lambda_k^{m_\alpha}$ holds for some integer m_α. This together with $\lambda_k-\lambda_1\geqq 0$ shows (3.9). Therefore there exists a constant C_4 such that for any $\varepsilon>0$, $(x, y)\in D\times D$ and $t\geqq s\geqq 0$,

$$\Big|I_1\Big(\frac{t-s}{\varepsilon^2}\Big)(D_r^{s,\varepsilon})^*p\Big(\frac{t-s}{\varepsilon^2}, x, y\Big)\Big|\leqq C_4 \tag{3.10}$$

holds, where * denotes the formal adjoint operator. From (3.10), (3.7) follows immediately. The proof of (3.8) is similar so it is omitted.

Note also that we have estimates for any $(t, x, y)\in(0, 1]\times\bar{D}\times\bar{D}$, $1\leqq j\leqq d$

$$\Big|\frac{\partial p}{\partial x^j}(t, x, y)\Big|\leqq C^5 t^{-(d+1/2)}\exp\Big(-\frac{|x-y|^2}{C_5 t}\Big), \tag{3.11}$$

with some constant C_5 (c.f. [5]), Let $g(t, x, y)$ be a Gauss kernel in R^d with covariance $C_5 I$, namely

$$g(t, x, y)=(2\pi tC_5)^{-(d/2)}\exp\Big(-\frac{|x-y|^2}{C_5 t}\Big), \tag{3.12}$$

and define an integral operator G_g^ε by

$$G_g^\varepsilon f(t, x)=\int_0^t I_2\Big(\frac{t-s}{\varepsilon^2}\Big)(t-s)^{-(1/2)}ds\int_D g\Big(\frac{t-s}{\varepsilon^2}, x, y\Big)f(s, y)dy$$

Step 3. *There exists a constant* C_6 *such that for any smooth nonnegative function* f *vanishing at* ∂D

(3.13) $$|\overline{G_r^{2,\varepsilon}f}(t, r)| \leqq C_6 \varepsilon G_g^{\varepsilon} f(t, r)$$

holds for any $\varepsilon>0$ *and* $t \in (0. T]$.

Since f vanishes at ∂D, we see

$$G_r^{2,\varepsilon} f(t, x)=\int_0^t \int_D I_2\left(\frac{t-s}{\varepsilon^2}\right)(D_r^{s,\varepsilon})^* \hat{p}\left(\frac{t-s}{\varepsilon^2}, x, y\right) f(s, y) dy ds .$$

Moreover (3.11) and (3.12) imply

$$\left|I_2\left(\frac{t-s}{\varepsilon^2}\right)(D_r^{s,\varepsilon})^* p\left(\frac{t-s}{\varepsilon^2}, x, y\right)\right| \leqq C_7 \varepsilon I_2\left(\frac{t-s}{\varepsilon_2}\right)(t-s)^{-(1/2)} g\left(\frac{t-s}{\varepsilon^2}, x, y\right)$$

with some constant C_7 for any ε, $t \geqq s$, (x, y). From this estimate and the rotation invariance of g, (3.13) follows easily.

Step 4. *There exists some constant* C_8 *such that*

$$|\overline{u_1^{\varepsilon}}(t, r)| \leqq C_8 \quad \textit{for any} \quad (t, r) \in [0, T] \times [0, 1] .$$

First note an identity

$$u_1^{\varepsilon}=u_2^{\varepsilon}+G_r^{1,\varepsilon} u_1^{\varepsilon}+G_r^{2,\varepsilon} u_1^{\varepsilon}+G_{\theta}^{\varepsilon} u_1^{\varepsilon} .$$

Step 1 implies, averaging both sides in θ,

(3.14) $$\overline{u_1^{\varepsilon}}=\overline{u_2^{\varepsilon}}+\overline{G_r^{1,\varepsilon} u_1^{\varepsilon}}+\overline{G_r^{2,\varepsilon} u_1^{\varepsilon}}$$

Since $\{u_1^{\varepsilon}\}$ is a bounded set in $L^2([0. T] \times D)$, it follows from Step 2 that there exists some constant $C_9>0$ such that

$$\overline{u_1^{\varepsilon}} \leqq \overline{u_2^{\varepsilon}}+C_9+\overline{G_r^{2,\varepsilon} u_1^{\varepsilon}} .$$

By the positivity of u_1^{ε} and Step 3, we have

(3.15) $$\overline{u_1^{\varepsilon}} \leqq \overline{u_2^{\varepsilon}}+C_9+C_6 \varepsilon G_g^{\varepsilon}(\overline{u_1^{\varepsilon}}) .$$

Noting the positivity of the operator G_g^{ε}, we get by the N-times iteration of (3.15)

(3.16) $$u_1^{\varepsilon} \leqq \sum_{k=0}^N (C_6 \varepsilon G_g^{\varepsilon})^k (\overline{u_2^{\varepsilon}}+C_9)+(C_6 \varepsilon G_g^{\varepsilon})^N \overline{u_1^{\varepsilon}} .$$

Here, it is clear that there exists a constant C_{10} such that for any $\varepsilon>0$

$$\| G_g^{\varepsilon} \|_{C([0, T] \times \bar{D}) \to C([0, T]) \times \bar{D})} \leqq C_{10} .$$

Let us fix ε such that $\varepsilon C_6 C_{10}<1$. Then when we get

(3.17) $$\lim_{N \to \infty} (C_6 \varepsilon G_g^{\varepsilon})^N \overline{u_1^{\varepsilon}}=0 .$$

(3.16) and (3.17) imply

(3.18)
$$\overline{u_1^\varepsilon} \leqq \overline{\lim_{N\to\infty}} \sum_{k=0}^{N} (C_6 \varepsilon G_g^\varepsilon)^k (\overline{u_2^\varepsilon} + C_9)$$
$$\leqq \frac{1}{1-\varepsilon C_6 C_{10}} \|\overline{u_2^\varepsilon} + C_9\|_{C([0,T]\times\bar{D})} .$$

It is clear that $u_2^\varepsilon \longrightarrow u_2$ everywhere as $\varepsilon \downarrow 0$ boundedly. This and (3.18) conclude the assertion of Step 4.

Step 5. Combining above consideration, we can show the lemma. By (3.14), for each fixed $\delta>0$,

(3.19)
$$\|\overline{u_1^\varepsilon} - \overline{u_1^{\varepsilon'}}\|_{C([\delta,T]\times\bar{D})}$$
$$= \|\overline{u_2^\varepsilon} - \overline{u_2^{\varepsilon'}}\|_{C([\delta,T]\times\bar{D})} + \|\overline{G_r^{1,\varepsilon} u_1^\varepsilon} - \overline{G_r^{1,\varepsilon'} u_2^{\varepsilon'}}\|_{C([\delta,T]\times\bar{D})}$$
$$+ \|\overline{G_r^{2,\varepsilon} u_1^\varepsilon} - \overline{G_r^{2\varepsilon'} u_1^{\varepsilon'}}\|_{C([\delta,T]\times\bar{E})} \qquad \text{for any} \quad 0<\varepsilon'<\varepsilon .$$

From Lemma 3.2. and Step 2 it follows that

(3.20)
$$\|\overline{G_r^{1,\varepsilon} u_1^\varepsilon} - \overline{G_r^{1,\varepsilon'} u_1^{\varepsilon'}}\|_{C([\delta,T]\times\bar{D})} = o(1) \qquad \text{as} \quad \varepsilon \downarrow 0 .$$

On the other hand, Step 4 and the inequality (3.13) imply

(3.21)
$$\|\overline{G_r^{2,\varepsilon} u_1^\varepsilon}\|_{C([\delta,T]\times\bar{D})} \leqq C_6 C_8 C_{10} \varepsilon .$$

However it is easily seen that $u_2^\varepsilon \longrightarrow u_2$ in $C([\delta, T]\times\bar{D})$, hence combining (3.20) and (3.21), we can conclude that $\{\overline{u_1^\varepsilon}\}$ is Cauchy in $C([\delta, T]\times\bar{D})$ for any $\delta>0$. Sinse $\overline{u_1^\varepsilon}$ converges to $\overline{u_1}$ in $L^2([0, T]\times D)$ by Lemma 3.2., $\overline{u_1^\varepsilon}$ has the limit $\overline{u_1}$ in $C([\delta, T]\times\bar{D})$. Observing $\overline{u_1^\varepsilon}(t, 0)=u_1(t, 0)$ and $\overline{u_1}(t, 0)=u_1(t, 0)$ (See the defition (3.5).), we can completete the proof.

4. A proof of the thorem

In view of Lemma 2.1. and Lemma 3.3, all we need for the proof of the theorem is to show that

$$u_1(T, 0) = f_1(0)\int_D f_1(x)dx \exp\left(\int_0^T L(\phi_t, \dot{\phi}_t)dt\right).$$

By Lemma 3.2. u_1 satisfies the following equation,

$$u_1(t, x) = Cf_1(x) + \int_0^t \int_D f_1(x) f_1(y) (Z^{s,0}) u_1(s, y) dy ds$$

where $C=\int_D f_1(y)dy$,

$$Z^{s,0} = -\frac{1}{6} R_{ikjl}(T-s, 0) x^k x^l \frac{\partial^2}{\partial x^i \partial x^j} + \frac{\partial \hat{b}^i}{\partial x^j}(T-t, 0) x^j \frac{\partial}{\partial x^i}$$
$$-\frac{1}{2}\sum_{k=1}^{d} \hat{b}^k(T-t, 0)^2 \quad \text{and}$$

$$\hat{b}^k(t, x) = b^k(t, x) - \frac{1}{2} g^{ij}(t, x) \Gamma_{ij}^k(t, x) - \dot{\phi}^k(t) + \varepsilon^k(t, x) .$$

So we can put $u_1(t, x)=C(t)f_1(x)$ for some function of t, and $C(t)$ satisfies

$$C(t)=C+\int_0^t C(s)\int_D f_1(y)\,(Z^{s,0})f_1(y)dyds\,. \tag{4.1}$$

In order to calculate the right-hand side of (4.1), we prepare the following identities:

$$\int_D y^i f_1(y)\frac{\partial f_1(y)}{\partial y^j}dy=-\frac{1}{2}\delta^{ij} \tag{4.2}$$

$$\int_D y^k y^l f_1(y)\frac{\partial^2 f_1(y)}{\partial y^i \partial y^j}dy=-\frac{1}{2}(\delta^{ik}\delta^{jl}+\delta^{il}\delta^{kj})+a^{ijkl} \tag{4.3}$$

where $\{a^{ijkl}\}$ is invariant under any permutation of i, j, k, l. Indeed (4.2) is easily obtained by the integration by parts and the boundary condition $f_1|_{\partial D}=0$. Similarly, we get

$$\int_D y^k y^l f_1(y)\frac{\partial^2 f_1(y)}{\partial y^i \partial y^j}dy=\frac{1}{2}(\delta^{ik}\delta^{jl}+\delta^{il}\delta^{kj})-\int_D y^k y^l \frac{\partial f_1(y)}{\partial y^i}\frac{\partial f_1(y)}{\partial y^j}dy\,.$$

However f_1 is rotation invariant. So putting $f_1(y)=f_1(r)$ we have

$$\int_D y^k y^l \frac{\partial f_1(y)}{\partial y^i}\frac{\partial f_1(y)}{\partial y^j}dy=\int_D y^k y^l f_1'(r)\frac{y^i}{r}f_1'(y)\frac{y^j}{r}dy$$

$$=\text{constant}\int_0^1 r^{n+1}f_1'(r)^2 dr\int_{\partial D}\theta^i\theta^j\theta^k\theta^l d\theta\,,$$

whish completes the proof of (4.3).

We return to the caluculation of (4.1). First we compute the second order term of $Z^{s,0}$:

$$-\frac{1}{6}R_{ikjl}(T-s,\,0)\int_D y^k y^l f_1(y)\frac{\partial^2 f_1(y)}{\partial y^i \partial y^j}dy$$

$$=-\frac{1}{6}R_{ikjl}(T-s,\,0)\Big\{\frac{1}{2}\delta^{ik}\delta^{jl}+\frac{1}{2}\delta^{il}\delta^{jk}+a^{ijkl}\Big\}$$

$$=-\frac{1}{12}R(\phi(T-s))\,.$$

Here we have used the definition of the scalar curvature $R=R_{ikjl}g^{il}g^{kj}$ ($=R_{ikjl}\delta^{il}\delta^{kj}$ in this case) and the fact that $R_{ikjl}=-R_{kijl}$.

Next we proceed to the calculation of the first order term of $Z^{s,0}$. From Lemma 1.1 and 1.2 it follows that

$$\frac{\partial \hat{b}^i}{\partial x^j}(t,\,0)=\frac{\partial b^i}{\partial x^j}(t,\,0)-\frac{1}{6}\{R_{kiul}(t)\delta^{uj}+R_{iklu}(t)\delta^{uj}\}\delta^{kl}\,.$$

Therefore by combining this with (4.2), we have

$$\frac{\partial \hat{b}^i}{\partial x^j}(T-s,\,0)\int_D y^i f_1(y)\frac{\partial f_1(y)}{\partial y^j}dy$$

$$=-\frac{1}{2}\operatorname{div}(b)(\phi(T-s))+\frac{1}{6}R(\phi(T-s))\,.$$

Finally, noting that $\hat{b}^k(t, 0)=b^k(t, 0)-\dot{\phi}^k(t)$, the 0-th order term becomes

$$-\frac{1}{2}\sum_{k=1}^{d}\hat{b}^k(t-s, 0)^2\int_D f_1(y)^2 dy=-\frac{1}{2}|b(\phi(T-s))-\dot{\phi}(T-s)|^2_{\phi(T-s)}.$$

Consequently, solving the equation (4.1), we get

$$C(T)=C\exp\Big(\int_0^T\Big\{-\frac{1}{2}|b(\phi(s))-\dot{\phi}(s)|^2_{\phi(s)}-\frac{1}{2}\operatorname{div}(b)(\phi(s))+\frac{1}{12}R(\phi(s))\Big\}ds.$$

Since $u_1(T, 0)=C(T)f_1(0)$, we finish the proof of the theorem.

5. A remark

First we remark here that even very small perturbation of the Riemannian metric ρ changes the exponent L of the right-hand side. Namely, let $C_\varepsilon(p)$ be a smooth real valued function for each $\varepsilon>0$ on M, satisfying

$$C_\varepsilon(p)=1+a(p)\varepsilon^2+0(\varepsilon^3) \qquad \text{as} \quad \varepsilon\downarrow 0$$

uniformly on each compact set in M including every its derivative of at least second order. Then we have the following asymptotic behavior;

$$P_q\{\rho(X_t, \phi_t)\leqq\varepsilon C_\varepsilon(\phi_t) \quad \text{for any} \quad t\in[0, T]\}$$
$$=f_1(0)\int_D f_1(y)dy\exp\Big(-\frac{\lambda_1 T}{\varepsilon^2}+\int_0^T\hat{L}(\phi_s, \dot{\phi}_s)ds+o(1)\Big) \quad \text{as} \quad \varepsilon\downarrow 0,$$

where $\hat{L}(p, v)=L(p, v)+2\lambda_1 a(p)$.

This can be proved by the same method as in the case of $a(p)\equiv 0$, if we use the fundamental solution for $\Big\{\frac{1}{2}-\varepsilon^2 a(\phi(T-t))\Big\}\Delta$ in place of that of $\frac{1}{2}\Delta$.

If, in particular, $C_\varepsilon(x)$ is chosen so that $\frac{V_\varepsilon(q)}{V_\varepsilon(p)}=1+0(\varepsilon^3)$ at each point $p, q \in M$, where $V_\varepsilon(x)$ denotes the Riemannian volume of the ball around x with radius $\varepsilon C_\varepsilon(x)$, then $a(p)=\frac{1}{6d(d+1)}R(p)$. Therefore this procedure cannot make the scalar ourvature term vanish.

Finally, we remark that if X_t starts with a smooth initial distribution $\eta(x)dx$ (dx: the Riemannian measure of M), the following estimate holds

$$\mu_\varepsilon^\eta(\phi)=\varepsilon^d\eta(\phi(0))\Big\{\int_D f_1(y)dy\Big\}^2\exp\Big(\int_0^T L(\phi_t, \dot{\phi}_t)dt\Big)\exp\Big(-\frac{\lambda_1 T}{\varepsilon^2}\Big).$$

This estimate is also easily obtained by the same method.

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